Sensitivity Analysis for the Configuration of a Multi-Purpose Machines Workshop

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Abstract: In this paper, the multi-purpose machines problem denoted $\text{MPM|split|C}_{\text{max}}$ is considered. In such a problem, each machine is qualified to process a subset of product types but not necessarily all the product types. The set of the qualifications machine/product represents the configuration of the multi-purpose machines workshop. Uncertainties on the demand are considered and a question related to sensitivity analysis is answered to find the neighbourhood of a forecasted demand in which the demands are completed by the configuration before a given deadline. The stability radius of a configuration is moreover computed.

Keywords: production systems, scheduling, configuration management, uncertainty, sensitivity analysis

1. INTRODUCTION

Many industrial production systems can be modelled as a Multi-Purpose Machines (MPM) problem (Brucker et al., 1997). In such a model each machine is able to process a subset of products but not necessarily all the products. To be able to process one type of products, the machine must undergo a specific adjustment. When a machine is set up to process one type of products, the machine is said to be qualified for this product and all the qualifications machine/product represent the configuration of the workshop.

A basic problem (considering that all the data are perfectly known and constant) is to find the configuration which minimises the setup costs while ensuring that a load-balanced production plan exists. This basic problem has already been studied in (Aubry et al., 2008). However, the authors have proved that enforcing a strict load-balancing is only relevant for uniform machines.

That is the reason why we propose to consider in this paper a more classical criterion that remains relevant for unrelated machines: the makespan. Moreover, in actual industrial context, considering that all the data are perfectly known and constant is becoming hard to justify. In this context the demand is variable and the completion-time is naturally sensitive to these perturbations.

This paper precisely focuses on the sensitivity analysis (Hall and Posner, 2004) of the configuration of a MPM workshop under demand uncertainties to evaluate which magnitude of perturbations can be dealt with by the configuration such that the completion-time remains less than a given deadline.

In section 2, we present the problem with the proposed model and the type of disturbances that are considered.

Then, in section 3, sensitivity analysis is presented and the studied sensitivity analysis problem is defined. The solving approach is highlighted in section 4. A stability radius is defined and calculated in section 5. Finally, the section 6 aims at illustrating all the theoretical results through an academic example.

2. A MODEL FOR THE MPM WORKSHOP AND THE ATTACHED SCHEDULING PROBLEM

2.1 The MPM workshop

In a multi-purpose machines (MPM) workshop, $m$ machines and $n$ types of products are considered, but all the machines cannot necessarily process all the product types: e.g., any machine can only process a subset of product types. The set $\{1, \ldots, m\}$ of the machines is denoted $J$. The set $\{1, \ldots, n\}$ of the product types is denoted $I$.

The machines. Multi-purpose machines are classically modelled as follows (Brucker et al., 1997): every product type $i$ can be processed by a set $O_i$ of machines, where $O_i$ is a subset of $J$. In this paper, we use a slightly different but equivalent model. The workshop machines are regarded as parallel machines and some technological constraints are used to prevent assigning products to a machine that cannot process them for technological reasons. All the technological constraints are modelled by a $n$-by-$m$ binary technological matrix $T$. Moreover, to be able to process one type of products, the machine must be qualified for the product type by undergoing a specific adjustment. The set of qualifications is modelled by a $n$-by-$m$ binary configuration matrix $Q$. $Q$ is defined as follows: $Q(i,j) = 1$ if machine $j$ is qualified to process product type $i$ (there is no qualification constraint between machine $j$ and product type $i$), otherwise $Q(i,j) = 0$. A non-qualification may be the result of:
(1) Technology: for some technological or physical rea-
sons, machine $j$ cannot process any product of type
$i$. This information is given by $T$ when $T(i,j) = 0$
$(T(i,j) = 0 \Rightarrow Q(i,j) = 0)$.
(2) Cost control: for financial reasons, all the machines
may not be qualified for all the product types because
this would be too expensive, and often useless (Aubry
et al., 2008).

Note that the multi-purpose machines problem generalises
classical parallel machine problems. Parallel machines
are multi-purpose machines with no technological con-
straint and no non-qualification: e.g., $T(i,j) = Q(i,j) = 1$,
$\forall (i,j) \in I \times J$.

To process the different product types, machines may have
different speeds. A $n$-by-$m$ real speed matrix $V$ is defined.
$V(i,j)$ is the number of products of type $i$ that machine
$j$ is able to process during one unit of time. Machines are
assumed to be unrelated.

The demand. The product types are the jobs to be
processed by the machines. They are also referred to as
the demand. The actual demand is modelled as a real $n$-
column vector $N$. $N(i)$ is the total amount of products of
type $i$ to be processed by the workshop for all $i$ in
$I$. It is also assumed that a forecasted demand $Nref$ is
available. The actual demand $N$ is regarded as the result of
a variation $\Delta N$ around $Nref$: $N = Nref + \Delta N$. We
define a neighbourhood $P_{a_1,\ldots,a_n}(Nref)$ around $Nref$ by:

$$P_{a_1,\ldots,a_n}(Nref) = \left\{ N | N = Nref + \sum_{k \in I} a_k \alpha_k, \epsilon_k \wedge \sum_{k \in I} \alpha_k \leq 1 \wedge \forall \epsilon_k \in I \right\}$$

(1) with $(\epsilon_k)_{k \in I}$ the canonical basis of $\mathbb{R}^n_+$ and $\alpha_k$ the positive
variation magnitude on the product type $k$.

We define the demand $N_{a\epsilon}$ by:

$$N_{a\epsilon} = Nref + a_k \epsilon_k$$

(2)

Production plan. Scheduling the production in the work-
shop requires assigning the products to the machines.
The corresponding scheduling problem is dealt with in the
next section. The result of the scheduling problem defines a
production plan that can be modelled by a $n$-by-$m$ real
matrix $R$. $R(i,j)$ is the total amount of time that machine
$j$ spends processing products of type $i$. Furthermore, pre-
emption and splitting are assumed: several machines may
be working on the same product type at the same time.
The order of the jobs on the machines has no importance,
only the affectation matters.

2.2 The scheduling problem

Given a demand $N$, the speed data $V$, and the con-
figuration $Q$, the scheduling problem consists in finding a
production plan $R$ such that the makespan $C_{max}$ is
minimised. Using the three fields notation introduced by
(Graham et al., 1979) and extended by (Blazewicz et al.,
1996; T’Kindt and Billaut, 2002), this problem can be
stated as $RMPM|split|C_{max}$. $RMPM$ stands for unrelated
multi-purpose machines, $split$ means that splitting
is allowed. $C_{max}$ means that the makespan (or the max-
imum completion-time of the machines) must be min-
imised. This problem is an easy problem (in the sense of
the $NP$-completeness theory) as it can be solved by
the following linear program denoted LP which is inspired
from the linear program of Lawler and Labetoulle (1978)
used to solve $R|pmtn|C_{max}$:

$$\begin{align*}
\min (C_{max}) \\
\sum_{i \in I} V(i,j) \times R(i,j) = N(i) \forall i \in I \quad (a) \\
\sum_{j \in J} R(i,j) \leq C_{max} \forall j \in J \quad (b) \\
(1 - Q(i,j)) \times R(i,j) \geq 0 \forall (i,j) \in I \times J \quad (c) \\
R(i,j) \geq 0 \forall (i,j) \in I \times J \quad (d)
\end{align*}$$

The speed $V(i,j)$, the configuration matrix $Q(i,j)$ and
the demand $N(i)$ are given data whereas $R(i,j)$ and $C_{max}$
are decision variables. The set of constraints (3a) enforces
that the demand is exactly met for any type of products
wheras the set of constraints (3b) enforces that $C_{max}$ is
greater than or equal to the completion-time of any ma-
chine in the shop. The set of constraints (3c) ensures that
product type $i$ is assigned to machine $j$ (e.g., $R(i,j) > 0$
if and only if the machine is effectively qualified for this
product type (e.g., $Q(i,j) = 1$ ). The last constraints define
the variation domains of the variables $R(i,j)$.

In the following, we will denote the optimal value of the
objective function of $LP(V,Q,N)$ by $C_{max}^Q(V,N)$. If
$(R, C_{max}^Q(V,N))$ is an optimal solution for $LP(V,Q,N)$
then the following equality holds:

$$C_{max}^Q(V,N) = \max_{j \in J} \left( \sum_{i \in I} R(i,j) \right)$$

(4)

3. SENSITIVITY ANALYSIS

3.1 Definition of sensitivity analysis

In (Mahjoub et al., 2005), sensitivity analysis is defined as
follows.

Definition 1. (Mahjoub et al., 2005). Given an optimisation
problem $P$ and an instance $I$ of $P$, let $S_2^I$ be an
optimal solution for $I$ and let $z_2^I$ be the optimal value
of $S_2^I$. A sensitivity analysis on $P$ and $I$, consists in answering
at least one of the following questions:

(1) In what neighbourhood of $I$, $S_2^I$ (resp. $z_2^I$) remains
optimal?
(2) In what neighbourhood of $I$, $S_2^I$ remains
admissible with an acceptable performance?
(3) Given $I'$ a neighbour of $I$, is $S_2^I$ still admissible for
$I'$? If it is, what is the performance degradation?
(4) Given $I'$ a neighbour of $I$, what is the new optimal
solution (resp. value)?

Moreover, replacing $S_2^I$ by any solution in the questions
(2) and (3) permits to obtain two other questions related
to sensitivity analysis.

3.2 Definition of the sensitivity analysis problem

The sensitivity analysis problem presented in this paper
consists in answering the question (2) of the sensitivity

15832
analysis defined in definition 1. In our context, a solution \( S \) is a configuration \( Q \), a particular instance \( I \) is the forecasted demand \( N_{\text{ref}} \) and the performance is measured by \( C_{\text{max}}^{Q,V}(N_{\text{ref}}) \). So, our sensitivity analysis problem consists in answering the following question: “given a forecasted demand \( N_{\text{ref}} \), a configuration \( Q \) and the speed data \( V \), in what neighbourhood of \( N_{\text{ref}} \) does \( Q \) remain admissible with an acceptable performance?”

Here we have to define the so-called acceptable performance. The configuration \( Q \) leads to an acceptable performance if it let the shop completing the actual demand by a given deadline \( d \). More formally, that means that the sensitivity analysis question can be formulated as follows: “Given a forecasted demand \( N_{\text{ref}} \), a configuration \( Q \) and the speed data \( V \), what is the neighbourhood \( P_{\text{ref}}^{Q_{1,\ldots,a_{n}}}(N_{\text{ref}}) \) such that \( \forall N \in P_{\text{ref}}^{Q_{1,\ldots,a_{n}}}(N_{\text{ref}}), C_{\text{max}}^{Q,V}(N) \leq \hat{d} \) ”

To ensure that \( P_{\text{ref}}^{Q_{1,\ldots,a_{n}}}(N_{\text{ref}}) \) is not empty, we assume that the following inequality holds:

\[
C_{\text{max}}^{Q,V}(N_{\text{ref}}) \leq \hat{d} \tag{5}
\]

4. RESOLUTION OF THE SENSITIVITY ANALYSIS PROBLEM

4.1 Preliminary results

Lemma 2. For any couple of demands \( (N_1, N_2) \), the following property holds:

\[
C_{\text{max}}^{Q,V}(N_1 + N_2) \leq C_{\text{max}}^{Q,V}(N_1) + C_{\text{max}}^{Q,V}(N_2) \tag{6}
\]

Proof. Let \( R_1 \) and \( R_2 \) be two optimal production plans for \( N_1 \) and \( N_2 \) respectively. An admissible production plan (but not necessarily optimal) for the demand defined by \( N_1 + N_2 \) can be built as follows: \( R_3 = R_1 + R_2 \). As \( R_3 \) is not necessarily optimal, its maximal completion-time defined by \( \max_{j \in J} \left( \sum_{i \in I} R_3(i,j) \right) \) is an upper bound for \( C_{\text{max}}^{Q,V}(N_1 + N_2) \).

Then the following inequality holds:

\[
C_{\text{max}}^{Q,V}(N_1 + N_2) \leq \max_{j \in J} \left( \sum_{i \in I} R_3(i,j) \right) = \max_{j \in J} \left( \sum_{i \in I} (R_1(i,j) + R_2(i,j)) \right) \leq C_{\text{max}}^{Q,V}(N_1) + C_{\text{max}}^{Q,V}(N_2) \tag{7}
\]

What is exactly the inequality (6). □

Lemma 3. For any demand \( N \) and any positive real number \( \alpha \), the following property holds:

\[
C_{\text{max}}^{Q,V}(\alpha.N) = \alpha \cdot C_{\text{max}}^{Q,V}(N) \tag{8}
\]

Proof. First let us show that if \( (R, C_{\text{max}}^{Q,V}(N)) \) is an optimal solution for \( LP(V, Q, N) \) then \( (\alpha.R, \alpha.C_{\text{max}}^{Q,V}(N)) \) is an admissible solution for \( LP(V, Q, \alpha.N) \).

The demand \( \alpha.N \) can be processed by the production plan \( \alpha.R \). Indeed, for all \( i \) in \( J \) the following equalities hold:

\[
\sum_{j \in J} (\alpha.R(i,j) \times V(i,j)) = \alpha \sum_{j \in J} (R(i,j) \times V(i,j)) \tag{9}
\]

Moreover, the makespan associated to the production plan \( \alpha.R \) can be written as:

\[
\max_{j \in J} \left\{ \sum_{i \in I} \alpha.R(i,j) \right\} = \alpha \cdot \max_{j \in J} \left\{ \sum_{i \in I} R(i,j) \right\} \tag{10}
\]

Then let us show by a reductio ad absurdum that this solution is optimal for \( LP(V, Q, \alpha.N) \). Let us assume that there exists a couple \( (R^*, C_{\text{max}}^{Q,V}(N)) \), solution for \( LP(V, Q, \alpha.N) \), such that:

\[
C_{\text{max}}^{Q,V}(N) < \alpha \cdot C_{\text{max}}^{Q,V}(N) \tag{11}
\]

\( \frac{R^*}{\alpha} \) is an admissible production plan for \( N \). In fact, for all \( i \) in \( I \), \( N(i) \) is processed according to this production plan as follows:

\[
\sum_{j \in J} \left( \frac{R^*(i,j)}{\alpha} \times V(i,j) \right) = \frac{1}{\alpha} \sum_{j \in J} (R^*(i,j) \times V(i,j)) \tag{12}
\]

Then:

\[
\tau = \max_{j \in J} \left\{ \sum_{i \in I} R^*(i,j) \right\} \tag{13}
\]

\[
\tau = \frac{1}{\alpha} \cdot C_{\text{max}}^{Q,V}(N) \tag{14}
\]

Finally, the following inequality holds:

\[
\tau < C_{\text{max}}^{Q,V}(N) \tag{15}
\]

Thus, there exists a production plan for \( N \) such that the makespan is strictly lower than the optimal makespan, which is contradiction. So the assumption (11) is false and \( (\alpha.R, \alpha.C_{\text{max}}^{Q,V}(N)) \) is an optimal solution for \( LP(V, Q, \alpha.N) \). □

Definition 4. Let \( P_d(Q) \) be the set of the demands that can be completed by the configuration \( Q \) in less than \( d \) units of time. More formally \( P_d(Q) \) can be defined as follows:

\[
P_d(Q) = \left\{ N \left| C_{\text{max}}^{Q,V}(N) \leq d \right. \right\} \tag{16}
\]

Theorem 5. \( P_d(Q) \) is a convex set.

Proof. Let \( N_1 \) and \( N_2 \) be two demands belonging to \( P_d(Q) \), and let \( \alpha \) and \( \beta \) be two non-nil real positive numbers such that:

\[
\alpha + \beta = 1 \tag{17}
\]

We have to prove that the demand defined by \( \alpha.N_1 + \beta.N_2 \) belongs to \( P_d(Q) \) and thus that \( C_{\text{max}}^{Q,V}(\alpha.N_1 + \beta.N_2) \leq d \).
As N1 and N2 belong to P̃d(Q), the following inequalities hold:
\[ C_{Q,V} \max (N_1) \leq \tilde{d} \quad (a) \]
\[ C_{Q,V} \max (N_2) \leq \tilde{d} \quad (b) \]

By using lemma 2 and 3 the following inequality holds:
\[ C_{Q,V} \max (\alpha.N_1 + \beta.N_2) \leq C_{Q,V} \max (\alpha.N_1) + C_{Q,V} \max (\beta.N_2) \]
\[ \Rightarrow C_{Q,V} \max (\alpha.N_1 + \beta.N_2) \leq \alpha.\tilde{d} + \beta.\tilde{d} \]
\[ \Rightarrow C_{Q,V} \max (\alpha.N_1 + \beta.N_2) \leq \tilde{d} \quad (c) \]

4.2 Building a neighbourhood \( P^Q_{a_1,\ldots,a_n}(N_{ref}) \) included in \( P_d(Q) \)

Definition 6. For all \( k \) in \( I \), the linear program \( LP_k \) is defined as follows:
\[ \min_{a(z)} \sum_{i,j} \ V(i,j) \times R(i,j) = N_{ref} \quad (a) \]
\[ \sum_{i,j} \ V(i,j) \times R(i,j) = a_k = N_{ref} \quad (b) \]
\[ \sum_{i} \ R(i,j) \leq \tilde{d} \quad (c) \]
\[ (1 - Q(i,j)) \times R(i,j) = 0 \quad (d) \]
\[ R(i,j) \geq 0 \quad (e) \]

\( V(i,j), N_{ref} \), \( Q(i,j) \) and \( \tilde{d} \) are given data whereas \( a_k \) and \( R(i,j) \) are decision variables. The sets of constraints (19a) and (19b) enforce that the demand \( N_{ref}^{a_k} \) is satisfied. The set of constraints (19c) enforces that \( \tilde{d} \) is greater than or equal to the completion-time of any machine in the shop. The set of constraints (19d) ensures that product type \( i \) is assigned to machine \( j \) (e.g., \( R(i,j) > 0 \)) if and only if the machine is effectively qualified for this product type (e.g., \( Q(i,j) = 1 \)). The last constraints define the variation domains of the variables \( R(i,j) \).

Solving \( LP_k \) to optimality is equivalent to answer the following question: “given a configuration \( Q \), a forecasted demand \( N_{ref} \) and a deadline \( \tilde{d} \), what is the maximal additional number \( a_k \) of products of type \( k \) that can be completed by the configuration \( Q \) without missing the deadline \( \tilde{d} \)?”

In the following of the paper, we will denote the optimal value of the objective function of \( LP_k \) \( (Q, V, N_{ref}, \tilde{d}) \) by \( a_k (Q, V, N_{ref}, \tilde{d}) \).

Lemma 7. The neighbourhood \( P^Q_{a_1,\ldots,a_n}(N_{ref}) \), defined by the family \( \{a_k (Q, V, N_{ref}, \tilde{d})\}_{k \in I} \) is included in \( P_d(Q) \).

Proof. Let \( N \) be in \( P^Q_{a_1,\ldots,a_n}(N_{ref}) \). There is a family \( \{a_k\}_{k \in I} \) of positive real numbers such that
\[ N = N_{ref} + \sum_{k \in I} \alpha_k a_k (Q, V, N_{ref}, \tilde{d}) \leq 1. \]

Let \( \beta \) be the real positive number such that \( \beta = 1 - \sum_{k \in I} \alpha_k \).

\( N \) can be written as follows:
\[ N = (\beta + \sum_{k \in I} \alpha_k) (Q, V, N_{ref}, \tilde{d}) \]
\[ \Rightarrow N = \beta N_{ref} + \sum_{k \in I} \alpha_k (Q, V, N_{ref}, \tilde{d}) \]
\[ \Rightarrow N = \beta N_{ref} + \alpha_k (Q, V, N_{ref}, \tilde{d}) \]

By definition of the family \( \{a_k (Q, V, N_{ref}, \tilde{d})\}_{k \in I} \), each demand \( N_{ref}^{a_k (Q, V, N_{ref}, \tilde{d})} \) belongs to \( P_d(Q) \). Moreover, according to inequality (5), \( N_{ref} \) belongs to \( P_d(Q) \) too. \( N \) is thus a convex combination of elements belonging to \( P_d(Q) \). According to the convex property of \( P_d(Q) \) (see theorem 5), \( N \) belongs to \( P_d(Q) \) and so \( P^Q_{a_1,\ldots,a_n}(N_{ref}) \subseteq P_d(Q) \).

5. DEFINITION AND CALCULATION OF A STABILITY RADIUS

5.1 Definition of the stability radius of an \( \epsilon \)-approximate solution

Definition 8. An \( \epsilon \)-approximate solution \( S \) for an instance \( T \) relatively to a performance criterion \( z \) to be minimised is a solution that satisfies the following inequality:
\[ z_T(S) \leq (1 + \epsilon)z_T^* \]

with \( z_T^* \) the optimal value of the objective function \( z \) on \( T \).

Definition 9. (Sotskov et al., 1998). Given an optimisation problem \( P \), an instance \( I \) of \( P \), the optimal value \( z_T^* \) and an \( \epsilon \)-approximate solution \( S \) for \( I \), the stability radius \( \rho_S(T) \) of the \( \epsilon \)-approximate solution \( S \) is the maximal radius of a ball with centre \( I \) in which the solution \( S \) does not go away from more than \( \epsilon \) percents of \( z_T^* \).

5.2 Stability radius of an \( \epsilon \)-approximate configuration \( Q \)

For any demand \( N \), the configuration \( Q = T \) is the one that leads to the minimum completion-time. \( C_{Q,V}^{max}(N) \) is thus the minimal makespan for any demand \( N \). An \( \epsilon \)-approximate configuration for a forecasted demand \( N_{ref} \) is a configuration \( Q \) such that the following inequality holds:
\[ C_{Q,V}^{max}(N_{ref}) \leq (1 + \epsilon)C_{Q,V}^{T,V}(N_{ref}) \]

Finding the stability radius of an \( \epsilon \)-approximate configuration \( Q \) is finding the maximal radius of a ball \( B \) with centre \( N_{ref} \) such that:
\[ C_{Q,V}^{max}(N) \leq (1 + \epsilon)C_{Q,V}^{T,V}(N_{ref}), \quad \forall N \in B \]

Let \( \tilde{d} \) be equal to \( (1 + \epsilon)C_{Q,V}^{T,V}(N_{ref}) \).
Theorem 10. The stability radius of an ϵ-approximate configuration Q is given by:

\[ \rho_{Q}(N_{\text{ref}}) = \min_{k \in I} \left\{ a_{k} \left( Q, V, N_{\text{ref}}, \tilde{d} \right) \right\} \]  

Proof. First we prove that the ball B with centre \( N_{\text{ref}} \) and with radius \( \rho_{Q}(N_{\text{ref}}) \) is such that the inequality (23) holds.

We define \( a_{0} \) as \( a_{0} = \min_{k \in I} \left\{ a_{k} \left( Q, V, N_{\text{ref}}, \tilde{d} \right) \right\} \). Moreover, \( k_{0} \) is the integer of I such that \( a_{k_{0}} = a_{0} \).

\( B \) can be formally defined as:

\[ B = \left\{ N \mid N = N_{\text{ref}} + a_{0} \sum_{k \in I} a_{k} e_{k} \land \|a_{k}\| \leq 1 \right\} \]  

Let us show that \( B \) is a subset of \( P_{\tilde{d}}(Q) \).

Let \( N \) be in \( B \). There is a family \( \{a_{k}\}_{k \in I} \) of positive real numbers such that \( N = N_{\text{ref}} + a_{0} \sum_{k \in I} a_{k} e_{k} \) and \( \|a_{k}\| \leq 1 \).

Let \( \beta \) be the real positive number such that:

\[ \beta = 1 - \|a_{k}\| \]  

For all \( k \) in \( I \), the following inequality holds:

\[ N(k) = N_{\text{ref}}(k) + a_{0} a_{k} \rightarrow N(k) \leq N_{\text{ref}}(k) + a_{0} \|a_{k}\| \rightarrow N(k) \leq N_{\text{ref}}(k) + a_{k} \left( Q, V, N_{\text{ref}}, \tilde{d} \right), \|a_{k}\| \]  

Let \( N^{+} \) be the demand defined by:

\[ N^{+} = N_{\text{ref}} + \sum_{k \in I} \left( a_{k} \left( Q, V, N_{\text{ref}}, \tilde{d} \right), \|a_{k}\| e_{k} \right) \]  

According to equation (26), \( N^{+} \) can be written as follows:

\[ N^{+} = \beta N_{\text{ref}} + \sum_{k \in I} \left( a_{k} \left( Q, V, N_{\text{ref}}, \tilde{d} \right), \|a_{k}\| e_{k} \right) \]  

By definition of the family \( \{a_{k} \left( Q, V, N_{\text{ref}}, \tilde{d} \right)\}_{k \in I} \), each demand \( N_{\text{ref}}^{a_{k}} \left( Q, V, N_{\text{ref}}, \tilde{d} \right) \) belongs to \( P_{\tilde{d}}(Q) \). Moreover, according to inequality (5), \( N_{\text{ref}} \) belongs to \( P_{\tilde{d}}(Q) \) too. \( N^{+} \) is thus a convex combination of elements belonging to \( P_{\tilde{d}}(Q) \).

According to the convex property of \( P_{\tilde{d}}(Q) \) (see theorem 5), \( N^{+} \) belongs to \( P_{\tilde{d}}(Q) \). Thus the following inequality holds:

\[ C_{\text{max}}^{Q,V}(N^{+}) \leq \tilde{d} \]  

According to equation (27), each element of the demand \( N^{+} \) is greater than or equal to the one of the demand \( N \), thus the following inequality holds:

\[ C_{\text{max}}^{Q,V}(N) \leq C_{\text{max}}^{Q,V}(N^{+}) \leq \tilde{d} \]  

By definition of \( P_{\tilde{d}}(Q) \), \( N \) belongs to \( P_{\tilde{d}}(Q) \). So, \( B \) is a subset of \( P_{\tilde{d}}(Q) \) and the following inequality holds:

\[ C_{\text{max}}^{Q,V}(N) \leq \tilde{d}, \forall N \in B \]  

By definition of \( \tilde{d} \), the inequality (32) is equivalent to the inequality (23).

Now, we prove that \( a_{0} \) is the maximal radius. By definition, \( a_{0} \) is the maximal additional number of products of type \( k_{0} \) that can be completed by the configuration \( Q \) without exceeding \( \tilde{d} = (1 + \epsilon) C_{\text{max}}^{T,V}(N_{\text{ref}}) \). \( N_{\text{ref}}^{a_{0}} \) is thus a member of the frontier of \( B \) and each demand \( N \) defined by \( N = N_{\text{ref}}^{a_{0} + \delta e_{k_{0}}} \) with \( \delta \) any positive real number, is such that \( C_{\text{max}}^{Q,V}(N) > (1 + \epsilon) C_{\text{max}}^{T,V}(N_{\text{ref}}) \). So, \( a_{0} \) cannot increase anymore. □

6. EXAMPLE

This section aims at illustrating all the theoretical results through a simple example. This example considers two product types and two machines. The technological matrix \( T \), the speed matrix \( V \), the configuration matrix \( Q \) and the forecast demand \( N_{\text{ref}} \) are given below:

\[ T = V = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \ Q = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \ N_{\text{ref}} = \begin{bmatrix} 5 \\ 4 \end{bmatrix} \]  

6.1 Scheduling problem

Solving \( LP(V, Q, N_{\text{ref}}) \) to optimality yields to:

\[ R = \begin{bmatrix} 0 & 5 \\ 4 & 0 \end{bmatrix}, \ C_{\text{max}}^{Q,V}(N_{\text{ref}}) = 5 \]  

It can be noted that the machines do not share the same completion-time. This is due to the non-qualification of machine 1 for the products of type 1.

Moreover, solving \( LP(V, T, N_{\text{ref}}) \) to optimality yields to \( C_{\text{max}}^{T,V}(N_{\text{ref}}) = 4.5 \).

6.2 Sensitivity analysis

The aim of sensitivity analysis is to determine which magnitude of perturbations can be dealt with by the configuration without passing a deadline \( \tilde{d} \).

For our example the deadline is set to \( \tilde{d} = 133\% C_{\text{max}}^{T,V}(N_{\text{ref}}) = 6 \). This is equivalent to considering that the completion-time deviation cannot exceed \( \epsilon = 33\% \) of \( C_{\text{max}}^{T,V}(N_{\text{ref}}) \).

It can be noted that \( Q \) is an \( \epsilon \)-approximate configuration as it verifies \( C_{\text{max}}^{Q,V}(N_{\text{ref}}) \leq (1 + \epsilon) C_{\text{max}}^{T,V}(N_{\text{ref}}) \) for \( k = 1 \) and \( k = 2 \).

\( a_{k} \) is the maximum amount of additional products of type \( k \) that can be completed by the workshop without passing the deadline \( \tilde{d} \).

Solving \( LP_{k}(Q, V, N_{\text{ref}}, \tilde{d}) \) for \( k = 1 \) and \( k = 2 \) yields to:

\[ \begin{align*}
    a_{1} = a_{1}(Q, V, N_{\text{ref}}, \tilde{d}) &= 1 \\
    a_{2} = a_{2}(Q, V, N_{\text{ref}}, \tilde{d}) &= 3
\end{align*} \]  

The figure 1 shows the set of demands resulting from an additive amount of products without passing the deadline \( \tilde{d} \).
Then, any demand $N$ defined by $N = N_{ref} + \alpha a_1 e_1 + \beta a_2 e_2$ with $\alpha + \beta \leq 1$ can be completed by the deadline.

It should be stressed that $P_{a_1, a_2}^{Q(N_{ref})}$ is included in $P_d^*(Q)$. Then, any demand in $P_{a_1, a_2}^{Q(N_{ref})}$ can be processed in less than $\bar{d}$ units of time. However, for any demand out of $P_{a_1, a_2}^{Q(N_{ref})}$, the completion-time must be computed with $LP$ in order to know if the deadline is met or not.

Let us illustrate this point on the following three demands denoted $N_1$, $N_2$ and $N_3$ defined by:

$$N_1 = \begin{bmatrix} 5.5 \\ 5 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 5.5 \\ 6 \end{bmatrix}, \quad N_3 = \begin{bmatrix} 5.5 \\ 7 \end{bmatrix}$$

The optimal production plans and optimal makespans associated with these demands are given below:

$$R_1 = \begin{bmatrix} 0 & 5.5 \\ 5 & 0 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 0 & 5.5 \\ 5.75 & 0.25 \end{bmatrix}, \quad R_3 = \begin{bmatrix} 0 & 5.5 \\ 6.25 & 0.75 \end{bmatrix}$$

$$C^{Q,V}_{\text{max}}(N_1) = 5.5, \quad C^{Q,V}_{\text{max}}(N_2) = 5.75, \quad C^{Q,V}_{\text{max}}(N_3) = 6.25$$

As $N_1$ is in $P_{a_1, a_2}^{Q(N_{ref})}$, it can be checked that its completion-time is less than 6 units of time. Although the demand $N_2$ is out of $P_{a_1, a_2}^{Q(N_{ref})}$, solving $LP$ allows to compute the completion-time that appears to be less than 6 units of time. Then, $N_2$ in in $P_d^*(Q)$. The demand $N_3$ is also out of $P_{a_1, a_2}^{Q(N_{ref})}$, and has a completion-time that is strictly greater than 6 units of time, this shows that $N_3$ is out of $P_d^*(Q)$.

### 6.3 Calculation of $\rho_Q(N_{ref})$

$\rho_Q(N_{ref})$ is the stability radius of the $\epsilon$-approximate configuration $Q$. It values the maximal magnitude of perturbations on $N_{ref}$ such that the completion-time by the configuration does not go away from more than $\epsilon$ percents of $C^{Q,V}_{\text{max}}(N_{ref})$.

Following the theorem 10, $\rho_Q(N_{ref})$ is valued by:

$$\rho_Q(N_{ref}) = \min_{k \in I} \left\{ a_k \left( Q, V, N_{ref}, \bar{d} \right) \right\} = a_1 = 1$$

### 7. CONCLUSION

We present in this paper a sensitivity analysis for the multi-purpose machines problem. The question “in which neighbourhood of a given demand, the configuration of the workshop remains admissible with an acceptable performance degradation” is answered by solving $n$ independent linear programs. Moreover, the stability radius of an $\epsilon$-approximate configuration is calculated.

It would be interesting in the future to measure the degradation of the makespan according to the deviation of the demand and to answer all the questions of the sensitivity analysis.

### REFERENCES


