Optimal Estimation over Channels with Limits on Usage
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Abstract: We consider several cases of a sequential estimation problem in which two decision making agents work together but with limited communication (between them) to minimize a performance criterion. One agent makes sequential observations about the state of an underlying, possibly vector valued, stochastic process for a fixed period of time. This observer agent upon observing the process decides whether or not to disclose some information about the process to the other agent, the estimator, and if yes, when and how. The constraint is that the observer may act only a limited, pre-specified number of times. For such problems, we develop the optimal observer-estimator policies first for the case when the source process is $n^{th}$ order Gauss-Markov, and then for the case when the source is a vector process.

1. INTRODUCTION
The emergence of networked real-time systems during the last decade has led to a number of challenging problems in control and communications, due to nontraditional requirements on control and communication policies and architectures in such settings, such as communication and coordination of various components of what used to be a centralized operation over digital and wireless networks under bandwidth, energy, and usage constraints [1]. In this paper we address one such constraint, channel usage limitation, in the context of estimation of stochastic processes. Even though the topic of estimation of random processes using noisy state information has been thoroughly studied in the literature and has long been a fairly mature field with a complete theory [2], with the kind of nontraditional (but realistic) limitation brought about by channel usage, the emerging new classes of estimation (as well as control) problems are quite novel and challenging. Such problems were introduced recently in [3, 4, 5, 6], and we build here on this earlier work. Specifically, we are interested in the problem of estimation of a discrete-time stochastic process over a decision horizon of length $N$ using only $M < N$ measurements. Measurement and estimation of the process are done sequentially by two different agents, called the observer and the estimator, respectively. Over the decision horizon of length $N$, the observer agent has exactly $M$ opportunities to disclose some information about the process to the estimator. These transmissions are assumed to be error and noise free. The general problem then is to jointly determine the optimal observation and estimation policies that minimize the average estimation error between the process output (input to the observer) and its estimate (output from the estimator).

As indicated above, problems of this type have various applications. Among them are monitoring and control of wireless sensor networks, scheduling of packet transmissions over time-allocation limited channels or any other situation in which power limitation is factor [7, 8, 9].

In Sect. 2 we formally define the problem and review what is known about its solution, as pertains to this paper. Sect. 3 discusses estimating an $n^{th}$ order Gauss-Markov scalar random sequence in which the dependence of the current state on the past reaches $n$ time units into the past. Sect. 4 considers optimal strategies for transmitting and estimating a set of $D$ identical, independent scalar random processes over $D$ identical channels. Sect. 5 includes extensions in which optimal transmission/estimation of non-uniform, independent processes over non-uniform channels (using each one incurs a different cost) is developed. Simulations and examples are presented in Sect. 6, and concluding remarks are made in Sect. 7.

2. PROBLEM STATEMENT AND DEVELOPMENT
2.1 Problem Definition
Building on [3], we treat the problem of optimal estimation with limited measurements in the framework of a communication system with limited channel uses. We consider the communication system of Figure 1. The source outputs some vector data $b_k$ for $0 \leq k \leq N-1$, to be communicated to a user over a channel, with the data $\{b_k\}$ generated according to some known stochastic process which may be i.i.d., or correlated as in a Markov process; further, it could be a scalar or a vector process. An encoder/observer and a decoder/estimator are placed after the source output and the channel output, respectively, to relay the data to the user optimally. Additionally, the encoder/observer may have access to a noise corrupted version of the source output: $z_k = b_k + v_k, 0 \leq k \leq N-1$, where $v_k$ is a noise process, independent of $b_k$ for all $k$ and with no correlation across time.

![Fig. 1. Communication with limited channel use.](image-url)
The primary constraint is that the encoder/observer can use the channel only a limited, $M < N$, number of times. For a vector process, we view the channel to be composed of several bands, each corresponding to a component of the process. All bands are used simultaneously when a transmission is made, and this counts as one channel use. We assume the channel to be memoryless and noiseless. The goal in these problems is to design an observer-estimator pair, $(\Theta, \mathcal{E})$, that will sequentially observe/encode the data measurements, $z_k$, and estimate/decode the channel output, $y_k$, while minimizing the error between the actual data, $b_k$, and estimated data, $\hat{b}_k$.

We define the estimation error by the standard mean square measure:

$$
epsilon_{(M,N)} = E \left\{ \sum_{k=0}^{N-1} \sum_{i=1}^{D} (b_{ki} - \hat{b}_{ki})^2 \right\}$$

where $D$ represents the dimension of the vectors, $b_{ki}$, $i$ is the subscript denoting the component of the vector, and $k$ is the subscript denoting the time step.

Using this framework, with the source, channel and estimation error defined, our problem is formally stated as

$$\min_{\Theta, \mathcal{E}} \left\{ E \left\{ \sum_{n=0}^{N-1} \sum_{i=1}^{D} (b_{ni} - E\{b_n|(s_n, t_n); x_n\})^2 \right\} \right\}$$

This minimization is carried out over all causal encoder-decoder (observer-estimator) pairs.

### 2.2 Problem Development

In [3] a special case of the defined problem, with a zero-mean i.i.d. scalar random sequence $b_k, 0 \leq k \leq N - 1$, was considered. The $b_k$'s are assumed to have a finite second moment, $\sigma^2_b$. Let $\mathcal{B}$ denote the range of the random variable $b_k$. At time $k$, the encoder/observer makes a sequential measurement of $b_k$ and decides whether to use one of $M < N$ channel transmissions. The channel is noiseless and even when the observer/encoder decides not to transmit, a 1-bit information may still be conveyed across the channel. Specifically, the channel input $x_k$ belongs to the set $\mathcal{X} := \mathcal{B} \cup NT$, where NT stands for "no transmission." Let $s_k$ denote the number of transmissions remaining at time $k$. Both the encoder and decoder can keep track of this by initializing $s_0 = M$ and decrementing it by 1 every time a transmission is made. The objective is to design an estimator/decoder $b_k = \mu_k(I_k^d)$, $0 \leq k \leq N - 1$, based on the information $I_k^d$ available at time $k$, which is a result of an outcome of decisions taken by the observer until time $k$. The observer's decision at time $k$ is denoted by $x_k = \mu_k(I_k^d)$, $0 \leq k \leq N - 1$, where $I_k^d$ is the information available to the observer at time $k$. The range of $\mu_k(\cdot)$ is $\mathcal{X}$ as defined above. We assume perfect recall and have

$$I_0^d = \{(s_0, t_0); b_0\}$$

$$I_k^d = \{(s_k, t_k); b_k; x_0^{k-1}\}, 1 \leq k \leq N - 1$$

where $tk$ and $sk$ denote respectively the number of time slots and transmissions left at time $k$. The channel output $y_k$ is $y_k = x_k$ if $s_k \geq 1$, and $y_k \in \emptyset$ (no information) if $s_k = 0$. So we can write the information $I_k^d$ available to the estimator at time $k$ as $I_k^d = \{(s_k, t_k); y_0^{k}\}, 0 \leq k \leq N - 1$.

In view of (1), we want to find an admissible policy that minimizes the $N$ stage estimation error. In [3], the problem was also considered for random processes which are Gaussian-Markov in nature.

To summarize, the problem is sequential in nature: at time $k$, $b_k$ is observed, then the observer decides whether or not to transmit some data. The estimator makes an estimate, incurring some (possibly zero) cost and we move on to the next time $k + 1$, and so on.

### 2.3 Relevant Results

Only the case $M < N$ is considered, because otherwise the result is trivial. For the zero mean i.i.d. problem it is shown in [3] that the optimal estimator is

$$\hat{\mu}_k(I_k^d) = \begin{cases} E\{b_k\} = 0 & \text{if } s_k = 0 \\ E\{b_k|(s_k, t_k); x_k\} & \text{if } s_k > 0 \end{cases}$$

It is also shown that the observer’s decision to use the channel to transmit a source measurement or not is based purely on the current observation, $b_k$, and its past actions only through $(s_k, t_k)$. As a result, the optimal observer policy must be of the form:

$$\hat{\mu}_k(I_k^d) = \begin{cases} b_k & \text{if } b_k \in \mathcal{F}(s_k, t_k) \\ NT & \text{if } b_k \notin \mathcal{F}(s_k, t_k) \end{cases}$$

where $\mathcal{F}(s_k, t_k)$ is a measurable set on $\mathcal{B}$, and is a function of $(s_k, t_k)$. The complement of this set is with respect to $\mathcal{B}$: $\mathcal{F}^c(s_k, t_k) = \mathcal{B} \setminus \mathcal{F}(s_k, t_k)$.

Two salient points emerge in these results. The first is that the optimal policy for the estimator/observer pair is to decouple the past from the present (and future). A related point is that the error incurred after time $k$, i.e.,

$$e_k = E \left( \sum_{n=k}^{N-1} \sum_{i=1}^{D} (b_{ni} - E\{b_n|(s_n, t_n); x_n\})^2 \right)$$

also depends only on $b_k$ and $(s_k, t_k)$.

Let $(s_k, t_k) = (s, t)$, and $e^*_{(s,t)}$ denote the optimal value of the estimation error when the decision horizon is of length $t$ and the observer is limited to $s$ channel uses, where $s \leq t$. Supposing the value of $b_k$ is unknown, the estimation error can be expressed recursively using a dynamic programming (DP) approach [10]: depending on what $b_k$ turns out to be, the remaining $(t - 1)$-stage estimation error is either $e^*_{(s-1,t-1)}$ or $e^*_{(s,t-1)}$. In the $D = 1$ case, therefore, we may write

$$e^*_{(s,t)} = \min_{\mathcal{F}(s,t)} \left\{ e^*_{(s-1,t-1)} \right\}$$

The second important point is that the optimization over the observer policies is equivalent to optimization over the sets $\mathcal{F}(s_k, t_k)$ for all $k$ such that $\max[0, M] \leq s_k \leq \min(t_k, M)$. The approach taken in [3] as well as in this paper is to restrict our search to sets $\mathcal{F}$ that are in the form of simple intervals, i.e., $\mathcal{F}^c = [a(s,t), b(s,t)]$. By rewriting the above integral in terms of only $\mathcal{F}(s,t)$, we can differentiate with respect to the endpoints of the integral to obtain their optimal values (in terms of $(s, t)$).
Similar results hold for a slight variant of this problem in which the source process is Markov, as indicated above that the optimal estimator has the form

$$\hat{\mu}(r, s, t; \eta_{N-t}) = \begin{cases} \eta_{N-t} & \text{if } y_{N-t} \in \mathcal{F}_{(r, s, t)} \\ \mathbb{E}[\eta_{N-t+1}|y_{N-t} \in \mathcal{F}_{(r, s, t)}] & \text{if } y_{N-t} \not\in \mathcal{F}_{(r, s, t)} \end{cases}$$

We take the last component of the vector $\mu$ to get the desired estimate, $\eta_{N-t}$. The quantity $\mathbb{E}[\eta_{N-t}|y_{N-t} \in \mathcal{F}_{(r, s, t)}]$ can be computed using the Kalman Filter. We will denote this quantity, the best estimate for $\eta_{N-t}$, by $\hat{\eta}_{N-t}(N-t)$. First we note that $\eta_{N-t}(N-t) = A'\eta_{N-t}$, where $\eta_{N-t}$ is known because it was transmitted $r$ time units ago. To determine the optimal observer structure and error recursion, we now make use of the following information, where $N$ denotes the Gaussian distribution:

$$b_{N-t} = y_{N-t} - N(CA'\eta_{N-t}, C\Sigma_{N-t}(N-t)(\)C')$$

Here $\Sigma_{N-t}(N-t)$ is the error matrix associated with estimating the vector $\eta_{N-t}$, using information until time $N - t - 1$. As verified below, this relevant information consists of the last transmission and the number of time units since that last transmission was made.

Claim 1. $\Sigma_{N-t}(N-t)$ can be expressed in terms of $A$, $r$ and $\sigma^2_w$.

This can be seen by analyzing the Kalman Filter expressions. Carrying out the necessary calculations reveals that

$$\Sigma_{N-t}(N-t) = \sum_{k=1}^{r} A_{k-1} \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma^2_w \end{pmatrix} (A')^k$$

Next we derive the error recursion:

$$e^*_{(r, s, t)} = \min_{\mathcal{F}_{(r, s, t)}} \left( e^*_{(s-1, t-1)} P_1 + e^*_{(r+1, s, t-1)} P_2 + \int_{b_{N-t} \in \mathcal{F}_{(r, s, t)}} \right)$$

where $P_1 = \mathbb{P}[b_{N-t} \in \mathcal{F}_{(r, s, t)}]$ and $P_2 = \mathbb{P}[b_{N-t} \in \mathcal{F}_{(r, s, t)}]$. Clearly $P_1 + P_2 = 1$. Substituting known quantities, we can simplify the above expression:

$$e^*_{(r, s, t)} = \left( e^*_{(s-1, t-1)} - (e^*_{(s-1, t-1)} + e^*_{(r+1, s, t-1)}) \right)$$

We conduct this optimization by restricting $\mathcal{F}_{(r, s, t)}$ to be in the form of a simple interval: $\mathcal{F}_{(r, s, t)} \equiv [a_{(r, s, t)}, b_{(r, s, t)}]$. We assume the simplifying symmetry $\alpha_{(r, s, t)} + \beta_{(r, s, t)} = 2CA'\eta_{N-t}$. The result of simply differentiating and obtaining optimal thresholds is
\[
\alpha = CA^t \eta_{N-t}\sqrt{e^{(1,s-1,t-1)} - e^{(r+1,s,t-1)}}
\]
\[
\beta = CA^t \eta_{N-t}\sqrt{e^{(1,s-1,t-1)} - e^{(r+1,s,t-1)}}
\]

Substitution of these optimum thresholds yields:
\[
\epsilon^*_{(r,t,t)} = \epsilon^{(1,s-1,t-1)} - \frac{\epsilon^{(r+1,s,t-1)} - \epsilon^{(1,s-1,t-1)}}{\epsilon^{(1,s-1,t-1)} - \epsilon^{(r+1,s,t-1)}} C \Sigma_{N-t} N_{(N-t-1)C^T}.
\]
\[
2 \Phi \left( \sqrt{\frac{\epsilon^{(1,s-1,t-1)} - \epsilon^{(r+1,s,t-1)}}{\epsilon^{(1,s-1,t-1)} - \epsilon^{(r+1,s,t-1)}} C \Sigma_{N-t} N_{(N-t-1)C^T} - 1} \right) - \frac{1}{2} \sqrt{C \Sigma_{N-t} N_{(N-t-1)C^T}}.
\]

This recursion is defined for \( r \geq 1 \) and \( 0 \leq s \leq t \) with boundary conditions given by
\[
\epsilon^*_{(r,t,t)} = 0
\]
\[
\epsilon^*_{(r,0,t)} = \sum_{i=r}^{t-r} \sum_{k=1}^{l} CA^{k-1} \left( \begin{array}{ccc}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0
\end{array} \right) (A^T)^{k-1} C^T.
\]

For \( n = 1 \), these results reduce to those of Sect. IV of [3], which corresponds to a one step Markov process for which we have one channel available for transmission. The generalization allows for \( n \) step Markov processes with the caveat that each transmission allows us to send a vector of \( n \) dimensions. If we have less than \( n \) channels, then the solution is no longer valid, and we cannot proceed with a DP approach. Note also that all optimal transmission/estimation problems dealing with a Markov process must begin with a transmission at the first time step.

4. Duplication Across D Channels

Here we discuss a series of related problems. Consider a scalar random process, for example a Gaussian i.i.d. process with bounded variance. Now suppose the observer makes measurements of \( D \) such i.i.d. processes. Optimal policies for the observer/estimator pair, \((\Theta, \mathcal{E})\), can be deduced by modeling the \( D \) processes as independent components of a vector random process. We first carry out the formal derivations for the Gaussian i.i.d. case. Then we reconsider the problem if each of the \( D \) processes is again Gaussian i.i.d., but also corrupted by noise. Finally we suppose that each of the processes is Gauss-Markov or \( n^{th}\) order Gauss-Markov. Note that the solution is not trivial - to minimize the error criterion, we should consider all components of the vector, i.e. applying the known solution in one dimension to the first component will not be optimal in general. Also, simply considering sequences that are \( D \) times as long as the original sequence (with \( D \) times the number of transmissions) will not work. There is the additional constraint that if a vector is sent at a particular time, we must send all components at once.

This problem can be applicable to network problems in which one may have a centralized scheduler at the source.

4.1 i.i.d. Gaussian Case

Suppose each of our \( D \) identical and independent zero mean Gaussian processes has variance \( \sigma^2\). Then we may represent the processes, \( b^1, \ldots, b^D\), as components of a vector process \( b \sim N(0, \Sigma_b) \) where \( \Sigma_b = \sigma^2 I \), since the processes are independent and of equal variance. We denote the corresponding pdf by \( f(b^1, \ldots, b^D) \). From the previous results, we see that the optimal estimator is
\[
\hat{\mu}_b(s,t; x_k) = \begin{cases}
E \{ b_k | b_k \in \mathcal{F}_{(s,t)} \} & \text{if } b_k \in \mathcal{F}_{(s,t)} \\
E \{ b_k | b_k \notin \mathcal{F}_{(s,t)} \} & \text{if } b_k \notin \mathcal{F}_{(s,t)}
\end{cases}
\]
where \( x_k \) is now a vector version of the same variable in Fig. 1 (it represents what is received by the estimator). As before, we assume a certain degree of symmetry in the decision region \( \mathcal{F}_{(s,t)} \). We allow the \( D \) dimensional region to be in the shape of a symmetrical hypercube. That is, for each axis \( i \), we allow \( \mathcal{F}_{(s,t),i} = [-\alpha_i, \alpha_i] \). Proceeding as above by developing the error recursion, we obtain:
\[
\epsilon^*_{(s,t)} = \epsilon^*_{(s-1,t-1)} + \min_{b_k} \left( - \epsilon^*_{(s-1,t-1)} - \epsilon^*_{(s-1,t-1)} \right).
\]
\[
\int_{-\alpha_i}^{\alpha_i} \cdots \int_{-\alpha_D}^{\alpha_D} f(b^1, \ldots, b^D) db^1 \ldots db^D
\]
\[
+ \int_{-\alpha_i}^{\alpha_i} \cdots \int_{-\alpha_D}^{\alpha_D} \epsilon^*_{(s-1,t-1)} db^1 \ldots db^D
\]
\[
+ \int_{-\alpha_i}^{\alpha_i} \cdots \int_{-\alpha_D}^{\alpha_D} \epsilon^*_{(s-1,t-1)} db^1 \ldots db^D
\]

where for now, \( w_1, \ldots, w_D \) are equal to 1. We want to minimize the quantity globally with respect to \( \mathcal{F}_{(s,t)} \). We first take partial derivatives with respect to \( \alpha_k \):
\[
\frac{d \epsilon^*_{(s,t)}}{d \alpha_k} = \frac{2 \epsilon^*_{(s-1,t-1)} - \epsilon^*_{(s-1,t-1)} + \alpha^2}{(2\pi\sigma_\alpha^2)^{D/2}} \prod_{j=1,j \neq k}^{D} \left( \int_{-\alpha_j}^{\alpha_j} \epsilon^*_{(s-1,t-1)} e^{-\frac{(y_j)^2}{2\sigma_\alpha^2}} db^j \right).
\]

Upon inspecting this quantity, we arrive at the following result of this subsection:

**Theorem 2.** \( \epsilon^*_{(s,t)} \) has a unique global minimizing decision region \( \mathcal{F}_{(s,t)} \) when restricted to sets of the form \( \mathcal{F}_{(s,t),i} = [-\alpha_i, \alpha_i] \) with \( 1 \leq i \leq D \) and \( \alpha_i > 0 \). This minimizer is given by \( \alpha_i = \alpha \), where \( \alpha > 0 \) satisfies
\[
0 = -\left( \epsilon^*_{(s-1,t-1)} - \epsilon^*_{(s-1,t-1)} \right) + \alpha^2
\]
\[
+ (D - 1) \left( \sigma_\alpha^2 - \frac{2\sigma_\alpha \alpha \epsilon^*_{(s-1,t-1)}}{\sqrt{2\pi}(2\Phi\left( \frac{\alpha}{\sigma_\alpha} \right) - 1)} \right)
\]
where \( \Phi \) is the standard Gaussian error function.

**Proof:** Using the usual rules of integration for Gaussian pdf's, it is easily shown that the given decision region is indeed a stationary point. This is done by noting that the choice causes all partial derivatives to be equal to zero. In the expression above, we focus on the part inside the
curly braces (the part multiplying it is always positive). Substituting $\alpha_i = \alpha$ in this expression, we get exactly the expression on the RHS above. To show that this admits a unique solution, and that this solution is the unique minimum sought, we proceed as follows. First note at the point $\alpha_k = 0$, the expression approaches a strictly negative value. However, as $\alpha_k$ becomes large, the expression becomes strictly positive. Finally, at any point where the expression is zero, moving an arbitrary amount to the right increases the quantity and moving to the left decreases the quantity. Hence we see that a solution exists and is unique, with all $\alpha_k$’s the same and positive. ♦

We may now wrap up the problem by rewriting the error recursion in terms of $\alpha$. This is done, as above, by substitution of the output thresholds into the recursion:

$$
e^{*(s,t)} = \left(2\Phi\left(\frac{\alpha}{\sigma_b}\right) - 1\right)^D + D\left(2\Phi\left(\frac{\alpha}{\sigma_b}\right) - 1\right)^{D-1}.
$$

The recursion is defined for $0 \leq s \leq t$ with boundary conditions given by $e^{*(t,t)} = 0$ and $e^{*(0,t)} = D\sigma_b^2 t$.

### 4.2 Noisy i.i.d. Gaussian case

Suppose each of the $D$ i.i.d. zero mean Gaussian processes has variance $\sigma^2_a$ and the observations are corrupted by i.i.d. zero mean Gaussian noise of variance $\sigma^2_v$. We may write: $z_k = b_k + v_k$ where $b_k$ is defined as in the previous subsection, $v_k$ is a zero mean Gaussian noise vector with covariance matrix $\sigma^2_v I$ and $b_k$ and $v_k$ are independent. The following distributions are also known:

$$f_{z_k} \sim N\left(0, (\sigma_b^2 + \sigma_v^2)I\right)$$

$$f_{b_k|z_k} \sim N\left(\frac{\sigma_b^2 z_k}{\sigma_b^2 + \sigma_v^2}, \frac{\sigma_v^2}{\sigma_b^2 + \sigma_v^2}\right)$$

After reasoning in the usual manner, we obtain the optimal estimator and observer values:

$$\hat{\mu}_k((s,t);x_k) = \left\{ \begin{array}{ll} \frac{\sigma_b^2 x_k}{\sigma_b^2 + \sigma_v^2} & \text{if } b_k \in F^F(c_t) \\ \hat{E}[b_k|z_k \in F^F(c_t)] & \text{if } b_k \notin F^F(c_t) \end{array} \right.$$

$$\hat{\mu}_k((s,t);z_k) = \left\{ \begin{array}{ll} \hat{\mu}_k((s,t);x_k) & \text{if } z_k \in F^F(c_t) \\ \frac{\sigma_v^2}{\sigma_b^2 + \sigma_v^2} & \text{if } z_k \notin F^F(c_t) \end{array} \right.$$

The optimal decision regions are given by $F^F(c)}) = [-\frac{\sigma^2_v + \sigma^2_a}{\sigma^2_a}, \frac{\sigma^2_v + \sigma^2_a}{\sigma^2_a}]$ where the subscript $i$ is used to index the components of our vector and $\alpha > 0$ is the solution to

$$0 = -\left(\alpha^{*(s-1,t-1)} - \alpha^{*(s,t-1)}\right) + \alpha^2 + (D-1)\left(\frac{\sigma^2_v}{\sigma_b^2 + \sigma_v^2}\right)\alpha^2.$$

Finally, the error recursion is given by

$$e^{*(s,t)} = e^{*(s-1,t-1)} + D\left(\frac{\sigma^2_v}{\sigma_b^2 + \sigma_v^2}\right)^2 - \left(e^{*(s-1,t-1)} - e^{*(s,t-1)}\right) + \left(2\Phi\left(\frac{\alpha}{\sigma_b}\right) - 1\right)^D + D\left(2\Phi\left(\frac{\alpha}{\sigma_b}\right) - 1\right)^{D-1}.$$

Non-uniform channel costs

We now consider a generalization of the given problem. Suppose that over each of $D$ channels we have a mean zero Gaussian process $b_i$ but each process has a possibly different variance, $\sigma^2_i$ for $1 \leq i < D$. Suppose also that we wish to weight the cost of using each channel differently, that is, the cost of using channel $i$ is $w_i$. In the above problems, we took $w_i = 1$ for all $i$. The problem is developed in the same way as in subsection 4.1. We again assume a certain degree of symmetry in the decision region $F^F(c)$ and allow the $D$ dimensional region to be in the shape of a symmetrical hypercube. That is, for each axis $i$, we allow $F^F(c;i) = [-\alpha_i, \alpha_i]$. Proceeding as above by developing the error recursion, we obtain (2) (this time weighting the channel costs, and again using $f$ for the corresponding joint pdf). Taking partial derivatives gives

$$\frac{\partial e^{*(s,t)}}{\partial \alpha_k} = \frac{2e^{-\frac{\alpha}{\sigma_b^2}}}{(2\pi)^{D/2}\sigma_{b1}...\sigma_{BD}} \prod_{j=1,j\neq k}^D \left(\int_{-\alpha_i}^{\alpha_i} e^{-\frac{(b_j - b_i)^2}{2\sigma_{b_j}^2}} db_j\right) + \sum_{i=1,i\neq k}^D w_i \int_{-\alpha_i}^{\alpha_i} e^{-\frac{(b_j - b_i)^2}{2\sigma_{b_j}^2}} db_j.$$
The recursion is defined for $0 \leq s \leq t$ with boundary conditions given by $e^{\ast}_{s,t}(t,t) = 0$ and $e^{\ast}_{s,t}(0,t) = t \sum_{i=1}^{D} w_{i} \sigma_{i}^{2}$. Note that this development simplifies to the duplication across $D$ channels case of subsection 4.1 when the weights $w_{i}$ are taken to be one and when the variances $\sigma_{i}$ (and subsequently decision thresholds $\alpha_{i}$) are independent of $i$.

6. SIMULATIONS AND EXAMPLES

6.1 A two step noiseless scalar Gauss-Markov process

Consider the following random process

$$b_{k+1} - 0.5b_{k} - 0.25b_{k-1} = w_{k}$$

where $w_{k}$ is i.i.d. Gaussian with $\sigma_{w}^{2} = 1$. This is a stable process, with the eigenvalues of the 2-dimensional matrix of its state space representation being 0.8431 and −0.5931. Implementation of the algorithm for an event horizon of $N = 30$ and varying $M$ over $[0, 30]$ yields an error curve resembling Fig. 5 from [3]. See the left side of Fig. 2 below.

6.2 Transmitting three i.i.d. Gaussian processes

Suppose we have three random processes, all mean zero i.i.d. Gaussian with variance $\sigma_{w}^{2} = 0.8$. Say that we are interested in the process over a horizon of $N = 40$ time steps and the observer, as defined above, is allowed $M < 40$ channel uses to transmit this information to an estimator agent. A channel use means transmitting the state of all three processes simultaneously. The right side of Fig. 2 shows the corresponding optimal error vs. number of allowed channel uses graph.

![Graph showing optimal error vs. number of allowed channel uses for three i.i.d. Gaussian processes.]

Fig. 2. Optimal 40-stage estimation error vs. the number of allowed channel uses under two scenarios.

Now suppose that rather than having three identical processes, we have 30 such processes. Plotting the graph again yields a much straighter curve, which seems to indicate a trend. As more processes are considered, the curve becomes a straight line connecting the points (0,32P) and (40,0) where $P$ is the number of processes. This is a topic that warrants further investigation.

7. CONCLUSION

We have obtained optimal transmission/estimation strategies for a general $n$-step Gauss-Markov process with limited channel usage. We have done this by considering a state space model and transmitting a vector process over the channel. We have also obtained the policies for sending vectors over a channel with limited transmissions, as long as the components of the vector are independent.

We can generalize the results of 4.1 to the $n^{th}$-order Gauss-Markov case where such a process is duplicated over $D$ channels (identical and independent as in the $0^{th}$ order case). The results have not been included due to space limitations. The form of the solution will be similar to the developments of 4.1. An important difference surfaces, however: the decision region is not centered around the origin, but around the most likely estimate of the process.

A number of open questions remain. An important problem is to determine optimal strategies when an $n^{th}$-order process is to be transmitted over fewer than $n$ channels. Another interesting case is to consider systems in which there is a Markov dependency as well as observation noise. Additionally, both [3] and this paper restrict the analysis to symmetric decision regions: a formal proof needs to be obtained to show that there is no loss of optimality in doing this. One might also consider optimal control problems in which actuators have limited transmissions to the plant, and when these transmissions have limited reliability, along the direction of the work in [5] and [11].

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REFERENCES