Finite-Dimensional $H_{\infty}$ Filter Design for Linear Systems with Measurement Delay

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Abstract: This paper presents the central finite-dimensional $H_{\infty}$ filter for linear systems with measurement delay, that is suboptimal for a given threshold $\gamma$ with respect to a modified Bolza-Meyer quadratic criterion including the attenuation control term with the opposite sign. In contrast to the results previously obtained for linear time delay systems, the paper reduces the original $H_{\infty}$ filtering problem to an $H_2$ (optimal mean-square) filtering problem, using the technique proposed in [1]. Application of the reduction technique becomes possible, since the optimal filtering equations solving the $H_2$ (mean-square) filtering problems have been obtained for linear systems with measurement delay. The paper presents the central suboptimal $H_{\infty}$ filter for linear systems with measurement delay, based on the optimal $H_2$ filter from [36], where the standard $H_{\infty}$ filtering conditions of stabilizability, detectability, and noise orthonormality are assumed. Finally, to relax the standard conditions, the paper presents the generalized version of the designed $H_{\infty}$ filter in the absence of the noise orthonormality. The proposed $H_{\infty}$ filtering algorithm provides a direct method to calculate the minimum achievable values of the threshold $\gamma$, based on the existence properties for a bounded solution of the gain matrix equation. Numerical simulations are conducted to verify performance of the designed central suboptimal filter for linear systems with state delay against the central suboptimal filter available for linear systems without delays. The simulation results show a definite advantage in the values of the noise-output transfer function $H_{\infty}$ norms in favor of the designed filter.

1. INTRODUCTION

Over the past two decades, the considerable attention has been paid to the $H_{\infty}$ estimation problems for linear and nonlinear systems with and without time delays. The seminal papers in $H_{\infty}$ control [1] and estimation [2–4] established a background for consistent treatment of filtering/controller problems in the $H_{\infty}$-framework. The $H_{\infty}$ filter design implies that the resulting closed-loop filtering system is robustly stable and achieves a prescribed level of attenuation from the disturbance input to the output estimation error in $L_2/L_2$-norm. A large number of results on this subject has been reported for systems in the general situation, linear or nonlinear (see [5]–[13]). For the specific area of linear time-delay systems, the $H_{\infty}$-filtering problem has also been extensively studied (see [14]–[31], [8,11,12]). The sufficient conditions for existence of an $H_{\infty}$ filter, where the filter gain matrices satisfy Riccati equations, were obtained for linear systems with state delay in [32] and with measurement delay in [33]. However, the criteria of existence and suboptimality of solution for the central $H_{\infty}$ filtering problems based on the reduction of the original $H_{\infty}$ problem to the induced $H_2$ one, similar to those obtained in [1,4] for linear systems without delay, remain yet unknown for linear systems with time delays.

This paper presents the central (see [1] for definition) finite-dimensional $H_{\infty}$ filter for linear systems with measurement delay, that are suboptimal for a given threshold $\gamma$ with respect to a modified Bolza-Meyer quadratic criterion including the attenuation control term with the opposite sign. In contrast to the results previously obtained for linear systems with state [32] or measurement delay [33], the paper reduces the original $H_{\infty}$ filtering problems to $H_2$ (mean-square) filtering problems, using the technique proposed in [1]. To the best authors’ knowledge, this is the first paper which applies the reduction technique of [1] to classes of systems other than conventional LTI plants. Indeed, application of the reduction technique makes sense, since the optimal filtering equations solving the $H_2$ (mean-square) filtering problems have been obtained for linear systems with state [34,35] or measurement [36] delays. Designing the central suboptimal $H_{\infty}$ filter for linear systems with measurement delay presents a significant advantage in the filtering theory and practice, since (1) it enables one to address filtering problems for LTV time-delay systems, where the LMI technique is hardly applicable, (2) the obtained $H_{\infty}$ filter is suboptimal, that is, optimal for any fixed $\gamma$ with respect to the $H_{\infty}$ noise attenuation criterion, and (3) the obtained $H_{\infty}$ filter is finite-dimensional and has the same structure of the estimate and gain matrix equations as the corresponding optimal $H_2$ filter. Moreover,
the proposed $H_\infty$ filtering algorithms provide direct methods to calculate the minimum achievable values of the threshold $\gamma$, based on the existence properties for a bounded solution of the gain matrix equation. The corresponding calculations are made for the designed filter.

It should be commented that the proposed design of the central suboptimal $H_\infty$ filters for linear time-delay systems with integral-quadratically bounded disturbances naturally carries over from the design of the optimal $H_\infty$ filters for linear time-delay systems with unbounded disturbances (white noises). The entire design approach creates a complete filtering algorithm of handling the linear time-delay systems with unbounded or integral-quadratically bounded disturbances optimally for all thresholds $\gamma$ uniformly or for any fixed $\gamma$ separately. A similar algorithm for linear systems without delay was developed in [1].

The paper presents the central suboptimal $H_\infty$ filter for linear systems with measurement delay, based on the optimal $H_2$ filter from [34], where the standard $H_\infty$ filtering conditions of stabilizability, detectability, and noise orthonormality (see [4]) are assumed. Finally, to relax the standard conditions, the paper presents the generalized version of the designed $H_\infty$ filter in the absence of the noise orthonormality, using the technique of handling non-orthonormal noises carried over from [33].

Numerical simulations are conducted to verify performance of the designed central suboptimal filter for linear systems with measurement delay against the central suboptimal $H_\infty$ filter available for linear systems without delays [4]. The simulation results show a definite advantage in the values of the noise-output transfer function norms in favor of the designed filter.

The paper is organized as follows. Section 2 presents the $H_\infty$ filtering problem statement for linear systems with state delay. The central suboptimal $H_\infty$ filter for linear systems with measurement delay is designed in Section 3. An example verifying performance of the $H_\infty$ filter designed in Sections 3 against the central suboptimal $H_\infty$ filter available for linear systems without delays is given in Section 4. The obtained results are generalized to the case of non-orthonormal noises in Section 5. Section 6 presents conclusions to this study.

2. $H_\infty$ FILTERING PROBLEM STATEMENT

Consider the following continuous-time LTV system with measurement delay:
\[ \mathcal{M}_1: \begin{align*}
\dot{x}(t) &= A(t)x(t) + B(t)\omega(t), \quad x(t_0) = x_0, \\
y(t) &= C(t)x(t-h) + D(t)\omega(t), \\
z(t) &= L(t)x(t),
\end{align*} \]

where $x(t) \in \mathbb{R}^n$ is the state vector, $z(t) \in \mathbb{R}^q$ is the signal to be estimated, $y(t) \in \mathbb{R}^p$ is the measured output, $\omega(t) \in L^2_{\omega}[0, \infty)$ is the disturbance input. $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, $D(\cdot)$, and $L(\cdot)$ are known continuous functions. $x_0$ is an unknown initial vector. The time delay $h$ is known.

For the system (1)-(4), the following standard conditions (\{4\}) are assumed:
- the pair $(A, B)$ is stabilizable; \((\mathcal{E}_1)\)
- the pair $(C, A)$ is detectable; \((\mathcal{E}_2)\)
- $D(t)B^T(t) = 0$ and $D(t)D^T(t) = I_m$. \((\mathcal{E}_3)\)

Here, $I_m$ is the identity matrix of dimension $m \times m$. As usual, the first two conditions ensure that the estimation error, provided by the designed $H_\infty$ filter, converge to zero (\{37\}). The last noise orthonormality condition is technical and corresponds to the condition of independence of the standard Wiener processes (Gaussian white noises) in the stochastic filtering problems (\{37\}).

Consider a full-order $\mathcal{H}_\infty$ filter in the following form:
\[ \mathcal{M}_2: \begin{align*}
\dot{x}_m(t) &= A(t)x_m(t) + K_m(t)[y(t) - C(t)x_m(t-h)], \\
z_m(t) &= L(t)x_m(t),
\end{align*} \]

where $x_m(t)$ is the filter state. The gain matrix $K_m(t)$ is to be determined.

Upon transforming the model (1)-(3) to include the states of the filter, the following filtering error system is obtained:
\[ \mathcal{M}_3 : \begin{align*}
\dot{e}(t) &= A(t)e(t) + B(t)\omega(t) - K_m(t)\tilde{y}(t), \\
\tilde{y}(t) &= C(t)(e(t-h) + D(t)\omega(t)), \\
\tilde{z}(t) &= L(t)e(t),
\end{align*} \]

where $e(t) = x(t) - x_m(t)$, $\tilde{y}(t) = y(t) - C(t)x_m(t-h)$, and $\tilde{z}(t) = z(t) - z_f(t)$.

Therefore, the problem to be addressed is stated similarly to the $H_\infty$ filtering problem from Section 2: develop a robust $\mathcal{H}_\infty$ filter of the form (4)-(5) for the LTV system with measurement delay ($\mathcal{M}_1$), such that the following two requirements are satisfied:

1. The resulting filtering error dynamics ($\mathcal{M}_3$) is robustly asymptotically stable in the absence of disturbances, $\omega(t) \equiv 0$.
2. The filtering error dynamics ($\mathcal{M}_3$) ensures a noise attenuation level $\gamma$ in an $\mathcal{H}_\infty$ sense. More specifically, for all nonzero $\omega(t) \in L^2_{\omega}[0, \infty)$, the inequality
\[ \|\tilde{z}(t)\|_2^2 \leq \gamma^2 \left\{\|\omega(t)\|_2^2 + \|x_0\|_{2, R}^2\right\} \]

holds for $\mathcal{H}_\infty$ filtering problem, where $\|f(t)\|_2^2 := \int_{t_0}^{\infty} f^T(t)f(t)dt$, $\|x_0\|_{2, R}^2 := x_0^T R x_0$, $R$ is a positive definite symmetric matrix, and $\gamma$ is a given real positive scalar.

3. FINITE-DIMENSIONAL $H_\infty$ FILTER DESIGN

The proposed design of the central $H_\infty$ filter (see Theorem 4 in [1]) for LTV systems with measurement delay is also based on the general result (see Theorem 3 in [1]) reducing the $H_\infty$ controller (in particular, filtering) problem to the corresponding $H_2$ (i.e., optimal linear-quadratic) controller (or mean-square filtering) problem. Then, the optimal mean-square filter of the Kalman-Bucy type for LTV systems with measurement delay (\{36\}) is employed to obtain the desired result, which is given by the following theorem.

**Theorem 1.** The central $H_\infty$ filter for the unobserved state (1) over the observations (2), ensuring the $H_\infty$ noise attenuation condition (9) for the output estimate $z_m(t)$, is given by the equations for the state estimate $x_m(t)$ and the output estimate $z_m(t)$
\[ x_m(t) = A(t)x_m(t) + P(t)\exp(-\int_{t-h}^{t} A^T(s)ds) \]

}\]
The condition $C^T(t)C(t) - \gamma^2 L^T(t)L(t) > 0$ assures boundedness of the filter gain matrix $P(t)$ for any finite $t$, and also as time goes to infinity. Apparently, if $C^T(t)C(t) - \gamma^2 L^T(t)L(t) < 0$, then the function $P(t)$ diverges to infinity for a finite time and the designed filter does not work. If the equality $C^T(t)C(t) - \gamma^2 L^T(t)L(t) = 0$ holds, then the estimation error is uniformly asymptotically stable, if the state dynamics matrix $A(t)$ itself is asymptotically stable.

Remark 3. According to the comments in Subsection V.G in [1], the obtained $H_\infty$ filter (10)–(12) has the suboptimality property, i.e., it minimizes the criterion

$$J = \|\hat{z}(t)\|_2^2 - \gamma^2 \left( \|\omega(t)\|_2^2 + \|\nu(t)\|_2^2 \right),$$

for such positive $\gamma > 0$ that the inequality $C^T(t)C(t) - \gamma^2 L^T(t)L(t) > 0$ holds.

Remark 4. Following the discussion in Subsection V.G in [1], note that the complementarity condition always holds for the obtained $H_\infty$ filter (10)–(12), since the positive definiteness of the initial condition matrix $R$ implies the positive definiteness of the filter gain matrix $P(t)$ as the solution of (13). Therefore, the stability failure is the only reason why the obtained filter can stop working. Thus, the stability margin
\[ \gamma = \sqrt{\|L^T(t)L(t)\|/\|C^T(t)C(t)\|} \] also defines the minimum possible value of \( \gamma \), for which the \( H_\infty \) condition (9) can still be satisfied.

4. EXAMPLE

This section presents an example of designing the central \( H_\infty \) filter for a linear state over linear observations with measurement delay and comparing it to the best \( H_\infty \) filter available for a linear system without delay, that is the filter obtained in Theorems 3 and 4 from [4].

Let the unmeasured state \( x(t) = [x_1(t), x_2(t)] \in \mathbb{R}^2 \) (a mechanical oscillator without delay) be given by

\[ \begin{align*}
\dot{x}_1(t) &= x_2(t), \\
\dot{x}_2(t) &= -x_1(t) + w_1(t),
\end{align*} \tag{19}\]

with an unknown initial condition \( x(0) = x_0 \), the delayed scalar observation process satisfy the equation

\[ y(t) = x_1(t - 5) + w_2(t), \tag{20}\]

and the scalar output be represented as

\[ z(t) = x_1(t). \tag{21}\]

Here, \( w(t) = [w_1(t), w_2(t)] \) is an \( L_2 \) disturbance input. It can be readily verified that the noise orthonormality condition (see Section 2) holds for the system (19)–(21).

The filtering problem is to find the \( H_\infty \) estimate for the linear state (19) over direct linear observations with measurement delay (20), which satisfies the noise attenuation condition (9) for a given \( \gamma \), using the designed \( H_\infty \) filter (10)–(12). Since the simulation in the interval \([0, 10]\) occurs to be insufficient to reveal the convergent properties of the output estimation errors, the filtering horizon is extended and set to \( T = 20 \).

The filtering equations (10)–(12) take the following particular form for the system (19),(20)

\[ \begin{align*}
\dot{x}_{m_1}(t) &= x_{m_2}(t) + (0.2837 P_{11}(t) + 0.9589 P_{12}(t)) y(t) - x_{m_1}(t - 5), \\
\dot{x}_{m_2}(t) &= -x_{m_1}(t) + (0.2837 P_{21}(t) + 0.9589 P_{22}(t)) y(t) - x_{m_2}(t - 5),
\end{align*} \tag{22,23}\]

with the initial condition \( x_0(t) = 0 \), where 0.2837 and 0.9589 are \((1, 1)\) and \((2, 1)\) entries of the integral of the reverse-time dynamics matrix for the state (19), \( \exp(-f_{0, t} \omega \mathbb{R}) \), and

\[ \begin{align*}
\hat{P}_{11}(t) &= 2 P_{11}(t) - (1 - \gamma^2) [0.0805 P_{11}^2(t) + 0.5444 P_{11}(t) P_{12}(t) + 0.9195 P_{12}^2(t)], \\
\hat{P}_{12}(t) &= -P_{11}(t) + P_{12}(t) - (1 - \gamma^2) [0.0805 P_{11}(t) P_{21}(t) + 0.272 P_{21}^2(t) + 0.1595 P_{12}(t) P_{22}(t)], \\
\hat{P}_{22}(t) &= 1 - 2 P_{12}(t) - (1 - \gamma^2) [0.0805 P_{21}^2(t) + 0.5444 P_{12}(t) P_{21}(t) + 0.9195 P_{22}^2(t)],
\end{align*} \tag{24,25}\]

with the initial condition \( P(0) = R^{-1} \), where the numerical values are the corresponding entries of the matrix \[ \exp(-f_{0, t} \omega \mathbb{R}) \] \( \exp(-f_{0, t} \omega \mathbb{R}) \).

The estimates obtained upon solving the equations (22),(23) are compared to the conventional \( H_\infty \) filter estimates, obtained in Theorems 3 and 4 from [4], which satisfy the following equations:

\[ \hat{m}_{K_1}(t) = m_{K_1}(t) + P_{11}(t) [y(t) - m_{K_1}(t - 5)], \tag{26}\]

\[ \hat{m}_{K_2}(t) = -m_{K_2}(t) + P_{12}(t) [y(t) - m_{K_2}(t - 5)], \tag{27}\]

with the initial condition \( m_{K_1}(0) = 0 \); \( k_{11}(t) = 2 P_{12}(t) - (1 - \gamma^2) P_{11}(t) \), \( k_{12}(t) = -P_{11}(t) + P_{12}(t) - (1 - \gamma^2) [0.0805 P_{11}(t) P_{21}(t) + 0.272 P_{21}^2(t) + 0.1595 P_{12}(t) P_{22}(t)], \)

\[ \hat{P}_{22}(t) = 1 - 2 P_{12}(t) - (1 - \gamma^2) [0.0805 P_{21}^2(t) + 0.5444 P_{12}(t) P_{21}(t) + 0.9195 P_{22}^2(t)], \tag{24,25}\]

with the initial condition \( P(0) = R^{-1} \), where the numerical values are the corresponding entries of the matrix \[ \exp(-f_{0, t} \omega \mathbb{R}) \] \( \exp(-f_{0, t} \omega \mathbb{R}) \).

The estimates obtained upon solving the equations (22),(23) are compared to the conventional \( H_\infty \) filter estimates, obtained in Theorems 3 and 4 from [4], which satisfy the following equations:

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\[ \hat{m}_{K_2}(t) = -m_{K_2}(t) + P_{12}(t) [y(t) - m_{K_2}(t - 5)], \tag{27}\]

with the initial condition \( m_{K_1}(0) = 0 \); \( k_{11}(t) = 2 P_{12}(t) - (1 - \gamma^2) P_{11}(t) \), \( k_{12}(t) = -P_{11}(t) + P_{12}(t) - (1 - \gamma^2) [0.0805 P_{11}(t) P_{21}(t) + 0.272 P_{21}^2(t) + 0.1595 P_{12}(t) P_{22}(t)], \)

\[ \hat{P}_{22}(t) = 1 - 2 P_{12}(t) - (1 - \gamma^2) [0.0805 P_{21}^2(t) + 0.5444 P_{12}(t) P_{21}(t) + 0.9195 P_{22}^2(t)], \tag{24,25}\]

with the initial condition \( P(0) = R^{-1} \), where the numerical values are the corresponding entries of the matrix \[ \exp(-f_{0, t} \omega \mathbb{R}) \] \( \exp(-f_{0, t} \omega \mathbb{R}) \).

It can be concluded that the central suboptimal multi-equational \( H_\infty \) filter (22),(23) provides reliably convergent behavior of the output estimation error, yielding a convincingly lesser value of the corresponding \( H_\infty \) norm, in comparison to the assigned threshold value \( \gamma = 1.1 \). The latter serves as an ultimate bound of the noise-output \( H_\infty \) norm as time tends to infinity. In contrast, the conventional central \( H_\infty \) filter (24),(25) provides divergent behavior of the output estimation error, yielding a much greater value of the corresponding \( H_\infty \) norm, which largely exceeds the assigned threshold. Thus, the simulation results show definite advantages of the designed central suboptimal \( H_\infty \) filter for linear systems with measurement delay, in comparison to the previously known conventional \( H_\infty \) filter.

5. GENERALIZATIONS

As shown in [33], the noise orthonormality condition (\( \mathcal{O}_3 \)), third standard condition from Section 2 (see also [1,4]), can be omitted. This leads to appearance of additional terms in all \( H_\infty \) filtering equations. The corresponding generalizations of the obtained \( H_\infty \) filters are given in the following propositions.

Corollary 1. In the absence of the noise orthonormality condition (\( \mathcal{O}_3 \)), the central \( H_\infty \) filter for the unobserved state (1)
over the observations (2), ensuring the $H_\infty$ noise attenuation condition (9) for the output estimate $z_m(t)$ is given by

$$
\dot{z}_m(t) = A(t)x_m(t) + [P(t)\exp(-\int_{t-h}^t A^T(s)ds)C^T(t) + B(t)D^T(t)\gamma(t) - C(t)x_m(t - h)],
$$

with the initial condition $x_m(t_0) = 0$, and the equation for the filter gain matrix $P(t)$

$$
dP(t) = (P(t)A(t) + A(t)P(t) + B(t)B^T(t) + \gamma^{-2}P(t)\exp(-\int_{t-h}^t A^T(s)ds)L^T(t)\times L(t)\exp(-\int_{t-h}^t A(s)ds)P(t) - [P(t)\exp(-\int_{t-h}^t A^T(s)ds)C^T(t) + B(t)D^T(t)]T^{-1}(t)\times [C(t)\exp(-\int_{t-h}^t A(s)ds)P(t) + D(t)B^T(t)]) dt,
$$

with the initial condition $P(0) = R^{-1}$, where the matrix $T(t)$ is defined as

$$
T(t) = D(t)D^T(t) + C(t)\exp(-\int_{t-h}^t A(s)ds)\times \int_{t-h}^t \exp\left(\int_{\tau}^t A(s)ds\right)B(\tau)B^T(\tau)\exp\left(\int_{\tau}^t A^T(s)ds\right)dt\times \exp(-\int_{t-h}^t A^T(s)ds)C^T(t).
$$

**Proof.** The proof is straightforwardly delivered using the technique of handling the $H_\infty$ filtering problems for systems with non-orthonormal noises, which can be found in [33].

**Remark 5.** Since the $H_\infty$ filter designed in Corollary 1 is based on the corresponding $H_2$ mean-square filter, which is optimal with respect to mean square criteria, Remarks 2–4 remain valid.

### 6. CONCLUSIONS

This paper designs the central finite-dimensional $H_\infty$ filter for linear systems with measurement delay, that is suboptimal for a given threshold $\gamma$ with respect to a modified Bolza-Meyer quadratic criterion including the attenuation control term with the opposite sign. The central suboptimal $H_\infty$ filter for linear systems with measurement delay is obtained. Finally, the generalized version of the filter is designed in the absence of the standard noise orthonormality condition.

In the example based on a model of a mechanical oscillator, the numerical simulations are run to verify performance of the designed central suboptimal filter for linear systems with measurement delay against the central suboptimal $H_\infty$ filter available for linear systems without delays. The simulation results show a definite advantage in the values of the noise-output transfer function $H_\infty$ norms in favor of the designed filter. In particular, the estimation errors given by the obtained filter converge to zero, whereas the estimation error of the conventional filter diverges. This significant improvement in the estimate behavior is obtained due to the more careful selection of the filter gain matrix in the designed filter. Although this conclusion follows from the developed theory, the numerical simulation serves as a convincing illustration.

The proposed design of the central suboptimal $H_\infty$ filter for linear time-delay systems with integral-gradually bounded disturbances naturally carries over from the design of the optimal $H_2$ filter for linear time-delay systems with unbounded disturbances (white noises). The entire design approach creates a complete filtering algorithm of handling the linear time-delay systems with unbounded or integral-gradually bounded disturbances optimally for all thresholds $\gamma$ uniformly or for any fixed $\gamma$ separately. The presented approach would be applied in the future to obtain the central suboptimal $H_\infty$ filters for nonlinear polynomial and nonlinear polynomial time-delay systems.

### REFERENCES


Section 5. Simulation Results

Fig. 1. Above. Graph of the output $H_m$ estimation error $z(t) - z_m(t)$ corresponding to the estimate $x_m(t)$ satisfying the equations (22),(23), in the simulation interval [0,20]. Below. Graph of the noise-output $H_m$ norm corresponding to the shown output $H_m$ estimation error, in the simulation interval [0,20].

Fig. 2. Above. Graph of the output $H_m$ estimation error $z(t) - z_m(t)$ corresponding to the estimate $x_m(t)$ satisfying the equations (24),(25), in the simulation interval [0,20]. Below. Graph of the noise-output $H_m$ norm corresponding to the shown output $H_m$ estimation error, in the simulation interval [0,20].