Disturbance Decoupling with Preview for Two-Dimensional Systems

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1. INTRODUCTION

The notion of controlled invariance for 1-D systems introduced in (Basile and Marro, 1969) is the cornerstone of the so-called geometric approach to control theory for LTI systems. The most celebrated control application of this concept is the disturbance decoupling problem (DDP), solved for the first time in (Basile and Marro, 1969). Disturbance decoupling with the extra requirement of closed-loop stability was addressed for the first time in (Wonham and Morse, 1970). Many important extensions of the classic DDP were proposed in the literature in the last thirty years. The most relevant for this paper is the so-called DDP with PID control law, (Willems, 1982, Bonilla Estrada and Malabre, 1999, Barbagli et al., 2001). In the discrete-time case, this problem is also referred to as DDP with preview, since the control law is allowed to include – in addition to the standard proportional state feedback component – feedforward terms depending on ‘future’ values of the disturbance up to the present.

In the last two decades, many valuable results have been achieved in an attempt to develop a geometric theory for 2-D systems, (Conte and Perdon, 1988, Karamançoğlu and Lewis, 1992, Ntogramatzidis et al., 2007). In particular, a geometric approach for 2-D systems was introduced in (Conte and Perdon, 1988) to treat 2-D decoupling problems of nonmeasurable and measurable disturbances, but without a guarantee of stability. In (Ntogramatzidis et al., 2007), new geometric techniques for internal and external stabilisation of controlled invariant subspaces were developed. This led to a new solution for the two aforementioned decoupling problems, while achieving asymptotic stability of the closed-loop.

In this paper the DDP with preview is extended for the first time to 2-D causal systems. Its solution is carried out by recasting this problem into a full information problem. This contrivance enables the structural solvability condition to be easily stated in terms of the matrices of a suitably defined extended system. However, the stability condition must be addressed independently, and here it is captured in terms of the stability property of an output-nulling subspace of the original system.

Notation. The symbol $\mathbf{0}_n$ stands for the origin of the vector space $\mathbb{R}^n$. The $n \times m$ zero matrix is denoted by $\mathbf{0}_{n \times m}$. Given the subspace $\mathcal{S}$, the symbol $\mathcal{S}^2$ stands for the Cartesian product $\mathcal{S} \times \mathcal{S}$.

2. PROBLEM STATEMENT

Consider a Fornasini-Marchesini (FM) model

$$
\begin{align*}
x_{i+1,j+1} &= A_1 x_{i+1,j} + A_2 x_{i,j+1} + B_1 u_{i+1,j} + B_2 u_{i,j+1} + H_1 w_{i,j+1} + H_2 w_{i,j+1}, \\
y_{i,j} &= C x_{i,j} + D u_{i,j} + G w_{i,j},
\end{align*}
$$

(1)

where for all $i, j \in \mathbb{Z}$, $x_{i,j} \in \mathbb{R}^n$ is the local state, $u_{i,j} \in \mathbb{R}^m$ is the control input, $w_{i,j} \in \mathbb{R}^d$ is a disturbance to be decoupled from the output $y_{i,j} \in \mathbb{R}^p$. The matrices appearing in (1) have sizes compatible with these signals. We identify the system $(A_1, A_2, \{B_1, H_1\}, \{B_2, H_2\}, C, [D G])$ with the symbol $\Sigma$. For $k \in \mathbb{Z}$, we define the separation sets $\mathcal{S}_k \triangleq \{(i,j) \in \mathbb{Z} \times \mathbb{Z} \mid i + j = k\}$, along with the so-called global state on $\mathcal{S}_k$ as $\mathcal{X}_k \triangleq \{x_{i,j} \mid (i,j) \in \mathcal{S}_k\}$, see (Fornasini and Marchesini, 1978). Similarly, we can define the global control $\mathcal{U}_k \triangleq \{u_{i,j} \mid (i,j) \in \mathcal{S}_k\}$, the global disturbance $\mathcal{W}_k \triangleq \{w_{i,j} \mid (i,j) \in \mathcal{S}_k\}$ and the global output $\mathcal{Y}_k \triangleq \{y_{i,j} \mid (i,j) \in \mathcal{S}_k\}$ on the separation sets. The boundary conditions usually associated with (1) take the form $x_{i,j} = b_{i,j}$ for $(i,j) \in \mathcal{S}_0$ for some constants $b_{i,j} \in \mathbb{R}^n$ for $(i,j) \in \mathcal{S}_0$. This uniquely defines $\mathcal{X}_k$ for all $k > 0$ given $\mathcal{U}_h$ and $\mathcal{W}_h$ for all $0 \leq h < k$.
Given a subspace $S$, by a $S$-valued boundary condition we intend $x_{i,j} \in S$ for all $(i,j) \in S_0$. By defining $\|X_r\| \triangleq \sup_{n \in \mathbb{Z}} \|x_{r-n,n}\|$, we recall that system (1) – and therefore, with a slight abuse of nomenclature, the pair $(A_1, A_2)$ – is asymptotically stable if, for finite $\|X_0\|$ and with both inputs set to zero, the sequence $\{\|X_r\|\}_{r=0}^{\infty}$ converges to zero. A simple sufficient condition that can be used to check asymptotic stability of the pair $(A_1, A_2)$ is the one proposed in (Kar and Sigh, 2003): The pair $(A_1, A_2)$ is asymptotically stable if two symmetric positive definite matrices $P_1$ and $P_2$ exist such that:

$$\begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} - \begin{bmatrix} A_1^T \\ A_2^T \end{bmatrix} (P_1 + P_2) \begin{bmatrix} A_1 & A_2 \end{bmatrix} > 0. \tag{2}$$

**Problem 2.1.** (Disturbance decoupling with preview)

Given $N, M \in \mathbb{N}$, find matrices $F \in \mathbb{R}^{m \times n}$ and $S_{k,l} \in \mathbb{R}^{m \times d}$, for $(k,l) \in [0, N] \times [0, M]$, so that the system

$$x_{i+1,j+1} = (A_1 + B_1 F) x_{i,j} + (A_2 + B_2 F) x_{i,j+1} + B_1 \varphi_{i,j} + B_2 \varphi_{i,j+1} + H_1 w_{i+1,j} + H_2 w_{i,j+1}, \tag{3}$$

$$y_{i,j} = (C + D F) x_{i,j} + D \varphi_{i,j} + G w_{i,j},$$

obtained by imposing the control action

$$u_{i,j} = F x_{i,j} + \varphi_{i,j}, \tag{4}$$

with $\varphi_{i,j} \triangleq \sum_{k=0}^{N} \sum_{l=0}^{M} S_{k,l} w_{i+k,j+l}$, on the system dynamics (1), yields a global output sequence $\{y_{i,j}\}_{i=0}^{\infty}$ with elements that converge to zero for any global-state boundary condition $X_0$ and any global disturbance $\{W_i\}_{i=0}^{\infty}$.

By linearity, Problem 2.1 is equivalent to requiring that

- with the boundary conditions set to zero, the output generated by (3) satisfies $y_{i,j} = 0$ for all $i + j \geq 0$ and for any global disturbance $\{W_i\}_{i=0}^{\infty}$;
- the pair $(A_1 + B_1 F, A_2 + B_2 F)$ be asymptotically stable, to ensure dissipation of the effect of non-zero boundary conditions on the output.

3. GEOMETRIC BACKGROUND FOR 2-D SYSTEMS

We now introduce some preliminaries for 2-D systems, which are taken from (Ntogramatzidis et al., 2007). We begin by considering the autonomous FM system

$$x_{i+1,j+1} = A_1 x_{i+1,j} + A_2 x_{i,j+1}. \tag{5}$$

The subspace $\mathcal{J}$ of $\mathbb{R}^{n \times n}$ is $(A_1, A_2)$-invariant if $A_1 \mathcal{J} \subseteq \mathcal{J}$ and $A_2 \mathcal{J} \subseteq \mathcal{J}$. If $\mathcal{J}$ is $(A_1, A_2)$-invariant, by choosing a nonsingular matrix $T = [T_1 \ T_2] \in \mathbb{R}^{n \times n}$ where the columns of $T_1$ span $\mathcal{J}$, we find that (5) can be written in the new coordinates described as $[x_{i,j}'] = T^{-1} x_{i,j}$:

$$x_{i+1,j+1}' = A_1^{(1,1)} x_{i,j+1}' + A_1^{(1,2)} x_{i+1,j}' + A_2^{(1,1)} x_{i,j+1}' + A_2^{(1,2)} x_{i,j+1}' \tag{6}$$

$$x_{i+1,j+1}' = A_1^{(2,1)} x_{i,j+1}' + A_1^{(2,2)} x_{i+1,j}' + A_2^{(2,1)} x_{i,j+1}' + A_2^{(2,2)} x_{i,j+1}' \tag{7}$$

Given an $(A_1, A_2)$-invariant subspace $\mathcal{J}$ for (5), any $\mathcal{J}$-valued boundary condition gives rise to a local state trajectory such that $x_{i,j} \in \mathcal{J}$ for all $i + j \geq 0$. Asymptotic stability of (5) can be “split” into two parts with respect to the invariant subspace $\mathcal{J}$. The $(A_1, A_2)$-invariant subspace $\mathcal{J}$ is said to be inner stable if $(A_1^{(1,1)}, A_1^{(1,2)})$ is asymptotically stable and outer stable if $(A_1^{(2,1)}, A_1^{(2,2)})$ is asymptotically stable.

Now, consider the nonautonomous FM system

$$x_{i+1,j+1} = A_1 x_{i+1,j} + A_2 x_{i,j+1} + B_1 u_{i+1,j} + B_2 u_{i,j+1}, \tag{8}$$

$$y_{i,j} = C x_{i,j} + D u_{i,j}. \tag{9}$$

The boundary conditions associated with (8-9) can still be assigned by specifying the global state over $S_0$. The subspace $\mathcal{V} \subseteq \mathbb{R}^{n}$ is output-nulling for (8-9) if $\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \mathcal{V} \subseteq (\mathcal{V}^2 \otimes \mathbf{0}) + \text{im} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$, (Conte and Perdon, 1988). Let $\mathcal{V}$ be a subspace of $\mathbb{R}^{n}$ and let $\mathcal{V}$ be a basis matrix of $\mathcal{V}$. The following are equivalent, (Ntogramatzidis et al., 2007):

- The subspace $\mathcal{V}$ is output-nulling for (8-9);
- There exist $X$ and $\Omega$ such that

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \mathcal{V} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} X + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \Omega \tag{10}$$

- There exist $F$ and $X$ such that

$$\begin{bmatrix} A_1 + B_1 F \\ A_2 + B_2 F \end{bmatrix} \mathcal{V} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} X, \tag{11}$$

Any $F$ such that (11) holds for some $X$ is called a friend of $\mathcal{V}$. Given a $\mathcal{V}$-valued boundary condition for (8-9), a control action $u_{i,j} = F x_{i,j}$, where $F$ satisfies (11), is such that $x_{i,j} \in \mathcal{V}$ and $y_{i,j} = 0$ for all $i + j \geq 0$. The output-nulling subspace $\mathcal{V}$ is said to be inner stabilisable (resp. outer stabilisable) if a friend $F$ exists such that $\mathcal{V}$ is an inner stable (resp. outer stable) $(A_1 + B_1 F, A_2 + B_2 F)$-invariant subspace. The set of friends of $\mathcal{V}$ are parameterised as the solutions of the linear equation $\Omega = -F \mathcal{V}$, where $\Omega$ satisfies (10) for some matrix $X$.

In particular, the solutions of $\Omega = -F \mathcal{V}$ can be written as $F = F_0 + \Lambda$, with $F_0 \triangleq -\Omega (\mathcal{V} \mathcal{V}^\top)^{-1} \mathcal{V}^\top$, where $\Omega$ satisfies (10) for some $X$ and $\Lambda$ is any matrix of suitable size such that $\Lambda \mathcal{V} = 0$, see (Ntogramatzidis et al., 2007). Writing the local state equation of the autonomous system obtained by applying $u_{i,j} = F x_{i,j}$, with $F = F_0 + \Lambda$, to (8) in a new basis given $T = [T_1 \ T_2]$ with $T_1 \mathcal{V} = \mathbf{0}$, yields

$$\begin{bmatrix} x_{i+1,j+1}' \\ x_{i+1,j+1}' \end{bmatrix} = \begin{bmatrix} M_1^{(1,1)} M_1^{(1,2)} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} x_{i+1,j}' \\ x_{i,j+1}' \end{bmatrix} + \begin{bmatrix} M_2^{(1,1)} M_2^{(1,2)} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} x_{i+1,j}' \\ x_{i,j+1}' \end{bmatrix},$$

where $M_1 \triangleq A_1 + B_1 F$. In (Ntogramatzidis et al., 2007) it is shown that in the pair $(M_1^{(1,1)}, M_1^{(1,2)})$ only depends on $F_0$ while the pair $(M_2^{(1,1)}, M_2^{(1,2)})$ only depends on $\Lambda$. Therefore, we can independently choose $F_0$ and $\Lambda$, so that – if $\mathcal{V}$ is inner stabilisable – $F_0$ stabilises the pair $(M_1^{(1,1)}, M_1^{(1,2)})$ and – if $\mathcal{V}$ is outer stabilisable – $\Lambda$ stabilises $(M_2^{(1,1)}, M_2^{(1,2)})$. By using the stability criterion (2) established in (Kar and Sigh, 2003), two procedures are derived in (Ntogramatzidis et al., 2007) for the inner and outer stabilisation of output-nulling subspaces. In particular, it is shown that the inner stabilisation of the controlled invariant subspace $\mathcal{V}$ requires...
the solution of a simple LMI; the outer stabilisation
requires the solution of a bilinear matrix inequality. For
its solution, different techniques may be employed. For
example, in (Ntogramatzidis et al., 2007) the so-called
sequential linear programming matrix method (SLPMM)
developed in (Leibfritz, 2001) is exploited for this purpose.

We end this section by recalling that, as in the 1-D case,
the set of output-nulling subspaces of (8-9) is closed under
subspace addition, and the largest output-nulling subspace
is denoted by \( V^* \). This subspace can be computed in finite
terms as the \((n-1)\)-th term of the monotic sequence
\[ \nu_0 = \mathbb{R}^n \text{ and } \nu_i = \left( A_1 \right)^{-1} \left( (2^{i-1} \times 0_p) + \text{im} \left[ \begin{array}{c} B_1 \\ B_2 \\ \vdots \\ B_n \end{array} \right] \right) \text{ for } i > 0, \]
see Theorem 2 in (Ntogramatzidis et al., 2007).

4. SOLUTION OF PROBLEM 2.1

In the 1-D case, the solution of the decoupling prob-
lem with preview can be expressed in terms of output-
nulling and input-containing subspaces of the original sys-
tem, (Willems, 1982, Bonilla Estrada and Malabre, 1999,
Barbagli et al., 2001). In the 2-D case, this does not seem
to be possible. We now analyse the possibility of solving
Problem 2.1 by turning it into a decoupling problem of
measurable input signals. In fact, the input \( w_{i,j} \) in (1)
can be thought of as being generated by a 2-D system \( \Delta \), whose
input is \( \hat{w}_{i,j} = w_{i+N,j+M} \) and whose output is \( y_{i,j} \). See
Figure 1, where the system ruled by (1) is denoted by \( \Sigma \),
the system \( \Delta \) is simply a shift by \( N \) and \( M \) of the signal
indexes \( i \) and \( j \), respectively, and \( \Sigma \) denotes the extended

\[ \hat{w}_{i,j} = \hat{F} \hat{z}_{i,j} + S \hat{w}_{i,j}, \]  

(12)

By partitioning \( \hat{F} = [F_x \ F_y] \), conformably with \( \left[ \begin{array}{c} x \\ \xi \end{array} \right] \),
the feedback matrix \( F \) of the original system can be taken
to be equal to \( F_x \). To find the matrices \( S_{k,l} \) in (4), we
can solve the system of the measurable signal decoupling
problem (12) with the input structure (4) imposed for
our problem. In other words, the matrices \( S_{k,l} \) can be
derived by matching (4) with (12). For this to be possible,
a particular FM realisation is required for the system
\( \Delta \). Another problem is how to accomplish the stability
requirement. This corresponds to a requirement that \( F_x \)
stabilises \( \Sigma \), which is quite different to requiring that
\( \hat{F} = [F_x \ F_y] \) stabilises \( \Sigma \). For the moment, we concentrate
on achieving decoupling and constructing an appropriate
realisation for \( \Delta \).

The 2-D decoupling problem with full information, solved
in (Conte and Perdon, 1988, Ntogramatzidis et al., 2007),
can be stated for system \( \Sigma \) as follows. Find \( \hat{F} \) and \( S \) such
that \( u_{i,j} = \hat{F} z_{i,j} + S \hat{w}_{i,j} \) decouples \( \hat{w} \) from \( y \). This problem
is solvable if

\[ \text{im} \left[ \begin{array}{c} \hat{h}_1 \\ \hat{h}_2 \\ \hat{h}_3 \end{array} \right] \subseteq \hat{V}^* \times \hat{V}^* \times 0_p + \text{im} \left[ \begin{array}{c} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{array} \right] \]  

(13)

holds, where \( \hat{V}^* \) is the largest output-nulling of the system
\( A_1, A_2, B_1, B_2, C, D \). This solvability condition is con-
structive. In fact, if (13) is satisfied, there exist matrices
\( \Phi_1, \Phi_2 \) and \( \Psi \) such that (Ntogramatzidis et al., 2007)

\[ \left[ \begin{array}{c} \hat{H}_1 \\ \hat{H}_2 \end{array} \right] = \left[ \begin{array}{c} V \ 0 \\ \hat{v} \end{array} \right] \left[ \begin{array}{c} \Phi_1 \\ \Phi_2 \end{array} \right] + \left[ \begin{array}{c} \hat{B}_1 \\ \hat{B}_2 \end{array} \right] \Psi, \]  

(14)

where \( \hat{V} \) is a basis matrix for \( \hat{V}^* \). If we take any friend
\( \hat{F} \) of \( \hat{V}^* \), the input \( u_{i,j} = \hat{F} z_{i,j} + S \hat{w}_{i,j} \) achieves exact
decoupling without stability. Indeed, by substituting this
control input in (1) we obtain

\[ z_{i+1,j+1} = (\hat{A}_1 + \hat{B}_1 \hat{F}) z_{i+1,j} + (\hat{A}_2 + \hat{B}_2 \hat{F}) z_{i,j+1} + \hat{V} \Phi_1 \hat{w}_{i,j} + \hat{V} \Phi_2 \hat{w}_{i,j+1}, \]

\[ y_{i,j} = (\hat{C} + \hat{D} \hat{F}) z_{i,j}, \]

which clearly disturbance decoupled, since, given any
\( \hat{V}^* \)-valued boundary condition over the separation set \( S_0 \),
we get \( z_{i,j} \in \hat{V}^* \) and \( y_{i,j} = 0 \) for all \( i,j \) such that \( i+j \geq 0 \).

In order to find the matrices \( S_{k,l} \), the expressions (4)
and (12) must be matched. To this end, it suffices to ensure
that the local state \( x_{k,l} \) of \( \Delta \) incorporates the values of the disturbance \( w \) for indexes in the rectangle
\( B_{k,l} \triangleq \{(k, l) \in \mathbb{Z} \times \mathbb{Z} \mid k \leq N, j \leq M \}, \)
which can be directly used as an input of the compensator.
This is achieved by finding a realisation for \( \Delta \) of order \( q \triangleq d(N + 1)(M + 1) - 1 \), where \( d \) is the
dimension of the disturbance, so that its local state is given
by the values of \( w \) on \( B_{k,l} \). A realisation meeting this requirement is given as follows. For \( P \in \mathbb{N} \), define

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**Fig. 1. Block diagram of the compensation scheme.**

The system obtained by the series connection of \( \Sigma \) and \( \Delta \). Let

\[ \xi_{i+1,j+1} = A_{1}^\Delta \xi_{i+1,j} + A_{1}^\Delta \xi_{i,j+1} + B_{1}^\Delta \hat{w}_{i+1,j} + B_{2}^\Delta \hat{w}_{i,j+1}, \]

\[ y_{i,j} = C_{\Delta}^\xi_{i,j} = \hat{w}_{i,j-M} = w_{i,j}, \]

denote a FM realisation for the \((N, M)\)-shift system \( \Delta \),
whose local state is denoted by \( \xi \). The DDP with preac-
tion can be turned into a measurable signal decoupling
problem, where the plant is given by the series connection
\( \hat{\Sigma} \), with input \( \left[ \begin{array}{c} u \\ w \end{array} \right] \), output \( y \) and local state \( z_{i,j} \triangleq \left[ \begin{array}{c} x \\ \xi \end{array} \right] \).

The corresponding system matrices are \( \hat{A}_k = \left[ \begin{array}{cc} A_k & H_k C_{\Delta}^A \\ 0 & A_k^\Delta \end{array} \right], \)

\[ \hat{B}_k = \left[ \begin{array}{c} B_k \\ 0 \end{array} \right], \hat{H}_k = \left[ \begin{array}{c} 0 \\ B_k \end{array} \right] \text{ for } k = 1, 2, \hat{C} = \left[ \begin{array}{cc} C & G C_{\Delta} \end{array} \right], \]

\[ \hat{D} = D, \hat{G} = G = 0. \] Problem 2.1 can be recast as a decoupling
problem of the measurable signal \( \hat{w}_{i,j} \). In fact, suppose we
are able to decouple the signal \( \hat{w}_{i,j} \) from the output \( y_{i,j} \)
by means of the control

\[ u_{i,j} = \hat{F} \left[ \begin{array}{c} \xi_{i,j} \\ \xi_{i,j} \end{array} \right] + S \hat{w}_{i,j}, \]

(12)
where \( N'_p, N''_p \in \mathbb{R}^{d \times P_d} \) and \( V_p \in \mathbb{R}^{P \times d} \). The matrices

\[
A^\Delta_1 = \begin{bmatrix}
N'_M & 0 & 0 & \ldots & 0 \\
0 & N_{M+1}' & 0 & \ldots & 0 \\
0 & 0 & N_{M+1}'' & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & N_{M+1}''
\end{bmatrix}, \quad B^\Delta_1 = \begin{bmatrix}
V_M \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix},
\]

\[
A^\Delta_2 = \begin{bmatrix}
0 & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & N''_{M+1}
\end{bmatrix}, \quad B^\Delta_2 = \begin{bmatrix}
V_{M+1} \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix},
\]

\[
C^\Delta = \begin{bmatrix} 0 & 0 & \ldots & 0 & I_d \end{bmatrix}
\]

are a realisation of \( \Delta \). If the decoupling problem of \( \hat{w}_{i,j} \) is solvable for \( \hat{\Sigma} \), a control function having the structure (12) can be found to achieve perfect decoupling. By partitioning

\[ F_\xi = \begin{bmatrix} F_{\xi}^1 & F_{\xi}^2 & \ldots & F_{\xi}^{N_{M+N+M}} \end{bmatrix} \]

conformably with \( \xi = \begin{bmatrix} \xi^1 & \xi^2 & \ldots & \xi^{N_{M+N+M}} \end{bmatrix} \), by comparing (4) with (12), it follows that

\[ F = F_x, S_{N,M} = S, S_{N-1,M} = F_{\xi}^{M+1}, S_{N,M-1} = F_{\xi}^1, S_{N-2,M-1} = F_{\xi}^{M+2}, S_{N-2,M-1} = F_{\xi}^{2M+3}, \ldots \]

solve Problem 2.1.

Now we turn our attention to the stability requirement. Requiring that \( \hat{\nu}_i^* \) is inner and outer stabilisable, as one might expect at first sight due to the analogy with the measurable signal decoupling problem, is not correct in this case, since \( F_\xi \) cannot be used to stabilise \( \hat{\Sigma} \). The stability condition required for the solution of Problem 2.1 can be stated in terms of the stabilisability of the largest output-nulling subspace \( \nu^* \) of the system \((A_1, A_2, B_1, B_2, C, D)\).

We first present the following lemma, where the relation between \( \nu^* \) and \( \hat{\nu}_i^* \) is established.

**Lemma 4.1.** The following identity holds:

\[
\hat{\nu}_i^* \cap \text{im} \begin{bmatrix} I_n \\ 0_{q \times n} \end{bmatrix} = \nu^* \times q.
\]

**Proof:** First, we show that the subspace on the left-hand side of (15) contains that on the right-hand side, i.e., \( \hat{\nu}_i^* \supseteq \nu^* \times q. \) Consider the two sequences of subspaces \( \{ \nu_i \}_{i \in \mathbb{N}} \) and \( \{ \hat{\nu}_i \}_{i \in \mathbb{N}} \) converging respectively to \( \nu^* \) and to \( \hat{\nu}_i^* \). By induction, suppose that \( \hat{\nu}_{i-1}^* \supseteq \nu_{i-1} \times q \). Take \( [\xi^i_0] \in \nu_{i-1} \times q \). Since \( x \in \nu_{i-1} \) we find that there exist \( \xi_1, \xi_2 \in \nu_{i-1} \) and \( \omega \in \mathbb{R}^n \) such that

\[
\begin{bmatrix} A_1 & A_2 \\ C & \end{bmatrix} x = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \omega.
\]

Now, we show that \( [\xi^i_0] \in \hat{\nu}_i \). In fact

\[
A^\Delta \begin{bmatrix} x \\ V \end{bmatrix} = \begin{bmatrix} A_1 x \\ A_2 x \\ A_{\Delta_1} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} V + \begin{bmatrix} 0 \\ 0 \end{bmatrix} D u,
\]

lies in \( (\nu_{i-1} \times q_o) \times q \). From the inductive assumption \( \nu_{i-1} \times q_o \subseteq \hat{\nu}_{i-1} \) it follows that \( \begin{bmatrix} \tilde{\xi}_1 \\ \tilde{\xi}_2 \end{bmatrix} \subseteq (\hat{\nu}_{i-1} \times q_o) \), and hence \( [\xi^i_0] \in \nu_{i-1} \times q_o \). Now, we prove the opposite inclusion. Let

\[
\nu_{i-1} \cap \text{im} \begin{bmatrix} I_n \\ 0_{q \times n} \end{bmatrix} \subseteq \nu_{i-1} \times q.
\]

To prove that the same is true for \( \hat{\nu}_i \), let \( \begin{bmatrix} x \\ 0 \end{bmatrix} \in \hat{\nu}_i \cap \text{im} \begin{bmatrix} I_n \\ 0_{q \times n} \end{bmatrix} \), so that

\[ \begin{bmatrix} A^\Delta_1 & F^\Delta \end{bmatrix} \begin{bmatrix} x \\ \omega \end{bmatrix} \in (\hat{\nu}_{i-1} \times q_o) + \text{im} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \]

on the other hand, by (16) we have

\[ \begin{bmatrix} A^\Delta_1 & F^\Delta \end{bmatrix} \begin{bmatrix} x \\ \omega \end{bmatrix} \in (\hat{\nu}_{i-1} \times q_o) \]

which leads to

\[ \begin{bmatrix} A^\Delta_1 & F^\Delta \end{bmatrix} \begin{bmatrix} x \\ \omega \end{bmatrix} \in (\nu_{i-1} \times q_o) \times q + \text{im} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \]

This in turn implies \( \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} x \in (\nu_{i-1} \times q_o) \times q + \text{im} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \) and so \( x \in \nu_{i-1} \times q \). Armed with Lemma 4.1, we can now provide a complete solution to Problem 2.1.

**Theorem 4.1.** Problem 2.1 is solvable if

(i) \( \text{im} \begin{bmatrix} \tilde{\xi}_1 \\ \tilde{\xi}_2 \end{bmatrix} \subseteq (\hat{\nu}^* \times \hat{\nu}^*) + \text{im} \begin{bmatrix} \tilde{\xi}_1 \\ \tilde{\xi}_2 \end{bmatrix} \text{ker } D; \)

(ii) \( \hat{\nu}^* \) is inner and outer stabilisable.

**Proof:** First, observe that the structural condition (i) is just a simplified way of writing (13), due to the fact that \( \hat{G} \) is zero. Now we show (ii). By virtue of Lemma 4.1, it follows that \( \hat{\nu}^* \) can be written as

\[
\hat{\nu}^* = \text{im} \begin{bmatrix} V \nu^* \\ 0 \nu^* \end{bmatrix}.
\]
From (18) we find the two identities $V_3X_{31} = 0$ and $V_3X_{41} = 0$, which lead to $X_{31} = 0$ and to $X_{41} = 0$ since $V_3$ is full column-rank. From the identities $A_1^TV_3X_{32} = V_3X_{32}$ and $A_2^TV_3X_{42}$, which follow from (18), we find that $V_3$ is an $(A_1^T, A_2^T)$-invariant subspace. Let us now write (10) for the output-nulling $\hat{V}^*$ in the partitioned form

$$
\begin{bmatrix}
A_1 H_1 C^\Delta + B_1 F_\xi \\
A_2 H_2 C^\Delta \\
C + D F_\xi \\
\end{bmatrix}
\begin{bmatrix}
V \\
V_2 \\
V_3 \\
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}
$$

From (18) we find the two identities $V_3X_{31} = 0$ and $V_3X_{41} = 0$, which lead to $X_{31} = 0$ and to $X_{41} = 0$ since $V_3$ is full column-rank. From the identities $A_1^TV_3X_{32} = V_3X_{32}$ and $A_2^TV_3X_{42}$, which follow from (18), we find that $V_3$ is an $(A_1^T, A_2^T)$-invariant subspace. Let us now write (10) for the output-nulling $\hat{V}^*$ in the partitioned form

$$
\begin{bmatrix}
A_1 H_1 C^\Delta \\
A_2 H_2 C^\Delta \\
C + G C^\Delta \\
\end{bmatrix}
\begin{bmatrix}
V \\
V_2 \\
V_3 \\
\end{bmatrix}
= 
\begin{bmatrix}
X_{11} X_{12} \\
X_{31} X_{32} \\
X_{41} X_{42} \\
\end{bmatrix}
\begin{bmatrix}
B_1 \\
B_2 \\
\end{bmatrix}
+ 
\begin{bmatrix}
\Omega_1 \\
\Omega_2 \\
\end{bmatrix}
$$

Since $\nu^*$ is inner and outer stabilisable, we can find a friend $F_\nu$ of $\nu^*$ such that $(A_1 + B_1 F_\nu) V = V X_{11}$. (Assume $X_{11}$ to be asymptotically stable; i.e., $F_\nu$ is inner and outer stabilises $\nu^*$.)

In Theorem 4.1, the structural condition is given in terms of $\nu^*$, while the stability condition is expressed in terms of the inner and outer stabilisability of $\nu^*$. Let us now write (18), we find that

$$(19)$$

$$\begin{bmatrix}
V_2 \\
V_3 \\
\end{bmatrix}
= 
\begin{bmatrix}
X_{11} X_{12} \\
X_{31} X_{32} \\
X_{41} X_{42} \\
\end{bmatrix}
\begin{bmatrix}
B_1 \\
B_2 \\
\end{bmatrix}
+ 
\begin{bmatrix}
\Omega_1 \\
\Omega_2 \\
\end{bmatrix}
$$

Now, the friend $\hat{F} = [F_x, F_\xi]$ of $\hat{V}^*$ can be computed as a solution of the equation $[\Omega_1 \Omega_2] = -[F_x, F_\xi] V_0 V_0^T = -[F_x, F_\xi] V_0 V_0^T$.

A direct check shows that the pair $(A_1 + B_1 F, A_2 + B_2 F)$ is asymptotically stable as it satisfies (2), so that $F$ stabilises $\nu^*$ externally as well. A model for system $\Delta$ is

$$
A_1^\Delta = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}, A_2^\Delta = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}, B_1^\Delta = \begin{bmatrix}
1 \\
0 \\
0 \\
0 \\
\end{bmatrix}, B_2^\Delta = \begin{bmatrix}
1 \\
0 \\
0 \\
0 \\
\end{bmatrix}
$$

A basis matrix for the subspace $\hat{V}^*$ can be expressed as in (17), where

$$
F_\xi = -(\Omega_2 + F_x V_2)(V_3 V_3^T)^{-1} V_3^T.
$$
\[
V_2 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0.9997 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\quad \text{and} \quad
V_3 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}.
\]

A direct check shows that conditions (i-ii) in Theorem 4.1 are satisfied, so that an input function in the form (4) with \( N = 2 \) and \( M = 1 \) exists such that the overall system is disturbance decoupled from the input \( \hat{w} \) to the output \( y \).

Let us exploit (19) for the computation of \( X_{12}, X_{22}, X_{32}, X_{42} \) and \( \Omega_2 \), so that (20) can be used to compute \( F_\xi \):

\[
F_\xi = \begin{bmatrix}
0 & 0 & 0 & -164 & -39.2662 \\
0 & 0 & 0 & -8 & -0.8910 \\
\end{bmatrix}
\]

As a result, the gain matrices of the FIR system are \( S_{0,0} = \begin{bmatrix}
-39.26624 \\
-0.8910 \\
\end{bmatrix} \), \( S_{0,1} = \begin{bmatrix}
-164 \\
-8 \\
\end{bmatrix} \), \( S_{1,0} = S_{1,1} = S_{2,0} = 0 \), while matrix \( S_{2,1} \) can be computed by solving equation (14) written with respect to \( \Sigma \) in \( \Psi \) and by taking \( S_{2,1} = -\Psi \). In this case, \( S_{2,1} = 0 \). It follows that the input \( u_{i,j} = F x_{i,j} + S_{0,0} w_{i,j} + S_{0,1} w_{i,j+1} \) solves the DDP. Clearly, the same result would have been found by choosing \( N = 0 \) and \( M = 1 \). This example shows that the possibility of enricbing the control law (4) with the previewed terms \( \varphi_{i,j} \) enlarges the possibilities of decoupling exactly the disturbance input \( w \) from the output \( y \).

Fig. 2. Disturbance \( w \) in the bounded frame \([0, 20] \times [0, 20]\).

Let \( \Sigma \) be subject to the randomly generated input \( w \) depicted within the interval \([0, 20] \times [0, 20]\) in Figure 2 and with randomly generated boundary conditions for \( \Sigma \). Asymptotic stability of the closed-loop guarantees that the output approaches zero as the index \((i,j)\) evolves away from the axes, see Figure 3.

Fig. 3. Output \( y_{i,j} \) in the interval \([0, 20] \times [0, 20]\) for nonzero boundary conditions.

In order to see that as the index \((i,j)\) evolves away from the axis the output \( y_{i,j} \) decreases in an exponential fashion, the first figure in Figure 4, shows the base 10 logarithm of \( |y_{i,j}| \) for \( i \in [0, 20] \).

If on the other hand we assume zero boundary conditions, the disturbance signal \( w \) in Figure 2 leads to the output \( y \) depicted in the second figure in Figure 4, which shows perfect decoupling (to within numerical noise).

Fig. 4. Logarithm of the output \(|y_{i,j}| \) for \( i \in [0, 20] \). Output \( y_{i,j} \) obtained with boundary conditions at zero. Note that \(|y_{i,j}| \sim 10^{-11}\).

REFERENCES


