Unfalsified Virtual Reference Adaptive Switching Control of Plants with Persistent Disturbances

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Abstract: This paper addresses virtual reference adaptive switching control whereby a data-driven supervisor aims at stabilizing an unknown time-invariant dynamic system by switching at any time in feedback with system one element from a finite family of candidate controllers. Under the only assumption of problem feasibility, viz. the controller family contains a stabilizing controller, the resulting switched system is shown to be stable against arbitrary exogenous persistent bounded disturbances.

1. INTRODUCTION

In many practical control applications, despite only a partial knowledge of the plant, it is asked to design a feedback controller which can ensure stability to the controlled system. This paper considers the unfalsified virtual reference adaptive switching control (VRASC) approach, whereby a data-driven “high-level” unit, called the supervisor, aims at stabilizing an unknown time-invariant dynamic system by switching at any time in feedback with system to be controlled one element from a set of candidate controllers. A VRASC system is depicted in Fig.1. There, II is the unknown plant to be controlled, and $K_{\sigma(t)}$ is the time-varying controller whose selection is carried out by the supervisor S based on the plant input $\delta u$ and output y.

Fig. 1. Basic VRASC scheme.

The supervisor performs in real-time the scheduling task (when to switch) and the routing task (which controller select) (Morse (1995)), by monitoring a purely data-driven cost function. When the current performance is not satisfactory, viz. the switched-on controller is falsified by measured data (Safonov and Tsao (1997); Mosca and Agnoloni (2003)), another candidate controller is selected so as to replace the previous one. Thanks to the use of the virtual reference tool (Safonov and Tsao (1997)), such a selection is carried out without directly checking the performance of all candidate controllers via their effective use in the feedback loop. Using past plant input-output records, the supervisor infers the performance of the potential loop made up by a candidate controller and selects the one corresponding to the minimal value of the inferred performance.

In the noiseless case, stability properties of the adaptive switched system can be ensured. In fact it can be proved that the switching mechanism stops in a finite time (Safonov et al. (2007); Mamelli et al. (2007))). However, apart from few exceptions, little attention has been so far devoted on how to deal with persistent disturbances: the issue considered in this paper. For other contributions to the topic, see e.g. also Morse (1997), Jun and Safonov (1999), Zhivoglyadov et al. (2000), Hespanha et al. (2003), Freidovich and Khalil (2005) and Fekri et al. (2006).

Sect.2 recalls some preliminary concepts and known stability results pertinent to the noiseless case. Sect.3 analyses, from the stability viewpoint, the problems which may arise in the presence of disturbances. Finally, Sect.4 shows how suitable choices of the cost functionals ensure that the switched system be stable in the face of arbitrary bounded disturbances.

2. PRELIMINARIES AND PROBLEM FORMULATION

The study is intentionally focused on the simplest non-trivial case of a strictly causal SISO plant consisting of a discrete-time time-invariant dynamic system II with input-increment $\delta u(t) := u(t) - u(t - 1)$ and output $y(t)$, $t \in \mathbb{Z}_+ = \{0, 1, \ldots \}$. The only assumption on II is that it belongs to a plant uncertainty set $\mathcal{P}$, which need not be known. Along with the uncertain plant II, a finite family $\mathcal{K}$ of one-degree-of-freedom causal and causally invertible (CCI) controllers $K_i$ is available, $\mathcal{K} = \{K_i, i \in \mathbb{N}\}$, $\mathbb{N} := \{1, 2, \ldots, N\}$. At any time $t$, the plant II is fed by the control-increment $\delta u(t)$ coinciding with the output...
of one, say \( K_{\sigma(t)} \), \( \sigma(t) \in \mathbb{N} \), among the \( N \) candidate controllers, viz., \( K_{\sigma(t)} \) is switched-on in feedback to the plant. Consequently,

\[
\begin{align*}
\{ y(t) &= \Pi(\delta u)(t) \\
\delta u(t) &= -K_{\sigma(t)}(y - r)(t)
\end{align*}
\]

where \( r \) denotes the reference to be tracked by the plant output \( y \). Here, \( K_{\sigma(t)}(y - r)(t) \) has to be introduced as a shorthand notation for \( K_{i}(y - r)(t) = \sigma(t) \). The latter means that the \( N \) candidate controllers are fed at all times by the tracking error \( \epsilon := r - y \), and the output of the \( \sigma(t) \)-th controller is used as the input-increment to the plant at time \( t \). In (1) \( \Pi \) and all \( K_{i} \)'s are allowed to have arbitrary states at time zero. The switching mechanism is implemented by possibly bumpless control transfer techniques (Goodwin et al. (2001)) or common state multi-controller schemes (Morse (1995)) so as to reduce as much as possible the transients caused by switching from one controller to a different one. Plant input increments are customarily adopted for control design so as to ensure constant disturbance rejection and zero-offset. As anticipated, the high-level device responsible for orchestrating \( \sigma(t) \) is the switching supervisor, and (1), combined with such a device, is referred to as an ASC system denoted \( (\Pi/K_{\sigma(t)}) \).

2.1 Unfalsified control

The problem of controlling the unknown plant \( \Pi \) is dealt within a framework commonly referred to as unfalsified control (Safonov and Tsao (1997)). To this end, we recall some useful definitions available in literature (Safonov et al. (2007)).

Definition 1. A signal \( v \) is said to belong to \( l_{2r} \) if it is square summable over any bounded interval of time \( \{1, 2, \ldots, t\} \), \( t \in \mathbb{Z}_{+} \).

Definition 2. Assuming \( r(t) \) as (temporarily) the only exogenous variable, \( (\Pi/K_{\sigma(t)}) \) is said to be \( l_{2r} \)-gain stable relative to \( r \) if there exist finite nonnegative constants \( \alpha \) and, \( \beta \) such that:

\[
\| D(t) \| \leq \beta \cdot \| r(t) \|^2 \quad + \quad \alpha
\]

\( \forall t \in \mathbb{Z}_{+}, \forall r(t) \), where \( v_{i} := \{ v(t) \}_{k=0}^{t} \) is the time truncation of \( v(t) \) over the interval \( [0, 1, \ldots, t] \), \( D(k) := [\delta u(k) \quad y(k)]^{T} \) the plant I/O vector at time \( k \), the prime denotes transpose, \( \| D(t) \|^2 := \sum_{k=0}^{t} |D(k)|^2, \| \cdot \| \) denote Euclidean and, respectively, \( l_{2} \)-norm.

It should be emphasized that stability of the system (1) is unfalsified by an input-output pair \( (r, D) \), during a specific infinite-length experiment, if (2) holds. On the opposite, with “stability” we mean that (2) holds true for every possible input, viz., stability of (1) is unfalsified by any possible pair \( (r, D) \).

In a VRASC system, for each \( i \in \mathbb{N} \), the variable \( v_{i}(t) = y(t) + K_{i}^{-1}(\delta u(t)) \)

\( t \in \mathbb{Z}_{+} \), is uniquely computed in real-time provided that \( K_{i} \) is causal and causally stably invertible (CCSI). Note that if \( K_{i}^{-1} \) and \( K_{i} \) have the same initial condition, then

\[-K_{i}(y - v_{i})(t) = -K_{i}(y - y - K_{i}^{-1}(\delta u(t))(t) = \delta u(t) \]

In words, \( v_{i} \) equals the fictitious or virtual reference that, if injected into the feedback system \( (\Pi/K_{i}) \), would reproduce \( D(t), t \in \mathbb{Z}_{+} \), the I/O pairs of the uncertain plant \( \Pi \) in (1). In other terms, if \( (\Pi/K_{\sigma(t)}) \) is intended as the (time-varying) transformation \( (r, D) \), we find:

\[
D = (\Pi/K_{\sigma(t)})r = (\Pi/K_{i})v_{i}
\]

where \( (\Pi/K_{i}) \) is the (time-invariant) transformation consisting of the plant \( \Pi \) fed-back by the \( i \)-th candidate controller \( K_{i} \).

The introduction of the \( v_{i} \)'s makes it possible to causally compute from \( D \) (virtual) nonnegative causal loss functionals \( V_{i}(\cdot), i \in \mathbb{N} \). Given \( D^{t} \), the supervisor compares the \( N \) performance indices \( V(t) := \{ V_{i}(t) \}_{i=1}^{N} \) and selects, at every \( t \in \mathbb{Z}_{+} \), the controller index \( \sigma(t) \) via the following hysteretic switching logic:

\[
\sigma(t) = \begin{cases} i & \text{if } V_{i} < V_{j}(V) + h \\ \gamma(V) & \text{otherwise} \end{cases}
\]

where \( \gamma(V) \) denotes the least integer \( i \in \mathbb{N} \) for which \( V_{i} \leq V_{j}, \forall j \in \mathbb{N} \), and \( h \) a (typically small) positive real, is the so-called hysteresis constant.

2.2 Results in the noiseless case

In VRASC system analysis, a fundamental role is played by the next Hysteresis Switching Lemma which establishes the limiting behavior of \( (\Pi/K_{\sigma(t)}) \).

Let \( S \) denote the class of all possible switching functions \( s : \mathbb{Z}_{+} \rightarrow \mathbb{N} \) giving rise to the switched system \( (\Pi/K_{\sigma(t)}) \). Consider the assumptions:

A1. For each \( s \in S \) and \( i \in \mathbb{N} \), the performance index \( V_{i}(t) \) admits a limit (even infinite) as \( t \rightarrow \infty \).

A2. There is at least one integer \( m \in \mathbb{N} \) such that \( V_{m}(\cdot) \) is bounded on \( \mathbb{Z}_{+} \) for each \( s \in S \).

Hysteresis Switching Logic (HSL) Lemma (Morse et al. (1992)): For any initial condition and reference \( r \), let \( D \) denote the \( I/O \) plant data to the supervisor, and \( \sigma \) the switching function resulting from (1), (4) and (5). Then, if A1 and A2 hold, there is a finite time \( t_{f} \) beyond which \( \sigma \) is constant as no more switching occurs. Moreover, \( V_{\sigma(t_{f})}(\cdot) \) is bounded.

In order to simplify the notation, from now on, the final controller index \( \sigma(t_{f}) \) will be denoted by \( f \), viz., \( K_{\sigma(t_{f})} = K_{f} \). A prerequisite for a VRASC system to yield stable and well-behaved final feedback loops \( (\Pi/K_{f}) \), \( \forall \Pi \in \mathcal{P} \) is that the set \( \mathcal{K} \) of candidate controllers be adequately chosen relatively to \( \mathcal{P} \). In this connection, the minimal requirement on \( \mathcal{K} \) is the so-called:

Problem Feasibility: For every \( \Pi \in \mathcal{P} \), there are indices \( i, \in \mathbb{N} \), such that \( (\Pi/K_{i}) \) is \( l_{2r} \)-gain stable.

Key properties which enable cost functions to reliably detect any instability exhibited by the adaptive system are as follows (Safonov et al. (2007)).

Definition 3. The pair \( (V, \mathcal{K}) \) is said to be cost-detectable if for every \( K_{\sigma(t)} \in \mathcal{K} \) with finitely many switching times, the following statements are equivalent: 1) \( V_{i} \) is bounded as \( t \rightarrow \infty \), 2) \( l_{2r} \)-gain stability of the system \( (\Pi/K_{\sigma(t)}) \) is unfalsified by \( (r, D) \).
Definition 4. Given the pair \((V, K)\), \(V\) is said to be \(l_2e\)-gain-related if for each \(D \in l_2e\) and \(K_i, i \in \rightarrow N\): 1) \(V_i(t)\) is monotone in \(t\), 2) the virtual reference signal \(v_i\) exists, and 3) for every \(K_i, i \in \rightarrow N\), and \(D \in l_2e\), \(V_i(t)\) is bounded as \(t \rightarrow \infty\) if and only if \(l_2e\)-gain stability of the system \((\Pi/K_i)\) is unfeasible by \((v_i,D)\).

If each \(K_i, i \in \rightarrow N\), is CCSI, the virtual reference \(v_i\) in (3) is well-defined. In such a case, cost detectability is equivalent to \(l_2e\)-gain-relatedness (see Safonov et al. (2007)). Therefore, one can focus on the latter in order to study the stability properties of the VRASC system. With this respect the following lemma holds that summarizes some of the results of Safonov et al. (2007).

Lemma 1. Let all the candidate controllers be CCSI, and the hysteresis switching logic (4)-(5) used. Then, provided that

B1. \(V\) is \(l_2e\)-gain-related and

B2. problem feasibility holds,

the HSL holds and the resulting VRASC system \((\Pi/K_{\sigma(i)})\), \(\Pi \in P\), is \(l_2e\)-gain stable relatively to \(r\).

Remark 1. For every index \(s\) corresponding to an internally stabilizing controller, \(l_2e\)-gain stability of the system \((\Pi/K_s)\) is always unfeasible by the I/O pair \((v_s,D)\) regardless of the switching sequence \(\sigma(t), t \in Z_+\). As a consequence, under problem feasibility, there always are indices \(i, i \in \rightarrow N\), for which \(l_2e\)-gain stability of the system \((\Pi/K_i)\) is unfeasible by \((v_i,D)\). This together with the third property in Def. 4 ensures that there always exist indices \(i\) for which the cost \(V_i(t)\) remains bounded as \(t\) increases to infinity, thus allowing for the application of the HSL.

Existing related literature proves the existence of \(l_2e\)-gain-related performance indices \(V_i\). In the light of Lemma 1, under problem feasibility, such performance indices satisfy the HSL and yield stability (2) relatively to \(r\) to the adaptive system \((\Pi/K_{\sigma(i)})\) resulting from (1), (4) and (5).

For instance, a simple form for \(l_2e\)-gain-related \(V_i\)’s is as follows

\[ V_i(t) = \max_{\tau \leq t} \frac{\|\epsilon_i(t)\|^2 + \rho \|\delta \hat{u}^r\|^2}{m^2 + \|\epsilon_i(t)\|^2} \tag{6} \]

where \(t \in Z_+, \rho > 0\), and \(\epsilon_i(k) := v_i(k) - y(k)\) denotes the tracking error in the feedback loop \((\Pi/K_i)\) driven by the virtual reference \(v_i\). The positive scalar \(m^2 > 0\) appears so as to prevent the denominator from assuming values too close to zero. The form of (6) is a natural one in that it is an estimate of the performance of \((\Pi/K_i)\) expressed in terms of the \(l_2 - l_2\) induced gain of the map embodied by \((\Pi/K_i)\) from \(v_i\) to \(D\). Further, its evaluation requires, once data \(D^t\) are collected, only computation of \(v_i^t\). The adoption of performance indices \(V_i\) computed via the maximum operator is a way for ensuring A1.

The following stability result descends directly from Lemma 1 and from the fact that the \(V_i\)’s defined in (6) are \(l_2e\)-gain-related.

Theorem 1. (Safonov et al. (2007)) Let all the candidate controllers be CCSI, and the hysteresis switching logic (4)-(5) used along with the \(V_i\)’s as in (6). Under the only assumption of problem feasibility, for any initial condition and reference \(r\), the HSL holds and the resulting VRASC system \((\Pi/K_{\sigma(i)})\), \(\Pi \in P\), is \(l_2e\)-gain-stable relatively to \(r\).
between such blocks, so as to describe the standard formulation of the problem of input-output stability. In this scenario, for the supervisor, \(n_u\) and \(n_y\) play the role of process disturbances acting “inside” \(\Pi\).

The performance index (6) is not proper to reflect correctly the I/O configuration in the noisy case and, consequently, it may cause a lack of information to the supervisor. In particular, the switching mechanism need not stop. In order to focus on how this case can arise, let \((D, \sigma)\) denote the unique solution of (8), subject to the hysteresis switching logic (4)-(5), for any initial state and bounded sequences \(n_u, n_y\), where the performance indices \(V_i(t), i \in \mathbb{N}\), are given by (6). As a matter of fact, one cannot conclude that such functionals \(V_i\)’s remain bounded even for the indices related to the stabilizing controllers. For example, this happens in case \(r \in l_2\). Indeed, suppose that both the plant and controllers are linear. Further, suppose that at the generic time instant \(t\) a certain controller \(K_i\) is switched on in the loop. Let \(P(d) = B(d)/A(d)\) and \(C_i(d) = S_i(d)/R_i(d)\) be the transfer function of \(\Pi\), respectively, \(K_i\). As shown in Fig.3, according to (3), the data given by the \(i\)-th virtual loop are obtained as

\[
\begin{align*}
\varepsilon_i(t) &= \frac{A(d)R_i(d)}{X_i(d)} (v_i(t) - n_y(t)) - \frac{B(d)R_i(d)}{X_i(d)} n_u(t) \\
\delta u(t) &= C_i(d) \varepsilon_i(t)
\end{align*}
\]

with \(X_i(d) := A(d)R_i(d) + B(d)S_i(d)\) the characteristic polynomial of \((\Pi/K_i)\). Consequently, omitting the argument, one can write

\[
\begin{align*}
\left\| \rho^{1/2} \delta u \right\|^2 &= \min_{\omega \in [0, 0.2\pi]} \left( \frac{|R_i|^2 + \rho |S_i|^2}{|X_i|} \right) \cdot I^2 \cdot \|p_i\|^2 = \\
&= N^2 \cdot \|p_i\|^2
\end{align*}
\]

with \(p_i(t) := A(d)(v_i(t) - n_y(t)) - B(d)n_u(t), \forall i \in \mathbb{N}; I^2, 0 < l \leq 1,\) accounts for the truncation effects on the \(l_2\) norm. Further, \(N^2 > 0\) because \(R_i\) and \(S_i\) are coprime polynomials. Roughly, since \(\varepsilon_i\) and \(\delta u\) are obtained by filtering the disturbances, provided that such disturbances are persistently exciting sequences of sufficiently high order, the numerator of \(V_i(t)\) grows at least linearly with \(t - T\). On the opposite, since for the switched-on controller the virtual reference converges exponentially to the true reference \(r \in l_2\), the denominator of \(V_i(t)\) turns out to be bounded as long as \(K_i\) is kept in the loop. Consequently, it cannot be argued that assumption A2 in the HSL lemma holds, and, hence, that the switching mechanism stops.

### 3.1 A novel notion of stability

This state of affairs can be understood by noting that the convergence of the virtual reference to the true one, for the switched-on controller, yields no account of the disturbances. In fact the supervisor cannot distinguish between disturbances and some process dynamics, as \(n_u\) and \(n_y\) do not affect the virtual reference of the switched-on controller. Consequently, even if the supervisor preserves the capability of detecting any trend to unboundedness due to instability, its capability of holding in the loop for ever a (final) stabilizing controller is lost.

For this reason, the minimal requirement on the functionals \(V_i\)’s is their boundedness for the indices \(i \equiv s\) related to the stabilizing controllers \(K_s, s \in \mathbb{N}\).

\[
\lim_{t \to \infty} V_i(t) < \infty \quad \text{for any possible } \sigma(-) \quad (11)
\]

It is important to remark that the performance indices (6) do not satisfy (11) because they were chosen to be \(l_{2g}\)-gain-related. However, the notion of \(l_{2g}\)-gain stability introduced in Def. 2 is inconsistent with the noisy system (8): in the presence of disturbances, even for indices \(s\) corresponding to internally stabilizing controllers, the \(l_{2g}\)-gain stability of the system \((\Pi/K_s)\) can be falsified by the I/O pair \((v_s, D)\). Consequently, for system (8) the considerations of Remark 1 do not hold.

Therefore, a different notion of stability has to be considered that takes into account the presence of bounded but possibly persistent disturbances affecting the plant.

**Definition 5.** System (8) is said to be weakly \(l_{2g}\)-gain-stable relatively to \(r\) if, for every reference \(r \in l_{2g}\), every initial condition, and every bounded disturbances \(n_u\) and \(n_y\), there exist finite nonnegative constants \(\alpha, \beta, \gamma\) such that

\[
\|D^\beta\| \leq \beta \cdot \|r\| + \gamma \cdot \sqrt{t + 1} + \alpha
\]

for any \(t \in \mathbb{Z}_+\).

The parameter \(\gamma\) stands from the contribution of \(n_u\) and \(n_y\) to the data \(D\), while \(\alpha\) allows for consideration of systems with non-zero initial state.

### 4. CHOICE OF PERFORMANCE INDICES

On the basis of the novel stability notion introduced in Def. 5, one can consider novel key properties that the cost functions have to satisfy in order to reliably detect losses of stability of the adaptive system \((\Pi/K_{s(i)})\).

**Definition 6.** Given the pair \((V, K)\), \(V\) is said to be weakly \(l_{2g}\)-gain-related if for each \(D \in l_{2g}\) and \(K_i, i \in \mathbb{N}\): 1) \(V_i(t)\) is monotone in \(t\), 2) the virtual reference \(v_i\) exists, and 3) for every \(K_i, i \in \mathbb{N}\), and \(D \in l_{2g}\), \(V_i(t)\) is bounded as \(t \to \infty\) if and only if weak \(l_{2g}\)-gain stability of the system \((\Pi/K_i)\) is unfalsified by \((v_i, D)\).
Def. 6 and the theorem that follows indicate some guidelines on how to choose a performance index so as to achieve weak $l_2$-gain stability.

4.1 Performance indices

**Theorem 2.** Let all the LTI candidate controllers be CCSI, and the hysteresis switching logic (4)-(5) used. Then, provided that

\[ C1. \quad V \text{ is weakly } l_2\text{-gain-related and} \]

\[ C2. \quad \text{problem feasibility holds,} \]

the HSL lemma holds and the resulting VRASC system $(Π/K_{σ(1)})$, $Π ∈ ℙ$, is weakly $l_2$-gain stable relatively to $r$.

**Proof.** Suppose that bounded disturbances $ν_a$ and $ν_y$ enter the system as in (8). Thus, for any index $s$ corresponding to an internally stabilizing controller, weak $l_2$-gain stability of the system $(Π/K_{σ(s)})$ is always falsified by the $1/O$ pair $(ν_a, D)$ regardless of the switching sequence $σ(t)$, $t ∈ ℤ_+$. As a consequence, by virtue of the third property in Def. 6, one may conclude that $V_s(⋅)$ remains bounded as $t$ increases to infinity for every $s ∈ N$ such that $(Π/K_{σ(s)})$ is internally stable. Therefore, under problem feasibility, the HSL holds. Further, the cost function $V_f$ related to the final switched-on controller $K_f$ is bounded. This, along with the assumption $C1$, implies that the weak $l_2$-gain stability of $(Π/K_{σ(s)})$ is falsified by $(ν_a, D)$, i.e. there exist finite nonnegative constants $α, β$ and $γ$ such that

\[ \|D^f\| ≤ β \cdot \|v_f^2\| + γ \cdot \sqrt{t + 1} + α \]  \hspace{1cm} (13)

As the virtual reference $v_f$ converges exponentially to the true reference $r$, there exists a finite nonnegative $α$ such that $\|v_f^2\| ≤ \|r^2\| + α$, $∀ t ∈ ℤ_+$. Consequently, one can conclude that

\[ \|D^f\| ≤ β \cdot \|r^2\| + γ \cdot \sqrt{t + 1} + α \]  \hspace{1cm} (14)

where $α := β \cdot α + α$. As (14) holds for the data from every possible input, the adaptive system $(Π/K_{σ(1)})$ is weakly $l_2$-gain stable, relatively to $r$. □

A possible choice, amongst many alternatives, for a weakly $l_2$-gain-related performance index $V_f(τ)$ consists of modifying the denominator of the $V_s$’s in (6) so as to account for the effects of the additional noise inputs $ν_a, ν_y$. This may be achieved by considering

\[ w_τ(t) := \left(\frac{v_f^2(t) + m^2}{τ^2}\right)^{1/2}, \quad i ∈ N \]  \hspace{1cm} (15)

where $m^2 > 0$ is a positive real, which might be chosen as an a priori estimate of the RMS value of the disturbances.

The following theorem holds.

**Lemma 2.** Let $e_τ(t) := v_τ(t) - y(t)$, $i ∈ N$, with $v_τ$ as in (3). The performance index

\[ V_f(τ) := \max_{τ ≤ t} \frac{\|v_f^2\| + μ \cdot γ \cdot √{τ + 1} + μ \cdot α}{\|w_τ^2\|}, \quad i ∈ N \]  \hspace{1cm} (16)

with $w_τ(t)$ as in (15), is weakly $l_2$-gain-related.

**Proof.** The first two properties in Def. 6 are trivially verified. With regard to the third property, notice that $V_f(τ)$ bounded implies that, $∀ τ ∈ ℤ_+$, there exist indices $i, j ∈ N$, such that $\|D^f_{τ,i}\| ≤ M \cdot \|w_f^2\|$, being $D^f_{τ,i} := \left[\frac{ρ^{1/2} d_u}{ε_i}\right]$, and $M$ a nonnegative constant. By triangle inequality it follows that

\[ \|D^f\| ≤ \|v_f^2\| + M \cdot \|w_f^2\| \]  \hspace{1cm} (27)

for some bounded positive real $M$. Hence, $∀ τ ∈ ℤ_+$

\[ \|D^f\| ≤ (1 + M) \cdot \|v_f^2\| + n \cdot M \cdot √{t + 1} \]  \hspace{1cm} (17)

On the opposite, suppose that the weak $l_2$-gain stability of $(Π/K_{σ(s)})$ is falsified by $(ν_a, D)$, i.e. $\|D^f\| ≤ β \cdot \|v_f^2\| + α \cdot γ \cdot √{t + 1}$, for some nonnegative reals $α, β$ and $γ$ and $∀ τ ∈ ℤ_+$.

Consequently, by triangular inequality, $\|D^f_{τ,i}\|^2 ≤ \|v_f^2\|^2 + \|y^2\|^2 + ρ \cdot γ \cdot √{τ + 1} ≤ \|v_f^2\|^2 + (1 + ρ) \|D^f\|^2$. Finally,

\[ \|D^f_{τ,i}\|^2 ≤ \|v_f^2\|^2 + √{1 + ρ \cdot \|D^f\|^2} \]  \hspace{1cm} (18)

Hence, boundedness of $V_f(τ)$ follows directly from the definition of $w_τ(t)$. In fact,

\[ V_f(τ) ≤ \left((1 + μ \cdot β) \cdot \|v_f^2\| + μ \cdot γ \cdot √{τ + 1} + μ \cdot α\right)^2 \]  \hspace{1cm} (19)

where $μ := √{1 + ρ}$. □

**Remark 3.** Similar results can be achieved in order to extend Def. 6 to the general CCCI controller case. In particular, a simple weakly $l_2$-gain-related performance index can be obtained re-defining $v_i(τ) := (S_i(τ)/S_i(1)) \cdot v_i(t)$ as the $i$-th virtual reference. A simple choice for the performance indices is given by

\[ V_i(τ) := \max_{τ ≤ t} \frac{\|v_i^2\|^2 + ρ \cdot γ \cdot √{τ + 1} + μ \cdot α}{\|w_i^2\|^2}, \quad i ∈ N \]  \hspace{1cm} (20)

with $w_i(t) := (v_i^2(t) + m^2)^{1/2}$, and $ε_i(t) := v_i(t) - y(t)$.

4.2 Interpretation

In order to understand the meaning of (12) w.r.t. the adaptive system (1), it is interesting to give a detailed analysis of the switching mechanism generated by (4)-(5) along with (15)-(16).

Let problem feasibility be satisfied. Let $K_s, s ∈ N$, be a stabilizing controller. For the index related to such a stabilizing controller, one has

\[ \|v_s^2\|^2 + ρ \cdot γ \cdot √{τ + 1} ≤ \|v_s^2\|^2 + \|y^2\|^2 + ρ \cdot γ \cdot √{τ + 1} \]  \hspace{1cm} (21)

\[ M_i^2 + \|v_s^2\|^2 \]  \hspace{1cm} (22)

for some finite nonnegative constants $M_i, i = 1, 2, 3$, being $\|w_s^2\| ≥ \|v_s^2\|$, $∀ τ ∈ ℤ_+$. Consequently,

\[ V_s(τ) ≤ \max_{τ ≤ t} \left(M_i^2 + \frac{\|v_s^2\|^2 + \|n_s^2\|^2}{\|w_s^2\|^2}\right) \]  \hspace{1cm} (23)

for some finite nonnegative constant $M$. From (15), as $\|w_s^2\|^2 ≥ t \cdot m^2$, one gets

\[ m^2 \cdot V_s(τ) ≤ m^2 \cdot M_i^2 + \frac{\|v_s^2\|^2 + \|n_s^2\|^2}{\|w_s^2\|^2} \]  \hspace{1cm} (24)

where $\|w_s^2\|^2$ denotes the $l_∞$-norm. Therefore, (24) ensures that the second assumption stated by the HSL Lemma is satisfied, viz. for any switching sequences $σ(⋅)$ in $S$, there is a finite time, $t_f$, beyond which $σ$ is constant. Being $V_f$ bounded, it follows by triangle inequality that

\[ \|D^f\| ≤ \|v_f^2\| + L \cdot \|w_f^2\| ≤ \|r^2\| + α + L \cdot \|w_f^∞\| \]  \hspace{1cm} (25)

for some bounded positive real $L$. Notice that the second inequality in (25) takes into account the possible different
initial condition between the controller $K_f$ switched on in
the loop and the controller $K_{f}^{-1}$ used for the calculation of
(3). Hence, as $y_f(t)$ converges exponentially to the reference
$r(t)$, $\forall t \geq t_f$, there exists a finite nonnegative constant $c$
such that $||y_f||_{\infty,t \geq t_f} = ||r||_{\infty,t \geq t_f} + c$, where
$-\infty \leq \tau \leq \infty$ denotes the $\tau$-norm over $t \geq \tau$. Therefore, one
can conclude that $||w_f||_{\infty} = (||w_f||_{\infty}^2 + m^2)^{1/2}$ equals

$$||w_f||_{\infty} = \max \{k, (||r||_{\infty,t \geq t_f} + c)^2 + m^2\}^{1/2}$$

(26)

where $k := \max_{0 \leq \tau \leq \tau} \{w_f(\tau)\}$. Clearly, $k$ and $c$ depend on
the particular switching sequence $\sigma(\cdot) \in S$, i.e. they concern
the initial state of the adaptive system and the process
disturbances. Hence, we can now give an interpretation to

$$||Df|| \leq ||f|| + \gamma \cdot \sqrt{t + 1 + \alpha}$$

(27)

where $\gamma := L \cdot ||w_f||_{\infty}$. In the concept of weak $l_2$-gain
stability given by (27), the term $\alpha$ accounts for the
nonzero adaptive system state before the last switching
time-constant, while the term $\gamma$ takes into account the
vector of the inputs to the adaptive system. In fact, $||w_f||_{\infty}$
correctly depends on the disturbances only for $t < t_f$. It
also depends on the initial conditions of the adaptive
system and the reference $r(t)$, $\forall t \in \mathbb{Z}_+$. On the opposite,
by (24)-(25), $L$ explicitly depends on the bounds $||u_i||_{\infty}$
and $||y_i||_{\infty}$ on the disturbances over $t \in \mathbb{Z}_+$. 

4.3 Loss of $l_2$-gain relatedness in the noiseless case

It is advisable to analyze the effects of the performance
index (16) in the absence of disturbances. Using (16)
instead of (6), we can no longer deduce the properties as
in Th. 1, even if there are no disturbances. This is caused
by the fact that the interfering mechanism is equipped
with a cost function which turns out to be no longer
$l_2$-gain related. In fact, in this connection, the weak $l_2$-
gain stability of the adaptive system $(H/K_{\sigma(j)})$ becomes
a sufficient condition for boundedness of the performance
index $V(K_j, D, t)$. For instance, consider the case where
the initial state of $H$ is $x_0 \neq 0$, while all exogenous inputs
to the loop equals zero. Let $K_j$ be the controller which
is switched on in feedback to the plant at the initial time
$t_0$, i.e. $\sigma(t_0) = j$. Consequently, $V_j(t) < h$, $\forall t \geq t_0$
implies $\sigma(t) \equiv j$, i.e. such a controller will never be switched
off of the loop. More specifically, any initial controller
$K_j$ is kept in the loop, provided that it generates data

$$D_p := \rho^{1/2}x$$

such that

$$||D_p||^2 \leq (t + 1) \cdot \rho^2 \cdot h$$

(28)

Consequently, every controller which stabilizes the system
in the weak sense is a potential final controller. Notice
that such a selection is carried out by the supervisor based
only on experimental data, viz. with no prior knowledge
of a plant model. In other words, within the unfalsified
control framework, with a specific single infinite-duration
experiment one cannot distinguish the effect of internal
noise on data from an unknown offset or bias in some
internal parameter of the plant.

5. CONCLUSION

The unfalsified VRASC approach has been extended to
handle the case of persistent plant disturbances (akin
the noise configuration considered for internal stability
of feedback-systems). It has been pointed out that the
presence of persistent disturbances which corrupt the data
to the switching supervisor can have the effect of making
invalid the nice conclusions of convergence analysis of
noiseless VRASC systems based on the HSL lemma. In
order to recover the relevant convergence properties of
VRASC systems in the persistent disturbance case, viz.
finitive switching stopping time and boundedness, a novel
notion of stability has been introduced and constructively
used so as to select new schemes which are shown to
achieve the desired goals.

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