Finite-Time Tracking Control of a Nonholonomic Mobile Robot

Zhao Wang*, ** Shihua Li*, 1 Shumin Fei*

* Key Laboratory of Measurement and Control of Complex Systems of Engineering, Ministry of Education, School of Automation, Southeast University, Nanjing, P.R.China
** College of Information and Control Engineering, China University of Petroleum, Dongying 257061, P.R.China

Abstract: In this paper, the finite-time tracking problem is investigated for a nonholonomic wheeled mobile robot in a fifth-order dynamic model. We consider the whole tracking error system as a cascaded system. Two continuous global finite-time stabilizing controllers are designed for a second-order subsystem and a third-order subsystem respectively. Then finite-time stability results for cascaded systems are employed to prove that the closed-loop system satisfies the finite-time stability. Thus the closed-loop system can track the reference trajectory in finite-time when the desired velocities satisfy some conditions. In particular, we discuss the control gains selection for the third-order finite-time controller and give sufficient conditions by using Lyapunov and backstepping techniques. Simulation results demonstrate the effectiveness of our method.

1. INTRODUCTION

In the past decades, control problems for nonholonomic systems have been extensively studied (Kolmanovsky et al. 1995). Nonholonomic wheeled mobile robot(WMR) is one of the benchmark nonholonomic systems. Previous results on the control problem of nonholonomic mobile robots can be roughly categorized as either stabilization or tracking. It is known that stabilization of nonholonomic wheeled mobile robots is in general quite difficult. A well-known work of Brockett (1983) identifies that nonholonomic systems can not be stabilized via smooth or continuous time-invariant state feedback. To overcome this difficulty, researchers have proposed controllers that utilize smooth time-varying control laws (Samson, 1990; Pomet, 1992; Murray and Sastry, 1993; Tian and Li, 2002), discontinuous control laws (Wit and Sodalen, 1992; Astolfi, 1996) or hybrid control laws (Sordalen and Egeland, 1995; M’Closkey and Murry, 1997; Hespanha and Morse, 1999).

From a practical perspective, the tracking control problem is a more interesting problem. Thus, the tracking control problem of nonholonomic WMR systems has attracted growing attention in recent years. A linearization-based local tracking control scheme is introduced for a mobile robot with a kinematic model (Kanayama et al. 1990). A linear time-varying feedback law is presented in Walsh et al. (1994) to solve the local uniform tracking control problem. Both local and global tracking problems are solved via time-varying state feedback based on the backstepping technique (Jiang and Nijmeijer 1997). Considering the input saturation case, Jiang et al. (2001) solves both global stabilization and global tracking problem for the kinematic model of a wheeled mobile robot. The paper of Tian and Cao (2007) considers the nonholonomic systems in chained form with target signals that may exponentially decay to zero. Smooth time-varying controllers render the tracking-error dynamics globally κ-exponentially stable and are applied to the tracking problem of a mobile robot. The work of Tian and Cao (2007) is generalized to a more general class of nonholonomic dynamic systems in Cao and Tian (2007).

While most of current research are focused on kinematic models of nonholonomic mobile robots, little attention has been attracted by dynamic models. The latter is of more practical significance since it considers not only system kinematics but also system dynamics. One way to solve the control design for nonholonomic mobile robots in dynamic models is an “indirect” method. Kinematic controllers are first designed or just borrowed from previous literature on the kinematic model, then torque controllers are designed by using integrator backstepping techniques. In this case, to ensure the global stability, global Lyapunov functions and stability analysis for the whole dynamic systems have to be given (Fierro and Lewis 1995; Jiang and Nijmeijer 1997; Fukao et al. 2000). Another way is to construct the control law directly. A variable structure control law is proposed for a mobile robot and guarantees bounded tracking errors with initial bounded errors under bounded disturbances (Shim et al. 1995).

Finite-time control aims to make states reach their targets in finite time. Finite-time stability means finite-time convergence and Lyapunov stability. Finite-time stable systems might enjoy not only faster convergence but also better robustness and disturbance rejection properties (Bhat and Bernstein, 1998). In recent years, more and more attention has been paid to the finite-time control design and analysis based on continuous feedbacks. For the double
2. PRELIMINARIES AND PROBLEM FORMULATION

2.1 Preliminaries

Consider a time-varying system
\[ \dot{x} = f(t, x), \quad f(t, 0) = 0, \quad \forall t \geq t_0 \geq 0, \]
where \( f : R^{n_1} \times R^n \rightarrow R^n \) is continuous. The solution which starts from \( x_0 \) at time \( t_0 \) is denoted as \( x(t, x_0) \).

**Definition 1** (Bhat and Bernstein, 1997; Moulat and Perruquet, 2003): The origin is said to be a uniformly finite-time stable (UFTS) equilibrium of (1) if

(i) the origin is uniformly Lyapunov stable in a neighborhood \( U \subseteq U_0 \) of the origin;

(ii) (uniform finite-time attractively) the origin is finite-time convergent in \( U \). Namely, if there is a settling time function \( T : U \rightarrow [0, \infty) \), such that, \( \forall x_0 \in U \), the solution \( x(t, x_0) \) of system (1) with \( x_0 \) as the initial conditions is defined and \( x(t, x_0) \in U/\{0\} \) for \( t \in [0, T(x_0)] \) with the properties: \( \lim_{t \rightarrow T(x_0)} x(t, x_0) = 0 \) and \( \lim_{t \rightarrow 0} T(x) = 0 \).

When \( U = U_0 = R^n \), then the origin is a globally finite-time stable (UGFTS) equilibrium.

Consider the following time-varying cascade system
\[ \dot{x}_1 = f_1(t, x_1, x_2), \]
\[ \dot{x}_2 = f_2(t, x_2), \]
where \( x_1 \in R^n, x_2 \in R^m, f_1(t, x_1) = f(t, x_1, 0), \]
\[ g(t, x_1, x_2) = f(t, x_1, x_2) - f(t, x_1, 0), \]
which implies that \( g(t, x_1, 0) = 0, \forall (t, x_1) \in (R_{>0} \times R^n) \). Moreover, \( f(t, x_1, x_2), f_2(t, x_2) \) are locally Hölder continuous in \( x_1, x_2 \) and uniformly continuous in \( t \).

**Lemma 1.** (Li and Tian, 2007) The cascaded system (2)-(3) is UGFTS if assumptions A1, A2 and A3 hold:

A1: The subsystem \( \dot{x}_1 = f_1(t, x_1) \) is uniformly globally finite-time stable (UGFTS).

A2: The subsystem (3) is UGFTS.

A3: (Uniform forward completeness) For any fixed and bounded \( x_2 \), there exists a finite-time bounded (FTB) function \( B(t, x_1) : R_{>0} \times R^n \rightarrow R_{>0} \) positive definite, proper, radially unbounded and decrescent, which satisfies
\[ B(t, x_1) \leq \beta_0(B(t, x_1)), \quad \forall t \geq t_0 \geq 0, \]
where \( \beta_0 : R_{>0} \rightarrow R_{>0} \) is a nondecreasing function satisfying the following condition for some constant \( a > 0 \)
\[ \beta_0(a) \geq 0, \int_0^\infty ds/(\beta_0(s)) = \infty. \]

For convenience, let \( sgn^{\alpha} : = sgn(x)|x|^\alpha \) for \( \alpha > 0 \), where \( sgn(\cdot) \) is the sign function.

**Lemma 2.** (Bhat and Bernstein, 1997) The following control law
\[ u = -k_1 sgn^{\alpha_1} x_1 - k_2 sgn^{\alpha_2} x_2, \]
where \( k_1, k_2 > 0, 0 < \alpha_2 < 1, \alpha_1 = \alpha_2/(2 - \alpha_2) \), finite-time stabilizes the double integrator system \( \dot{x}_1 = x_2, \quad \dot{x}_2 = u. \)
2.2 Problem formulation

The problem we study here deals with a nonholonomic wheeled mobile robot with two-degrees-of-freedom. A simplified dynamic model of it is given by Jiang and Nijmeijer (1997)

\[
\begin{align*}
\dot{x} &= v \cos \theta, \quad \dot{y} = v \sin \theta, \quad \dot{\theta} = \omega, \\
\dot{v} &= u_1, \quad \dot{\omega} = u_2,
\end{align*}
\]

where \((x, y)\) are the Cartesian coordinates of the center of mass of the vehicle, \(\theta\) is the angle between the heading direction and the \(x\) axis, \(v\) is the linear velocity and \(\omega\) is the angular velocity of the mobile robot. \(u_1\) and \(u_2\) may be regarded as torques or generalized force variables of the two-degrees-of-freedom mobile robot.

The objective is the robot follows a reference robot, with position \((x_r, y_r, \theta_r)\) and inputs \(v_r\) and \(\omega_r\). The reference trajectory is given by the following equations.

\[
\begin{align*}
\dot{x}_r &= v_r \cos \theta_r, \quad \dot{y}_r = v_r \sin \theta_r, \quad \dot{\theta}_r = \omega_r.
\end{align*}
\]

Denoting the error coordinates by Kanayama et al. (1990)

\[
\begin{bmatrix}
x_e \\
y_e \\
\theta_e
\end{bmatrix} =
\begin{bmatrix}
\cos \theta & \sin \theta & 0 \\
-\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x - x_r \\
y - y_r \\
\theta - \theta_r
\end{bmatrix}
\]

which implies that

\[
\begin{align*}
\dot{x}_e &= \omega y_e - v + v_r \cos \theta_e, \\
\dot{y}_e &= -\omega x_e + v_r \sin \theta_e, \\
\dot{\theta}_e &= \omega_r - \omega, \\
\dot{v} &= u_1, \\
\dot{\omega} &= u_2.
\end{align*}
\]

3. FINITE-TIME TRACKING CONTROLLER DESIGN

We can treat the tracking error model (10) as a cascaded system consisting of two subsystems. One is the third-order subsystem with states \(x_e, y_e\) and \(v\). The other is the second-order subsystem with \(\theta_e\) and \(\omega\). First, a finite-time controller is designed for the second-order subsystem. Second, a finite-time controller is designed for the third-order subsystem. And then the sufficient conditions of finite-time stability for cascaded systems are verified so that the closed loop system is proved to be uniformly globally finite-time stable.

Considering a system with a chain of third-order integrator, the selection of control gains is also a very important problem, especially from the practical application aspect. To this end, we give sufficient conditions for selecting available control gains.

3.1 Finite-time control law for a triple integrator system

In this section, we aim at deriving a finite-time control law for a system with a chain of a triple integrator as:

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= x_3, \\
\dot{x}_3 &= u.
\end{align*}
\]

Theorem 3. The system (11) is globally finite-time stabilized by the following control law

\[
u = -l_3(x_3^{1/(2q_1)} - 1) + l_2^{1/(2q_1)}(x_2^{1/q_1} + l_1^{1/q_1} x_1))^{3q_2 - 2}.
\]

Remark 1: It should be pointed out that the controller structure (12) as well as the proof procedure of Theorem 1 are similar to that in Huang et al. (2005). The only difference is that here we do not restrict the control gains as well as the coefficients of Lyapunov functions to be constants. Hence, a sufficient condition for the selection of control gains can be extra obtained.

Remark 2: It can be observed from the proof that the fraction power \(q_1\) directly influences the convergence rate of the closed loop system. From the proof, we know that the states converge faster while \(q_1\) is smaller. Moreover, when \(q_1\) approaches 1, the response of the closed loop system approaches exponential convergence and when \(q_1\) approaches 2/3, the controller approaches a variable structure controller.

3.2 Discussion for the selection of control gains

The three nonlinear inequalities of Theorem 1 are complicated. To this end, we try to find some available methods to ensure the inequalities (13), (14) and (15) all have solutions for \(k_j(i = 1, 2)\) and \(l_j(j = 1, 2, 3)\). One way is to use a trial method. That is, first selecting the values according to the characteristics of the three inequalities, then testing...
them by substituting them into the three inequalities. It is not easy to take values according to them. This method requires patience and experience due to the complexity of inequalities. Moreover, though one can finally find a feasible solution, a good comprehension of the solution space for the control gains can not be obtained.

Another way is to use an exhaustive algorithm. First, $q_1$ is chosen. Second, we restrict the searching intervals for $l_1$, $l_2$, $l_3$, and $l_{3inf}$ respectively. Third, by using a nested circuit technique, we can get all the feasible solutions in the given intervals satisfying (13) and (14), and then the infimum of $l_3$, denoted as $l_{3inf}$ can thus be obtained by (15). Hence, a group value for $l_1$, $l_2$ and $l_{3inf}$ can be chosen from the saved data sets $S_{inf}$. The numerical feasible solution set $S$ in the given intervals can thus be expressed as follows

$$S = \{ (l_1, l_2, l_3) | (l_1, l_2, l_{3inf}) \in S_{inf}, l_3 > l_{3inf} \}.$$  

One can also find some further descriptions and results in the simulation section.

3.3 Finite-time tracking control for the dynamic model

In this section, we design a finite-time tracking controller for the dynamic model of a nonholonomic wheeled mobile robot. Let us consider the error dynamic model (10) and assume that the velocities $v$ and $\omega$ are subject to the following constraints:

**B1:** $\omega_r$, $\dot{\omega}_r$, $\ddot{\omega}_r$ exist and $0 < \omega_{rmin} \leq |\omega_r(t)| \leq \omega_{rmax}$, $|\omega_r(t)| < \omega_{rmax}^t$ for each $t \geq t_0 \geq 0$.

**B2:** $v_r$ and $\dot{v}_r$ exist and $|v_r(t)| \leq v_{rmax}$ for each $t \geq t_0 \geq 0$.

If we denote $\omega = \omega - \omega_r$, $\omega_r$, (10) can be rephrased as follows

$$\dot{x}_e = \omega_r y_e - v + \omega_r - \omega y_e + v_r (\cos \theta_e - 1),$$
$$\dot{y}_e = -\omega_r x_e + \omega x_e + v_r \sin \theta_e,$$
$$\dot{\theta}_e = \omega,$$
$$\ddot{\omega}_e = \ddot{\omega},$$

where $\ddot{\omega} = \omega_r - \omega$ and $\ddot{\omega}_e = \ddot{\omega}_r - u_2$. Consider a state transformation defined by

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_e \\ -\omega_r x_e \\ -\omega^2_r y_e + \omega_r (v - v_r) + \omega x_e / \omega_r \end{bmatrix},$$

$$X_2 = \begin{bmatrix} x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} \theta_e \\ \omega \end{bmatrix}.$$  

The derivatives of $X_1$ and $X_2$ are

$$\dot{X}_1 = f(X_1, u_1) + g(X_1, X_2),$$
$$\dot{X}_2 = \begin{bmatrix} x_5 \\ u_2 \end{bmatrix},$$

where

$$f(X_1, u_1) = \begin{bmatrix} x_2 \\ x_3 \\ u_1 \end{bmatrix},$$
$$g(X_1, X_2) = \begin{bmatrix} -x_2 x_3 / \omega_r + v_r \sin \theta_r x_4 \\ x_1 x_3 / \omega_r - \omega_r v_r (\cos \theta_r - 1) \\ x_2 x_3 / \omega_r - v_r \sin \theta_r x_4 + \omega \dot{x}_r x_3 - \omega_r v_r (\cos \theta_r - 1) \end{bmatrix}.$$

Thus, the tracking error model (10) is transformed to a cascaded system (18) and (19).

**Theorem 4.** If assumptions B1 and B2 hold, then the system (10) is uniformly globally finite-time stable, i.e., the system (7) can globally track the reference trajectory (8) in finite time, under the following control laws

$$u_1 = -\frac{1}{\omega_r} l_3 (x_3 / (2q_1 - 1) + l_2 / (2q_1 - 1) (x_2 / q_1 + l_1 / q_1 x_1))^{3q_1 - 2} - \ddot{\omega}_r x_3 + \dot{v}_r,$$

$$u_2 = \ddot{\omega}_r + l_4 \sin \theta^\alpha (x_4) + l_5 \sin \theta^\beta (x_5).$$

where $l_i (i = 1, \ldots, 5)$, $\alpha_1$ and $\alpha_2$ satisfy the conditions in Theorem 3 and Lemma 2 respectively.

**Proof:** It can be verified that subsystems $\dot{X}_1 = f(X_1, u_1)$ and (19) under control laws (20) and (21) will be finite-time stable by using Theorem 3 and Lemma 2. Next we will show that the closed loop system (18) and (19) will also satisfy the uniform forward completeness condition.

Consider the finite-time bounded function $R(X_1) = x_2^2 / 2 + x_3^2 / 2 + x_5^2 / 2$, which is obviously a decreasing function. Taking the derivative of $B(X_1)$ along solutions of (18) yields

$$\dot{B}(X_1) = x_1 (x_2 - x_2 x_5 / \omega_r + v_r \sin \theta_r x_3 + x_2 (x_3 - \omega_r x_5 + \omega_r v_r (\cos \theta_r - 1)) + x_3 (u_1 + \omega_r x_2 x_5 - \omega^2_r v_r x_3 + \omega \dot{x}_r x_3 + \omega x_5) + \omega_r x_2 x_5 + x_3 u_1 + \omega_r x_2 x_3 x_5 + \omega^2_r x_r x_3)$$

+ $\omega_r x_3 x_5 + \omega_r x_5 x_3.$

Substituting (12) into (22) yields

$$\dot{B}(X_1) \leq x_1 x_2 + (\omega - 1 / \omega_r) x_1 x_2 x_5 + |v_r x_3| + x_2 x_3$$

+ $|\omega_r v_r x_2| + \omega_r x_2 x_5 - l_3 x_3 x_3 / (2q_1 - 1) + \omega x_2 x_5 - l_2 (x_2 / q_1 + l_1 / q_1 x_1)^{3q_1 - 2} + \omega^2_r x_r x_3 + \omega r x_3 x_5 + \omega x_5 x_3.$

Since the subsystem (19) is uniformly globally finite-time stable, $x_5$ is bounded. We denote $|x_5(t)| \leq x_5^{max} > 0$ is a constant. Let $\eta_1 = \|X_1(t)\| = \sqrt{x_2^2 + x_3^2 + x_5^2} \geq \eta > 1$, then we have $|x_1(t)| \leq \eta_1 \leq \eta_1^2$ and $|x_2(t) x_3(t)| \leq \eta_1^2 / 2$, $i = 1, 2, 3$. With this in mind, we obtain

$$\dot{B}(X_1) \leq \frac{\eta_1^2}{2} + \frac{\eta_1^2}{2} |\omega - 1 / \omega_r| x_5^{max} + |v_r| \eta_1 + \frac{\eta_1^2}{2} + |\omega_r v_r| \eta_1$$

+ $\frac{\eta_1^2}{2} |\omega_r x_2^{max} + \omega x_5^{max} + |v_r| \eta_1 + \frac{\eta_1^2}{2} |\omega_r v_r| \eta_1^2 + l_3 / (2q_1 - 1) + l_2 / (2q_1 - 1) \eta_1^{3q_1 - 2}.$
Consider assumptions B1 and B2, then $\dot{B}(X_1)$ can be rewritten as

$$\dot{B}(X_1) \leq KB(X_1) \tag{23}$$

where

$$K = 2 + \left( \omega_r^{\max} + \frac{1}{\omega_r^{\min}} \right) v_r^{\max} + 2 \omega_r^{\max} v_r^{\max} + \omega_r^{\max} x_0^{\max}$$

$$+ 2 v_r^{\max} + 2 \left( \omega_r^{\max} \right)^2 v_r^{\max} + \omega_r^{\max} x_0^{\max} + 2 \alpha_1 v_r^{\max}$$

$$+ 2 \beta_3 \left( \frac{l_2^3}{(2q_1 - 1)} + \frac{l_2^3}{(2q_1 - 1)} \frac{l_4}{2q_1 - 2} \right).$$

On the other hand, if $\eta_1 < 1$, there exists a constant $L > 0$ such that $\dot{B}(X_1) \leq L$. Thus, we obtain

$$\dot{B}(X_1) \leq KB(X_1) + L.$$

Let $\beta_0(s) = Ks + L$. Obviously, the forward completeness condition is satisfied.

By virtue of Lemma 1, the closed loop system (18) and (19) is uniformly globally finite-time stable. Then under the control laws (20) and (21), the closed loop system (10) is uniformly globally finite-time stable, which means that the system (7) can globally track the reference trajectory (8) in finite time. Thus, the proof is completed.

4. SIMULATION RESULTS

In this section, we demonstrate the previous theoretical results by means of numerical simulation. The reference velocities are given by

$$\omega_r = 1 + 2t/(t+10) \text{ m/s}^{-1}, v_r = 1.5 - 1.5t/(t+10) \text{ rad/s}^{-1}.$$

For subsystem (19), we select $\alpha_2 = 3/5, \alpha_1 = \alpha_2/(2 - \alpha_2) = 3/7, l_1 = 10$ and $l_2 = 8$. For subsystem $f(X_1, u_1) = [x_2, x_3, u_1]^T$, we choose parameter value as $q_1 = 11/13$.

According to previous descriptions, we can use an exhaustive program to get feasible control gains. First, we restrict $k_1 \in [0.1, 4], k_2 \in [1, 300], l_1 \in [0.6, 1.5]$ and $l_2 \in [1, 10]$. The step sizes are all set to 0.1. By using a nested circulation technique, all the feasible solutions of $l_1$ and $l_2$ satisfying (13) and (14) in the restricted area can be found. Thus we obtain the corresponding inimums of $l_3$ from (15). To have a good comprehension of the feasible solution, we give a plot of $(l_1, l_2, l_3(n))$ in the case of fixing $k_1$ and $k_2$ as $k_1 = 2.91$ and $k_2 = 294.6$. It is shown in Fig. 1.

Thus, a group of control parameters is selected as $l_1 = 0.81, l_2 = 3.9$ and $l_3 = 210 > l_3(n) = 196.92$. Thus, two finite-time controllers are constructed as

$$u_1 = \frac{-210}{\omega_r} \left( x_3^{13/9} + 3.913^{13/9} x_2^{14/11} + 0.81^{13/11} x_1 \right)^{7/13}$$

$$+ \omega_r x_3 + \left( -\frac{\omega_r}{\omega_r^{2} x_3} + \omega_r + \frac{2}{\omega_r^{2} x_3} x_2 \right) x_2 - \frac{2}{\omega_r^{2}} \omega_r x_3 + \dot{v}_r \tag{24}$$

$$u_2 = \omega_r + 10 \sin 3/7 x_4 + 8 \sin 3/7 x_5. \tag{25}$$

From Theorem 2, the system (7) can globally track the reference trajectory (8) in finite time.

In order to get the states in the dynamic model, the following transformations are needed.

$$\theta = \theta_1 - x_4, x = x_r + \cos \theta \omega_r x_2 + x_1 \sin \theta,$$

$$y = y_r + \sin \theta \omega_r x_2 - x_1 \cos \theta, v = x_3 \omega_r + \omega_r x_1 + v_r - \frac{\omega_r x_2}{\omega_r^2},$$

$$\omega = \omega_r - x_5.$$

In simulation we take the initial conditions as $[x_1(0), x_2(0), x_3(0), x_4(0), x_5(0)] = [0.54, -0.25, -1.4, -0.5, 0.5]$ and $[x_r(0), y_r(0), \theta_r(0)] = [0, 0, -1]$.

We have simulated the nonholonomic WMR with the tracking controllers (24) and (25). The simulation results are shown in Fig. 2-4. Fig. 2 shows the time response of $x_r, y_r, \theta_r$. Fig. 3 shows the time history of control input $u_1$ and $u_2$. Fig. 4 shows the phase plot on $x - y$ plane.

5. CONCLUSION

A finite-time tracking control design method for the nonholonomic wheeled mobile robot in a dynamic model has been presented. The closed loop system can track the desired trajectory in finite time. A sufficient condition for the control gains of the third-order finite-time controller has been derived for the triple integrator subsystem. The results of simulations have validated the efficiency of the proposed control method.

REFERENCES


