Partial stabilization of a class of hydraulic mechanical systems with Casimir functions

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Abstract: This paper gives a new modeling and passivity based control of a class of fluid mechanical systems with Casimir functions. First, we propose two stabilization methods, a new dynamic asymptotic stabilization method and a new partial stabilization method. Second, we give a new model of a class of hydraulic mechanical systems using Casimir functions. Furthermore, the proposed two stabilization methods are applied to the derived model. Finally, the validity of our methods are confirmed by numerical simulation.

1. INTRODUCTION

This paper gives a new modeling and passivity based control of a class of hydraulic mechanical systems such as hydraulic robot arms, pneumatic robot arms and so on. It is well known that modeling and control of this class of hydraulic mechanical systems are much difficult than those of mechatornical systems. This is due to that the driving system is complex and consists of “compressible” fluid systems, which are nonlinear dynamical systems with unknown (or hard-to-be identified) parameters.

To solve these problems, this paper gives fundamental results about modeling and control of this class of hydraulic mechanical systems, especially of the driving system, by focusing and developing port-Hamiltonian systems and control theory.

Port-Hamiltonian systems (van der Schaft (2000)) are generalization of Hamiltonian systems in classical mechanics and can model many systems such as electro-mechanical systems, mechanical systems with nonholonomic constraints (Maschke and van der Schaft (1994)), distributed systems and their mixed systems (Macchelli and Melchiorri (2005)) as well as classical mechanical systems.

Passivity is the most important property of port-Hamiltonian systems and some passivity based control methods, originally from Takegaki and Arimoto (1981), were developed, such as, Energy-Casimir methods (van der Schaft (2000)), the generalized canonical transformations (Fujimoto and Sugie (2001)), IDA-PBC (Ortega and García-Cánseco (2004)) and IPC approach (Stramigioli et al. (1998)) and so on. These methods can give nonlinear robust controllers, for example, the generalized canonical transformations give nonlinear robust dynamic output feedback stabilizers (Sakai and Fujimoto (2005)).

Modeling and control of several fluid systems are already discussed in port-Hamiltonian form. For example, Ramkrishna and van. der. Schaft (2006) discuss infinite dimensional canal systems in three dimensional space and Johanson (2006) discuss the four-tank systems based on IDA-PBC. These fluid systems have free-surface and are incompressible. Riccardo et al. (2006) discuss the modeling of hydraulic arms and show some experimental results. Gernot and Schlacher (2005) discuss the control of the hydraulic arms in port-Hamiltonian form. In these approaches, the modeling is based on the standard procedure in port-Hamiltonian framework. Apart from these approaches, we discuss the modeling and control of the class of fluid mechanical systems, such as hydraulic robot arms, pneumatic robot arms, based on a new structural property, (natural) Casimir functions.

Advantages of port-Hamiltonian systems are from their structural properties, such as passivity, which do not exist in general nonlinear systems. In this paper, we focus on new structural properties of special port-Hamiltonian systems, that is, port-Hamiltonian systems with Casimir functions which do not exist in general port-Hamiltonian systems. In a word, a key point of this paper is that the class of hydraulic mechanical systems are found to be “special” port-Hamiltonian systems, that is, port-Hamiltonian systems with Casimir functions. The Casimir functions are used in the modeling and control as a new structural property of port-Hamiltonian systems in addition to the standard structural properties, such as passivity.

This paper is organized as follows. In Section 2, we refer port-Hamiltonian systems and their properties, especially Casimir functions. In Section 3, we propose two stabilization methods, a new dynamic asymptotic stabilization method and a new partial stabilization method. In Section 4, we give a new model of the class of hydraulic mechanical systems using Casimir functions. At the same time, a very fundamental coordinate of the class of hydraulic mechanical systems is discovered. In Section 5, the proposed two stabilization method are applied to the new model and a new passivity based control are proposed. In Section 6, the validity of our methods are confirmed by numerical simulation and finally we conclude this paper in Section 7.

In this paper, $I_n$ is $n \times n$ identity matrix, $R^m \times n$ is the real space of $m$ rows and $n$ columns matrix.
2. PORT-HAMILTONIAN SYSTEMS

2.1 Port-Hamiltonian systems

A port-Hamiltonian system with a Hamiltonian $H(x) \in \mathbb{R}$ is a system described by

$$
\begin{cases}
\dot{x} = J(x) \frac{\partial H(x)}{\partial x}^T + g(x)u \\
y = g(x)^T \frac{\partial H(x)}{\partial x}
\end{cases}
$$

(1)

with $u, y \in \mathbb{R}^m, x \in \mathbb{R}^n$ and a skew symmetric matrix $J(x)$, i.e. $-J(x) = J(x)^T$ holds. The zero-state detectability and the positive definiteness of the Hamiltonian assumed in Lemma 1 do not always hold for general port-Hamiltonian systems. In such a case, the generalized canonical transformation is useful.

2.2 Casimir functions

One of the properties of port-Hamiltonian systems are the existence of Casimir functions. Casimir functions (with respect to $J$) are defined as the solutions of the following PDE,

$$
\frac{\partial C(x)}{\partial x} J(x) \equiv 0.
$$

(3)

Casimir functions are the special first integrals, that is,

$$
\dot{C} \equiv 0
$$

(4)

for any Hamiltonian $H(x)$ at $u = 0$. Casimir functions are not bounded from below nor upper in general. Casimir functions do not always exist for port-Hamiltonian systems in general.

Note that in this paper we do not treat Casimir functions of the closed-loop systems but treat only Casimir functions of controlled systems (plants). The former Casimir functions are discussed in the Energy-Casimir method and are artificially designed functions. From this point of view, we call the latter “natural” Casimir functions in this paper.

3. DYNAMIC AND PARTIAL STABILIZATION FOR PORT-HAMILTONIAN SYSTEMS

3.1 Dynamic stabilization for port-Hamiltonian systems

In this subsection, we give a dynamic stabilization method for port-Hamiltonian systems. In the previous section, port-Hamiltonian systems can be stabilized by the static stabilizers. However, the dynamic stabilizers are also useful and will be applied later.

Theorem 1 Consider the following (mechanical) port-Hamiltonian systems

$$
\Sigma_{phm} : \begin{bmatrix}
\dot{q} \\
\dot{p}
\end{bmatrix} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H_m}{\partial q}^T \\ \frac{\partial H_m}{\partial p} \end{bmatrix} + \begin{bmatrix} 0 \\ G \end{bmatrix} u
$$

(5)

where $x = (q, p) \in \mathbb{R}^n$ are the position and the momentum, $G$ is nonsingular matrix, the Hamiltonian is $H_m = (1/2)(p^T M^{-1} p) + U(q)$ and $M = M^T > 0, U(q) \geq U(0) = 0$. Then, the following dynamic feedback

$$
\Sigma_{dd} : \begin{bmatrix}
\dot{r} & \dot{q} \\
\dot{p}
\end{bmatrix} = \begin{bmatrix} P(x)^T G & \frac{\partial H_m}{\partial q}^T \\ -D & \frac{\partial H_m}{\partial p} \end{bmatrix} \begin{bmatrix} \frac{\partial H_m}{\partial q}^T \\ \frac{\partial H_m}{\partial p} \end{bmatrix} - \begin{bmatrix} P(x)^T G P(x) \frac{\partial H_m}{\partial p} \\ -P(x)^T G T - D(x) \end{bmatrix} u
$$

(6)

makes $\Omega_0 = \{(q, p)|y = u = 0\}$ asymptotically stable, where $r \in \mathbb{R}^s, H_r = (1/2)r^T R r, P^T$ is tall matrix, $D = D^T > 0$ and $R = R^T > 0$.

Proof of Theorem 1

The closed-loop system is

$$
\Sigma_{cl} : \begin{bmatrix}
\dot{q} \\
\dot{p}
\end{bmatrix} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H_m + H_r}{\partial q}^T \\ \frac{\partial H_m + H_r}{\partial p} \end{bmatrix} + \begin{bmatrix} 0 \\ G \end{bmatrix} u
$$

(7)

and (may be non-canonical) Hamiltonian systems with a new Hamiltonian $H_m + H_r$ and dissipation. Since the time derivative of $H_m + H_r$ is

$$
\dot{H}_m + \dot{H}_r = -\frac{\partial H_r}{\partial r} \frac{\partial H_r}{\partial \dot{r}} \leq 0
$$

(8)

and $H_m + H_r$ is bounded from below,

$$
\frac{\partial H_r}{\partial \dot{r}} \rightarrow 0
$$

(9)

as $t \rightarrow \infty$ and the $\{u = 0\}$ is asymptotically stable. This implies $r \rightarrow 0$, that is, $\dot{r} \rightarrow 0$ because $H_r$ belongs to class $K$ with respect to $\|r\|$. From Equation (7) and the tall matrix $P(x)^T$, we have

$$
P(x)^T \frac{\partial (H_m + H_r)}{\partial p} = 0 \Rightarrow \frac{\partial H}{\partial p} = 0.
$$

(10)
This means that the set \( \{y = 0\} \) is asymptotically stable. In all, the set \( \{y = u = 0\} \) is asymptotically stable. (Q.E.D.)

Note that the proposed dynamic asymptotic stabilization is a generalization of the result of our previous result (Sakai and Fujimoto (2005)). It is easy to extend \( \Sigma_{phm} \) to more general port-Hamiltonian systems but \( \Sigma_{phm} \) will be applied directly later.

3.2 Partial stabilization for port-Hamiltonian systems with Casimir function

Advantages of port-Hamiltonian systems are from their structural properties, such as passivity, which do not exist in general nonlinear systems. In this subsection, we focus on structural properties of special port-Hamiltonian systems, that is, port-Hamiltonian systems with Casimir functions which do not exist in general port-Hamiltonian systems.

The following theorem gives a new partial stabilization method for port-Hamiltonian systems with Casimir functions.

**Theorem 2** Consider the following port-Hamiltonian systems with Casimir functions \( C(x) \)

\[
\Sigma_{phc} : \begin{cases} \dot{q} = J(q,p) \left[ \frac{\partial H}{\partial q} \right]^T + \begin{bmatrix} 0 \\ C \end{bmatrix}^T u \\ y = G^T \frac{\partial H}{\partial q} \end{cases} \tag{11}
\]

where \( x = (q,p)^T, G \) is nonsingular matrix. Suppose there exist a coordinate transformation \( x \mapsto \phi(x) = (x_r,C)^T \) such that

\[
H(\phi) = H_r(x_r) + H_c(C). \tag{12}
\]

where \( x_r \in \mathbb{R}^r, r = \text{rank}(J) \) and \( H_r \) is bounded from below. Then the feedback

\[
u = -D(x) y_r \tag{13}
\]

makes the set \( \{y_r = 0\} \) asymptotically stable where

\[
y_r = \begin{bmatrix} 0 \\ G^T \frac{\partial q}{\partial \phi} \begin{bmatrix} I_r \\ 0 \end{bmatrix} \frac{\partial H_r}{\partial x_r} \end{bmatrix} . \tag{14}
\]

**Proof of Theorem 2**

From the existence of Casimir functions \( C(x) \), there exist a coordinate transformation which convert the system (11) into

\[
\begin{bmatrix} \dot{x}_r \\ \dot{C} \end{bmatrix} = \begin{bmatrix} J_r(x_r,C) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H(x_r,C)}{\partial x_r} \end{bmatrix}^T + \begin{bmatrix} 0 \\ \frac{\partial \phi}{\partial x} \end{bmatrix}^T u
\]

\[
y = \begin{bmatrix} 0 \\ G^T \frac{\partial q}{\partial \phi} \frac{\partial H}{\partial C} \end{bmatrix} . \tag{15}
\]

where \( J_r = -J_r^T \in \mathbb{R}^{r \times r} \). The time derivative of the function \( H_c \) is given as

\[
\dot{H}_r = \frac{\partial H_r}{\partial x_r} \dot{x}_r + \frac{\partial H_r}{\partial C} \dot{C}
\]

\[
= \frac{\partial H_r}{\partial x_r} \begin{bmatrix} J_r \frac{\partial H(x_r,C)}{\partial x_r} \end{bmatrix}^T + \begin{bmatrix} I_r \\ 0 \end{bmatrix} \frac{\partial \phi}{\partial x} \begin{bmatrix} 0 \\ C \end{bmatrix}^T u
\]

\[
= \frac{\partial H_r}{\partial x_r} \begin{bmatrix} J_r \frac{\partial H_r}{\partial x_r} \end{bmatrix}^T + \begin{bmatrix} I_r \\ 0 \end{bmatrix} \frac{\partial \phi}{\partial x} \begin{bmatrix} 0 \\ C \end{bmatrix}^T u
\]

\[
due to the special Hamiltonian structure (12).

The system with input \( u \) and output \( y_r \) is passive (lossless) with respect to the storage function \( \dot{H}_r \), that is,

\[
\dot{H}_r = y_r^T u \tag{17}
\]

and the controller (13) makes the set \( \{y_r = 0\} \) asymptotically stable since

\[
\dot{H}_r = -y_r^T D y_r \leq 0 \tag{18}
\]

and \( H_r \) is bounded from below. (Q.E.D.)

Note that the above output \( y_r \) is different from the usual output \( y \) of port-Hamiltonian function and a new output based on the structural properties of special port-Hamiltonian systems, that is, port-Hamiltonian systems with Casimir functions. Furthermore, not all states, but only the partial state \( x_r \) can be stabilized in Theorem 2.

4. MODELING USING CASIMIR FUNCTIONS

In this section, modeling of a class of hydraulic mechanical system (such as Fig.1 and Fig.2) is discussed with Casimir functions. In this section, this class of hydraulic mechanical systems are discussed in practical way, that is, we start not from infinite dimensional model but from finite dimensional model and we take input not as torque but as spool displacement. This finite dimensional model with spool displacement input is well-known in practical control situations (Jelali and Kroll (2002)). In addition, it is also known that the multi-degree of freedom case can be separated into this one degree of freedom cases completely.

![Fig. 1. Hydraulic robot arms (example)](image-url)
4.1 Generalized continuity equation

A continuity equation for incompressible fluid is

\[ \dot{p}_i = \frac{E}{V} (Q_i^{in} - Q_i^{out}) \]  

(19)

where \( p_i \) is the pressure of chamber \( i \), \( Q_i^{in} \) is flow to chamber \( i \), \( Q_i^{out} \) is flow from chamber \( i \), \( E \) is the bulk modulus and \( V \) is fluid volume. For example, see Jelali and Kroll (2002) for this modeling assumptions in detail.

As the port-Hamiltonian systems are generalization of classical Hamiltonian system (with no inputs and outputs), we give the following generalization of continuity equation

\[
\Sigma_f:\begin{cases}
\dot{x}_f = \beta \alpha A \left( u_x + \left[ g_{p1} \right] u_f \right) \\
y_f = -\alpha A \left( \frac{\partial H_f}{\partial x_f} \right)^T
\end{cases}
\]

where

\[ H_f = \frac{1}{2V} E (x_{f1}^2 + x_{f2}^2) \]

(21)

\( x_f = (x_{f1}, x_{f2})^T \), \( x_{f1} = (V/E)p_i \), \( g_{p2} = g_{p2}(x_{f1}) \) are from Bernoulli’s equation (omitted from its uniqueness), \( u_f \) is the spool input displacement, \( A \) and \( \alpha A \) (\( \alpha \in (0, 1) \)) are the area in the chambers.

Note that the system \( \Sigma_f \) is NOT port-Hamiltonian systems and that the state is not pressure and different from that of the model by Gernot and Schlacher (2005).

4.2 Interconnection with generalized continuity equation

Now the mechanical system \( \Sigma_{phm} \) and the fluid system \( \Sigma_f \) are interconnected by the following

\[
\begin{align*}
   u_o &= y \\
u &= -y_f.
\end{align*}
\]

(22)

Then the interconnected system is

\[
\begin{bmatrix}
\dot{q} \\
p
\end{bmatrix} = \begin{bmatrix}
0 & I & 0 & 0 \\
-I & 0 & GA - \alpha GA & 0 \\
0 & -AG^T & 0 & 0 \\
0 & \alpha AG & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_{f1} \\
x_{f2}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{\partial H_{fm}}{\partial x_{f1}} \\
\frac{\partial H_{fm}}{\partial x_{f2}}
\end{bmatrix}
\]

(23)

where \( H_{fm} = H_f + H_m \) and \( G_f = [g_{p1} g_{p2}]^T \) and is easily confirmed to be port-Hamiltonian systems since \( J \) matrix part is again skew symmetric.

Note that the interconnected system of the fluid system \( \Sigma_f \) and the mechanical system \( \Sigma_{phm} \) is again a port-Hamiltonian system although the fluid system \( \Sigma_f \) is NOT port-Hamiltonian system.

This situation is different from that in classical mechanics where we should take only energy conservation in Navier-Stokes equations and, apart from this, take only mass conservation in continuity equation. Thanks to the formulation of the generalized continuity equation and the existing port-Hamiltonian framework, we can take not only energy conservation but also mass conservation into account simultaneously.

4.3 Modeling based on Casimir functions

In this section, we give the most important result about the modeling of the class of hydraulic mechanical systems.

Lemma 2 Consider the fluid-mechanical Hamiltonian systems \( \Sigma_{fm} \). Then there exist a Casimir function

\[ C_f = \alpha x_{f1} + x_{f2}. \]

(24)

Proof of Lemma 2.

It is confirmed that \( C_f \) satisfies the PDE (3) by a direct calculation, that is, \( C_f \equiv 0 \) holds for any Hamiltonian \( H_{fm} \) with zero-input. (Q.E.D.)

Theorem 3 Consider the fluid-mechanical Hamiltonian systems \( \Sigma_{fm} \). Then there exist a coordinate transformation \( \phi \) such that the transformed systems satisfy the condition (12).

Proof of Theorem 3.

Consider the following coordinate transformation,

\[
\begin{bmatrix}
    q \\
p \\
x_{f1} \\
x_{f2}
\end{bmatrix} = \phi(q, p, x_{f1}, x_{f2})
\]

\[
= \begin{bmatrix}
    1 & 0 & 0 & -\alpha \\
0 & \sqrt{1 + \alpha^2} & \sqrt{1 + \alpha^2} & \sqrt{1 + \alpha^2} \\
0 & \alpha & 1 & +1
\end{bmatrix}
\begin{bmatrix}
    q \\
p \\
x_{f1} \\
x_{f2}
\end{bmatrix}.
\]

(25)

It is calculated that the fluid-mechanical systems are transformed to a new port-Hamiltonian systems which satisfies
the conditions (12) as \( H_{fm} = H_r + (1/2)C_f^TC_f \) where \( H_r \) is given in the following partial dynamics corresponding to that by \( J_r \) part in Equation (15)

\[
\begin{bmatrix}
\dot{q} \\
\dot{p}
\end{bmatrix}
= \begin{bmatrix}
0 & I \\
-I & 0
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
\sqrt{1+\alpha^2A} \\
0
\end{bmatrix}
\begin{bmatrix}
\partial H_r \\
\partial q \\
\partial H_r \\
\partial p
\end{bmatrix} + \begin{bmatrix}
0 \\
g_r
\end{bmatrix} u_f \tag{26}
\]

where \( g_r \) is omitted because of its uniqueness and

\[
H_r = H + \frac{1}{2V}E \hat{x}_{fr}^2 \tag{27}
\]

is bounded from below. (Q.E.D.)

5. CONTROL USING CASIMIR FUNCTIONS

In this section, we give a new stabilization method for the class of hydraulic mechanical systems based on the previous results in this paper. The proposed controller in this section can stabilize only mechanical part, even if the parameters of fluid systems, the bulk modules \( E \), is unknown and the mechanical part has few damping effect.

**Lemma 3** Consider the fluid-mechanical systems \( \Sigma_{fm} \) and suppose that \( U(q) \) is the positive definite function. Then the feedback

\[
u_f = -D g_r \frac{\partial H_r}{\partial x_{fr}} \tag{28}
\]

makes the set \( \{(q,p) = 0\} \) asymptotically stable.

**Proof of Lemma 3.**

First, from Theorem 2 and Theorem 3, the set

\[
\{ y_r = g_r \frac{\partial H_r}{\partial x_{fr}} = 0 \} \tag{29}
\]

is asymptotically stable. Second, since the closed-loop system of (26) and (28) is equivalent to that of Theorem 1, the feedback makes the set

\[
\{ y = \frac{\partial H}{\partial p} = 0 \} \tag{30}
\]

asymptotically stable. This implies that the set \( \{(q,p) = 0\} \) is asymptotically stabilized due to the zero-state detectability. (Q.E.D.)

Note that the proposed controller stabilizes for any parameter \( E \). It is very important that \( x_{fr} \) is a new (natural) coordinate for fluid-mechanical system and never reported in existing results (Merrit (1967), Jelali and Kroll (2002)).

6. NUMERICAL SIMULATION

In this section, we confirm the validity of our methods by numerical simulation. All parameters of plants are normalized as 1.

Fig.3 shows the time response of a standard linear mechanical-spring SISO system with the stabilizer in Theorem 1. All states converts to the origin smoothly and the validity of Theorem 1 is confirmed.

Then we show the hydraulic arm in Fig.2 with the stabilizer in Lemma 3. Initial states are \((1, -1, 1, 1)\). Fig.4 shows the time responses of the state at \( D = 1/2 \). Only the state of mechanical systems \((q,p)\) converts to the origin smoothly. This implies that the validity of the partial stabilization methods in Theorem 2 is confirmed. In Fig.4 the settling time is about 20s.

Fig.5 shows the time responses of the state at \( D = 1/5 \). In this case, only the state of mechanical systems \((q,p)\) also converts to the origin smoothly but the settling time is about 12s even when the gain \( D \) is smaller than that in Fig.4. This is the same tendency as that by Sakai and Fujimoto (2005), where the authors gave a effective gain-tuning guideline based on the analysis of the linearized system. Fig.6 and Fig.7 show the time response of the discovered coordinate \( x_{fr} \) in the case of \( D = 1/2 \) and \( 1/5 \), respectively. The states convert to the origin. In all, the validity of our methods are confirmed.

![Fig. 3. Time response of the state of the linear system](image)

![Fig. 4. Time response of the all state \((D = 1/2)\)](image)
7. CONCLUSIONS

First, we propose two stabilization methods, a new dynamic asymptotic stabilization method and a new partial stabilization method. Second, we give a new model of a class of hydraulic mechanical systems using Casimir functions. At the same time, a very fundamental coordinate \( x_{fr} \) of the class of hydraulic mechanical systems is discovered. Third, the proposed two stabilization methods are applied to the new model of the class of hydraulic mechanical systems and a new passivity based control are proposed. Finally, the validity of our methods are confirmed by numerical simulation.

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