A Multi–Parametric Optimization Strategy for Multilevel Hierarchical Control problems

Nuno P. Faisca, Konstantinos I. Kouramas, Efstratios N. Pistikopoulos

Centre for Process Systems Engineering, Department of Chemical Engineering, Imperial College London, SW7 2AZ, UK (e-mail: e.pistikopoulos@imperial.ac.uk).

Abstract: In this work the three–level hierarchical control problem and the decentralised control problem are investigated and a general optimisation strategy is developed for solving these problems based on recent developments on multi–parametric programming. The main idea is to recast each optimisation subproblem in the multilevel hierarchy as a multi–parametric programming problem and then transform the multilevel problem into a single-level optimisation problem. This allows for the control policies (decisions) at each level of the multilevel optimisation problem to be obtained as explicit functions of the state of the dynamic systems involved in each level and the control policies of the higher levels. A three person dynamic optimisation problem is presented to illustrate the mathematical developments.

Keywords: Hierarchical control; decentralised control; multilevel and multi-follower optimization; multi–parametric programming.

1. INTRODUCTION

In optimisation and control of large–scale dynamic systems, hierarchical and decentralised control allow for the decomposition of the original problem into smaller, interconnected problems which are typical arranged into a multilevel hierarchy (Mesarovic et al. [1970], Cohen [1977], Morari et al. [1980], N.R. Sandell et al. [1978], Stephanopoulos and Ng [2000], Venkat et al. [2005]). Various applications of hierarchical and decentralised control arise in process systems engineering (Morari et al. [1980], Stephanopoulos and Ng [2000]), mechanical and power systems (Delaleau and Stanković [2004]), aeronautics (Tolani et al. [2004], Li et al. [2002]), traffic control (Shimizu et al. [1995]) and large-scale systems control (N.R. Sandell et al. [1978], Roberts and Becerra [2001]). In most of these problems the general formulations of the decomposed multilevel hierarchy is given in Figures 1 and 2. Examples of the hierarchical structure in Figure 1 can be found in Delaleau and Stanković [2004], Singh et al. [1975], Stanković and Šiljak [1989]. In Delaleau and Stanković [2004] the control problem of a PM synchronous motor is decomposed into a bi–level hierarchical control problem (similar to Figure 1 without the third level). A high–level controller, corresponding to the slow dynamics of the motor’s mechanical system, is designed to obtain the right set points for the low–level controller which controls the fast dynamics of the electrical system. In [Singh et al., 1975, Section 5] and Stanković and Šiljak [1989] the same hierarchical decomposition was used to deal with optimal LQG and optimal LQR control of sequentially (or serially) interconnected linear dynamical systems. Decentralised control of large-scale systems also yields a two–level structure where in the lower level more than one subproblems are considered (Figure 2) (N.R. Sandell et al. [1978], Venkat et al. [2005]).

* This work is supported by EPSRC (GR/T02560/01) and Marie Curie European Project PRISM (MRTN-CT-2004-512233).
Recent work by Pistikopoulos and co-workers (Pistikopoulos et al. [2002], Dua et al. [2002], and Faïse et al. [2007, 2008]) have proposed novel solution algorithms, based on multi-parallel programming theory (Fiacco [1983], Pistikopoulos et al. [2007],) which open the possibility to address general classes of multilevel optimization problems. In an optimization problem where the objective is to minimize (or maximize) a cost criterion subject to constraints and which includes a number of parameters varying between specified lower and upper bounds, multi-parallel programming is a method to obtain i) the objective function and the optimization variables as functions of the parameters and ii) the region in the parameter space where these functions are valid (Fiacco [1983], Dua et al. [2002], Pistikopoulos et al. [2002]). Multi-parallel programming has found many applications in model-based predictive control (Bemporad et al. [2002], Pistikopoulos et al. [2002, 2007]) where the on-line optimization problem is solved off-line by multi-parallel programming and the control policy is obtained as a set of explicit functions of the state measurements. The control action is then implemented on-line by function evaluations, reducing the complexity of the implementation of the controller and improving the computational speed.

Our approach is to recast each optimization subproblem as a multi-parallel programming problem, and hence obtain an analytical solution for the rational reaction set for each of the subproblems. Each of the control policies at each level of the three-level hierarchical control problem are then obtained as explicit functions of the states of the system/systems involved in each level and the policies on the higher levels, with the policy of the highest level obtained as an explicit function of the initial states only. In the decentralized control problem, the policy for each secondary controller is obtained as a function of the overall system and the policy of the central controller. The policy of the central controller is an explicit function of the initial state of the system. The policies at each level then of both problems can then be obtained as simple function evaluations. In the knowledge of the authors there is currently no relevant work on treating multilevel hierarchical control problem with multi-parallel programming, despite the vast research on these problems.

The paper is organised as follows. Section II discusses the methodology for solving three-level hierarchical multilevel control problems by multi-parallel programming techniques. The two-level decentralized control problem (leader-follower problem) is described and solved again by multi-parallel programming in Section III. An example is presented in Section IV that illustrates the two methodologies and the complexity analysis on and on-line multilevel optimization issues are also discussed. The paper finally concludes with the necessary conclusions.

2. THREE LEVEL HIERARCHICAL CONTROL

Consider the following three-level hierarchical control problem

\[ J_1 = \min_{U^T, U^1, U^2} \{ V_1(x, x_1^1, x_2^1, u_1^1, u_2^1, u_2^2) \} \] (1a)

\[ \text{s.t. } x_{i+1} = A x_i + B u_i + D x_i \] (1b)

\[ g(x_i, x_{i,1}^1, x_{i,2}^1, u_1^i, u_2^i) \leq 0 \] (1c)

\[ J_2 = \min_{U^1} \{ V_2(x, x_1^1, x_2^1, u_1^1, u_2^1) \} \] (1d)

\[ \text{s.t. } x_{i+1}^1 = A x_i^1 + B_1 u_1^i + D_1 x_i^1 \] (1e)

\[ g_2(x_i, x_{i,1}^1, x_{i,2}^1, u_1^i, u_2^i) \leq 0 \] (1f)

\[ J_3 = \min_{U^2} \{ V_3(x, x_1^1, x_2^1, u_1^1, u_2^1) \} \] (1g)

\[ \text{s.t. } x_{i+1}^2 = A x_i^2 + B_2 u_2^i + D_2 x_i^2 \] (1h)

\[ g_3(x_i, x_{i,1}^1, x_{i,2}^1, u_1^i, u_2^i) \leq 0 \] (1i)

\[ x_i \in \mathbb{R}^n_i, x_1^i \in \mathbb{R}^{n_1}, x_2^i \in \mathbb{R}^{n_2}, t = 1, \ldots, N \] (1j)

\[ u \in \mathbb{R}^m, u^1 \in \mathbb{R}^{m_1}, u^2 \in \mathbb{R}^{m_2}, t = 1, \ldots, N-1 \] (1k)

where \( V_1, V_2, V_3 \) are all convex quadratic functions of \( x, x_1^i, x_2^i \) and the control sequences (control policies) \( U = \{ u_0, \ldots, u_{N-1} \} \in \mathbb{R}^{mN}, U^1 = \{ u_0^1, \ldots, u_{N-1}^1 \} \in \mathbb{R}^{m_1N} \) and \( U^2 = \{ u_0^2, \ldots, u_{N-1}^2 \} \in \mathbb{R}^{m_2N} \). The matrices \( A, A_1, A_2, B_1, B_2, D_1 \) and \( D_2 \) are of appropriate dimensions while \( g, g_1, g_2 \) are vectors of linear functions of \( x_i, x_1^i, x_2^i, U, U^1 \) and \( U^2 \) defining a set of linear inequality constraints in (1c), (1f) and (1i). Finally, the vectors \( z = (x^T \, (x_1^1)^T \, (x_1^2)^T) \) and \( z^2 = (x^T \, (x_1^1)^2)^T \) represent the interconnection between the linear discrete-time dynamic system in the different hierarchical levels of problem (1). Problem (1) is a multilevel optimization problem of the form shown in Figure 1 where in each level a different dynamic system, objective and set of constraints are considered.

A special subcase of problem (1) is the multilevel hierarchical control of sequentially interconnected dynamical systems (Singh et al. [1975], Stanković and Šiljak [1989]) where \( z^1 = 0 \), \( z^2 = x \) and \( z = x^T \, x_1^2 \) i.e. the dynamical system at each level of (1) is independent of the ones in the higher levels (it does not depend on the states and inputs of the higher systems in the higher levels). In this case the constraints of the problem at each level are also independent of the states and inputs of the higher levels. The structure of the problem for sequentially interconnected dynamic systems is thus simpler than the one considered in (1). However, the more complex multilevel optimisation problem (1) is examined here to cover a more broad case of hierarchical control problems as those arising for example in Delaleau and Stanković [2004]. Finally, extension to smaller bi-level optimization problems (with no third optimisation level in (1)) should be easy.

The objective is, given the initial states \( x_0, x_0^1, x_0^2 \) of each linear system, to obtain the optimal control policies \( U^*, U_1^*, U_2^* \) for the three-level hierarchical control problem (1). Substituting the linear discrete-time model equations (1b), (1e) and (1h) into (1a), (1d), (1g) and (1c), (1f), (1i) the following multilevel optimisation problem is formulated
\[ J_1 = \min_{U, U^2, U^1} V_1(x_0, x_0', x_0^2, U, U^1, U^2) \] (2a)
\[ \text{s.t. } g(x_0, x_0', x_0^2, U, U^1, U^2) \leq 0 \] (2b)
\[ J_2 \quad \text{of the central controller. Only the central controller has access over the complete set of optimisation variables.} \]

\[ J_1 = \min_{U, U^2, U^1} V_1(x_0, x_0', x_0^2, U, U^1, U^2) \] (2a)
\[ \text{s.t. } g(x_0, x_0', x_0^2, U, U^1, U^2) \leq 0 \] (2b)
\[ J_2 \quad \text{of the central controller. Only the central controller has access over the complete set of optimisation variables.} \]

In Step 5 of Algorithm 1 the control policy \( U \) is obtained as an explicit function of the initial states \( x_0, x_0', x_0^2 \) of the linear systems (1b), (1e), (1h). Thus, when the initial states \( x_0, x_0', x_0^2 \) are given the optimal control policy \( U \) can be obtain by simple function evaluations from (9). The optimal control policy \( U \) for the second optimisation level is obtained from (8) given \( x_0, x_0', x_0^2, U \) and finally \( U^1 \) is obtained from (7). It can be noticed that the control policy at each hierarchical level is a function of the control policies of the higher levels. The rational reaction set for each level is thus obtained analytically.

**Remark 1.** It is possible that in Steps 4 and 5 overlapping regions might occur since quadratic objective functions are considered. In that case one can employ the comparison method described in Acevedo and Pistikopoulos (1997) to compute the explicit solutions in the overlapping of the regions.

### 3. DECENTRALISED HIERARCHICAL CONTROL

Consider the decentralised control scheme of Figure (2) that consist of a central controller (the leader) and \( m \) secondary controllers (the followers) on the second level (Başar and Selbuz [1994]). The objective is the optimal control of the overall linear discrete–time system

\[ x_{t+1} = A x_t + B_0 u_t + \sum_{i=1}^{s} B_i u_i \] (10)

where \( x_t \in \mathbb{R}^n, u_t \in \mathbb{R}^{n}, u_i \in \mathbb{R}^{m_i}, i = 1, \ldots, s \) are the states of the system, the input to the system controlled by the leader and the inputs to the system controlled by the followers respectively, with respect to the following quadratic optimal control problem

\[ J_0 = \min_{U, U^1, \ldots, U^s} V(x_t, U) \] (11a)

\[ = \min_{U} x_t^T Q_{KN} x_t + \sum_{i=0}^{N-1} \left[ x_i^T Q_{ii} x_i + u_i^T R_i u_i + \sum_{i=1}^{s} u_i^T R_{ii} u_i \right] \] (11b)

\[ \text{s.t. } x_{t+1} = A x_t + B_0 u_t + \sum_{i=1}^{s} B_i u_i \] (11c)

\[ g(x_t, U, U^1) \leq 0 \] (11d)

where \( U = \{u_0, \ldots, u_{N-1}\}, U^1 = \{u_0', \ldots, u_{S-1}'\}, i = 1, \ldots, s \) are the control policies of the leader and \( i \)-th follower respectively, \( g \) is a vector of linear functions corresponding to a set of linear inequalities (11c). It is assumed that the secondary controllers do not have access to the minimisation of (11a) but they select the optimal policies \( U^i \) based on the *announced* policy \( U \) of the central controller. Only the central controller has access over the complete set of optimisation variables.

---

6309
Each of the secondary controllers solves a local optimisation subproblem
\[ J_i = \min_{U_i} V_i(x_t, U, U^i, U^j) \] (12a)
\[ = \min_{U^i} x_t^T Q_i^0 x_t + \sum_{t=0}^{N-1} \left[ x_t^T Q^0_i x_t + u_t^T R^0_i u_t + \sum_{k=1}^{s} u_t^T R^k_i u_t \right] \]
\[ \text{s.t. } x_{t+1} = A_i x_t + B_0 u_t + \sum_{j=1}^{s} B_{ij} u_t^j \] (12b)
\[ g_i(x_t, U, U^i, U^j) \leq 0 \] (12c)
\[ Q^0_i \geq 0, R^0_i \geq 0, R^k_i > 0 \] (12d)
for the \( i \)-th secondary controller, \( i = 1, \ldots, s \). Only the \( i \)-th follower has access to the \( i \)-th minimisation problem (12) i.e. the minimisation in (12) is only with respect to \( U^i \).

Remark 2. It is possible that each of the secondary controllers could include its own local linear dynamic model. The problem can then be formulated by including all linear dynamic models altogether in (11)–(12) and by adding extra quadratic terms in the objective functions (11a) and (12a) to accommodate for the new states introduced. The objectives and the constraints in (11) and (12) are then all functions of all states and inputs.

The multilevel optimisation problem based on the leader–follower formulation of (11)–(12) has been treated for the case where no constraints are present N.R. Sandell et al. [1978], Başar and Selbst [1994], while in Venkat et al. [2005] a similar distributed control problem was treated with input constraints.

Here, the more general problem is treated where both state and input constraints are present as is the case in most engineering problems where any real actions is restricted by physical or operational constraints. The problem is solved to obtain the global optimum of the central controller and the best possible optimal policies for the \( i \)-th controller.

Replacing (11b) in (11), (12) and considering we seek for the optimal policies \( U^*, U^{i*} \) then the problem reformulates in the following multilevel optimisation problem

\[ J_0 = \min_{U, U^i, U^j} V(x_0, U, U^1, \ldots, U^p) \] (13a)
\[ \text{s.t. } g_i(x_0, U, U^1, \ldots, U^p) \leq 0 \] (13b)
\[ \{ J_i = \min_{U^i} V_i(x_0, x_t^0, U, U^1, \ldots, U^p) \} \]
\[ \text{s.t. } g_i(x_0, U, U^1, \ldots, U^j) \leq 0 \} \] (13c)

where in the second level (13c) the secondary controller optimisation problems are considered altogether. Since \( V, V_i \) are quadratic functions and \( g_i \) are linear functions of \( x_0, U, U^i, U^j \), both optimisation problems in both levels are quadratic programming problems with respect to the \( x_0, U, U^i \). The feasible and rational reaction set of each of the secondary controllers are respectively

\[ \Omega_i(x_0, U, U^1, \ldots, U^{i-1}, U^{i+1}, \ldots, U^p) = \{ U^i \in \mathbb{R}^{N_{0i}} \mid g_i \leq 0 \} \] (14)
\[ \Phi_i(x_0, U, U^1, \ldots, U^{i-1}, U^{i+1}, \ldots, U^p) = \{ U^i \in \mathbb{R}^{N_{0i}} \mid U^i \in \text{argmin}_i \{ V_i \mid U^i \in \Omega_i \} \} \] (15)

One can notice that the central controller problem as well as each of the secondary controller problems can be recast as multi–parametric programming problems where \( x_0 \) is the parameter and \( U^i, U^j \) are the optimisation variable. More specifically, each of the secondary controller programs are multi–parametric programming problems where the control policy \( U^j \) is the optimisation variable and \( x_0 \) and the control policies \( U, U^j, j \neq i \) are the parameters.

Each secondary controller has access to the optimisation problem of the rest of the secondary controllers either via the state \( x_t \) or via the announced policy of the central controller. In that case it is most natural to introduce an equilibrium concept between the secondary controllers.

\[ V(x_0, U^i, U^{i+1}, \ldots, U^p) \leq V(x_0, U, U^1, \ldots, U^p) \] (16a)
\[ V_i(x_0, U^i, U^{i+1}, \ldots, U^p) \leq V_i(x_0, U^i, U^1, U^2, \ldots, U^p) \] (16b)
\[ \vdots \]
\[ V_s(x_0, U^i, U^{i+1}, \ldots, U^p) \leq V_s(x_0, U^i, U^1, U^2, \ldots, U^{p-1}, U^p) \] (16c)
similar to the Nash equilibrium concept. The equilibrium can be computed by direct comparison as in Faïsca et al. [2007], Liu [1998].

As was shown previously each of the optimisation problems (11) and (12) can be recast as multi–parametric programming problems. Hence, one can solve the problem by employing multi–parametric programming techniques (Dua et al. [2002], Pistikopoulos et al. [2002]). This is given in detail in the following algorithm which applies multi–parametric programming to solve the leader–follower problem (11)–(12)

Algorithm 2:

1. Recast each of the secondary controller subproblems (12) as a parametric optimisation problem where \( U^i \) is the optimisation variable and \( x_0, U, U^j, j \neq i \) are the parameters.

2. Solve each of the multi–parametric programs using a multi–parametric programming algorithm and obtain the multi–parametric solution of each of the controllers as

\[ U^i = K_i x_0 + K_{0i} U + \sum_{j \neq i} K_{ij} U^j + c_i^j \]

\[ \text{if } L_{ij} x_0 + L_{0ij} U + \sum_{j \neq i} L_{ij} U^j \leq b_i^j \] (17)

\[ \ell = 1, \ldots, p' \]

where \( \ell \) is the number of critical regions for the \( i \)-th multi–parametric programming problem corresponding to the \( i \)-th secondary controller. The set of \( p' \) linear functions of \( U^j \) in (17) defines the rational reaction set for the \( i \)-th subproblem.

3. Compute the equilibrium point (16).

4. Substitute the expressions of \( U^j \) into the central controller problem (11) and recast the problem as a multi–parametric programming problem where \( U \) is the optimisation variable and \( x_0 \) is the parameter.

5. Solve the resulting multi–parametric programming problem and obtain the parametric solution as an explicit functions of \( x_0 \)

\[ U = K x_0 + c_i^j \]

\[ \text{if } L x_0 \leq b_i \] (18)

\[ \ell = 1, \ldots, p' \]

Algorithm 2 obtains the central controller control policy as an explicit function of the initial state \( x_0 \) and the secondary control policies as explicit functions of \( x_0 \) and the central controllers policy \( U \) i.e. obtains analytically the rational reaction set for the \( i \)-th controller. Each time \( x_0 \) is given then the central controller can obtain its optimal policy by function evaluation from (18).
The policy \( U \) is then announced to the secondary controllers which obtain their policy based in \( x_0 \) and \( U \).

4. EXAMPLE

In this section the two multilevel optimisation methods for hierarchical and decentralised control are illustrated by means of an example. For this, the multiple person dynamic linear quadratic optimisation problem presented in Nie et al. [2006] is considered, which involves the coordination of three controllers within a complex environment.

We assume first that the three controllers are structured as a three level hierarchical control problem ((1), Section II). The objectives (1a), (1d) and (1g) for each of the three levels in (1) are given respectively by

\[
J_1 = \min_{u_0, u_1, u_2} 4x_3 + 3x_1 + 2x_3^2 + \sum_{i=0}^{2} (u_i)^2 - (u_i)^2 - 2u_0x_1 + (u_i)^2
\]

(19)

\[
J_2 = \min_{u_0, u_1, u_2} 2x_3 + 3x_1^2 + \sum_{i=0}^{2} (2 - u_0u_1^2 + (u_1 + 1)^2 - (u_1 + 1)^2)
\]

(20)

\[
J_3 = \min_{u_0, u_1, u_2} x_1 + 2x_1^3 - \sum_{i=0}^{2} (15u_1 + (u_1 - 1)^2 - 2u_1u_2^2 + (u_2)^2)
\]

(21)

the corresponding linear systems (1b), (1e) and (1b) are

\[
\begin{align*}
&x_{i+1} = x_i + u_i - 2a_i + u_i, \quad 1 \\
&x_{i+1} = x_i + 2u_i, \quad 2 \\
&x_{i+1} = x_i + 2u_i, \quad 3 \\
&t = 0, 1, 2
\end{align*}
\]

respectively and the state and input constraints for each of the systems and hence for each of the subproblems in (1) are

\[
\begin{align*}
&-20 \leq u_i \leq 20, -10 \leq x_i \leq 10 \\
&-30 \leq u_i^2 \leq 30, -10 \leq x_i^2 \leq 10 \\
&-30 \leq u_i^3 \leq 30, -10 \leq x_i^3 \leq 10
\end{align*}
\]

(25) (26) (27)

where \( t = 0, 1, 2 \). By following the steps of Algorithm 1, we obtain the explicit multi–parametric solution to the three level hierarchical control problem which is given in Table 1. The policy \( U = \{u_0, u_1, u_2\} \) to the highest level of (1) for this problem is given by four linear functions, each defined in a different set of the space of \( x_0 \). The policies of the two other levels are given by a simple linear function of \( U \). Given a value for \( x_0 \), let say \( x_0 = 10 \) one can obtain the control policy \( U \) by looking in which critical region \( x_0 \) belongs i.e. critical region 4. Then the policy is obtained by evaluating the expressions of \( U \) in this region for the given value \( x_0 = 10 \)

\[
u_0 = -16.2732, u_1 = 20, u_2 = -20
\]

These values of the policy are then used to obtain the policies \( U^2 \) and \( U^3 \) of the lower levels from Table 1

\[
u_1 = -10.1306, u_1 = -12, u_1 = 8, \\
u_2 = -10.1306, u_2 = -12, u_2 = 8
\]

The same problem of the three controller coordination is formulated in the leader – follower structure (Figure 2, Section III). The optimisation problem of the leader (central controller) is formulated by considering (19) as the objective and (22) and (23) the constraints. Similarly, (20), (23), (26) and (21), (24), (27) form the optimisation problems of the two followers. One should notice that in this problem each of the secondary controllers also includes its own dynamics in the problem. Then, as we discussed in Remark 1, the linear models of all three systems (22), (23), (24) have to be included in the three optimisation problems (19), (20), (21) of the leader and the two followers respectively. The objective functions in (19), (20), (21) have already included terms corresponding to the states of each of the systems (22), (23), (24). The problem is then solved as a leader–follower multilevel optimisation problem by following the steps Algorithm 2. The results are given in Table 2. The policies of the central controller and the secondary controllers are again obtained as linear functions of the initial condition \( x_0 \). The solution yields only one critical region in which each of the policies of the central and the secondary controllers are explicit linear functions of the initial state \( x_0 \). For \( x_0 = 10 \) the control policies are \( u_0 = -9, u_1 = 2, u_2 = -5, u_0 = u_2 = 4, u_1 = u_2 = -7, u_1 = u_2 = 0 \).

4.1 Complexity analysis

In this section we briefly discuss a few complexity issues of the two algorithms introduced in this work for the solution of three–level hierarchical and decentralised control problems. It is important to notice that in both Algorithms 1 and 2, the number of critical regions depends on the size of the problem that we encounter. With a significant increase in the number of states and inputs of the systems and number of the constraints involved in the two problems (1) and (11)–(12), it is possible that the number of critical regions will also increase substantially, thus increasing the computational cost. The methodology described here involves convex objectives and linear systems and constraints. The complexity of the problem increases once nonlinear, nonconvex objectives, system dynamics and constraints are considered.

4.2 On–line multilevel optimisation

In many control applications it is rather possible that the hierarchical and decentralised control problems described by (1) and (11)–(12) will have to be solved on–line to maintain a constant level of control and supervision of the system under consideration. In that case, the methodology described in Section II and III moves the solution of the on–line multilevel optimisation problems off–line and solve by employing multi–parametric

\[
\begin{array}{|c|c|}
\hline
\text{Critical Region 1} & \text{Critical Region 2} \\
\hline
u_0 = 6.84615 - 0.76928x_0 & u_0 = -0.33333 - 1.8519x_0 \\
u_1 = -20 & u_1 = -1.33333 + 2.8148x_0 \\
u_2 = 15.2308 + 0.15588x_0 & u_2 = -2 - 2.44444x_0 \\
-10 \leq x_0 \leq -6.3161 & -6.3161 \leq x_0 \leq 7.36377 \\
\hline
\text{Critical Region 3} & \text{Critical Region 4} \\
\hline
u_0 = -1.53333 - 1.6889x_0 & u_0 = -9 - 0.7273x_0 \\
u_1 = 8.26667 + 1.5111x_0 & u_1 = 20 \\
u_2 = -20 & u_2 = -20 \\
7.36377 \leq x_0 \leq 7.76466 & 7.76466 \leq x_0 \leq 10 \\
u_0 = u_2 = -2 - 0.5u_0; u_1 = u_2 = -2 - 0.5u_1; u_2 = u_2 = -2 - 0.5u_2 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
\text{Critical Region 1} & \text{Critical Region 2} \\
\hline
u_0 = 1 - x_0; u_1 = -8 + x_0; u_2 = 5 - x_0 \\
u_1 = u_2 = -6 + x_0; u_1 = u_2 = 3 - x_0 \\
u_1 = u_1 = -10 + x_0; -10 \leq x_0 \leq 10 \\
\hline
\end{array}
\]

Table 2. Solution to the decentralised problem
programming techniques. The control policies, considered in each of the subproblems of the multilevel problem, are implemented on-line by performing function evaluations on the explicit expressions given in the Step 2,4 and 5 of Algorithm 1 and the Steps 2,3 and 5 of Algorithm 2, each time the initial states of the system are available. For example, if problem (1) represents model–based predictive control of a sequentially connected system where in each level a smaller subsystem of the overall system is considered (Stankovic and Siljak [1989]), then each time the control policies \( U^0, U^1, U^2 \) are obtained only \( u_0, u_1, u_2 \) have to be applied to the system.

5. CONCLUSIONS

In this work the three–level hierarchical control and multilevel decentralised control problems were treated. Two algorithms were developed that solve the problems by recasting each optimisation problem in each level as a multi–parametric programming problem. It was shown that the control policies at each level and the rational reaction sets can be obtained analytically as a set of explicit functions. There is currently non known work relevant to the solution of multi–level optimisation problem, such as the ones in Sections II and III, in the knowledge of the authors, despite the vast research in the area of multilevel optimisation and hierarchical control. This work attempts, possibly for the first time, to address the issue and motivate further developments on the problems of hierarchical control and multilevel optimisation.

6. ACKNOWLEDGEMENTS

The financial support of EU (Marie Curie European Project PRISM-MRTN-CT-2004-512233) and EPSRC (GR/T02560/01) is gratefully acknowledged.

REFERENCES


