Global exponential stability of delayed parabolic neural networks

Li Sheng ∗. Huizhong Yang ∗

∗School of Communication and Control Engineering, Jiaofan University, 1800 Lihu Rd., Wuxi, Jiangsu 214122, P.R. China (victory8209@yahoo.com.cn, yhz@jiangnan.edu.cn).

Abstract: The globally exponentially stable conditions for delayed parabolic neural networks with variable coefficients in this paper. We first derive the globally exponentially stable condition for delayed parabolic neural networks with variable coefficients based on delay differential inequality combining with Young inequality. Compared with the method of Lyapunov functionals as in most previous studies, our method is simpler and more effective for stability analysis.

1. INTRODUCTION

Neural networks have many applications in pattern recognition, image processing, association, etc. Some of these applications require that the equilibrium points of the designed network be stable (Zhang et al., 2005). Therefore, it is vital to study the stability of neural networks. In biological and artificial neural networks, time delays often arise in the process of information storage and transmission. In recent years, the stability of delayed neural networks (DNN) have been investigated by many researchers (Joy M. 1999; Liao and Wang 1999; Arik 2000; Cao 2001; Cao and Wang 2003; Cui and Lou 2006; Lou and Cui 2006).

On the other hand, parabolic (and hyperbolic) evolution equations describe processes that are evolving in time. For such an equation the initial state of the system is part of the auxiliary data for a well-posed problem. We also notice that parabolic equations play a special role in the mathematical modelling of polymerization-type chemical reaction phenomena (aggregation and fragmentation of clusters), in atmosphere physics, biology, and immunology. Recently, the existence, stability and oscillation of such systems have been widely studied (Brzychczy 2002; Leiva and Sequer 2003; Li and Cui 2001; Minchev and Yoshida 2003; Wang and Teo 2005).

Furthermore, real neural networks are more likely to be time-varying evolving networks, namely, the topology is changing with the time. In this paper, we further extend the parabolic models to describe the varying topology neural networks. Using the Green's formula and boundary condition, we can easily deal with the parabolic terms. To the best of our knowledge, this is the first time to introduce and study delayed parabolic neural networks with variable coefficients. The main purposes of this paper are firstly to present the model of delayed parabolic neural networks with variable coefficients; and secondly to discuss the stability of delayed parabolic neural networks by using delay differential inequality combining with Young inequality. One criterion is given to guarantee the global exponential stability for delayed parabolic neural networks with variable coefficients.

2. SYSTEM DESCRIPTION AND PRELIMINARY

In this paper, we obtain some conditions for delayed parabolic neural networks with variable coefficients of the form

\[
\frac{\partial u_i(x, t)}{\partial t} = a_i(t) \Delta u_i(x, t) + \sum_{k=1}^{s} b_{ik}(t) \Delta u_i(x, t - \rho_k(t)) - c_i(t) u_i(x, t) + \sum_{j=1}^{n} w_{ij}(t) f_j(u_j(x, t)) + \sum_{j=1}^{n} h_{ij}(t) f_j(u_j(x, t - \tau_j(t))) + I_i,
\]

for \( i = 1, 2, \ldots, n \), where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) with a piecewise smooth boundary \( \partial \Omega \), \( \Delta \) is the Laplacian in \( \mathbb{R}^n \) and

\[
\Delta u_i(x, t) = \sum_{r=1}^{m} \frac{\partial^2 u_i(x, t)}{\partial x_r^2},
\]

where \( u_i(x, t) \) is the state of the \( i \)th unit at time \( t \). \( f_j(\cdot) \) is the signal functions of the \( j \)th neurons at time \( t \) and in space \( x \). \( \sigma(t) \) is the rate at which the \( i \)th neuron will reset its potential to the resting state in isolation when disconnected from the network and external inputs at time \( t \). \( \tau_j(t) \) are time-varying delays of the neural network satisfying \( 0 \leq \tau_j(t) \leq \sigma \) (\( \sigma \) is a constant).

We assume throughout this paper that

(H1) \( a_i(t), b_{ik}(t) \in C([0, \infty); [0, \infty]), k = 1, 2, \ldots, s; \)

(H2) \( \rho_k(t) \in C([0, \infty); [0, \infty]), \lim_{t \to \infty} (t - \rho_k(t)) = \infty, k = 1, 2, \ldots, s; \)

(H3) The neurons activation functions \( f_j(\cdot) \) are bounded and Lipschitz-continuous, that is, there exist constants \( L_i > 0 \) such that
\[|f_i(\xi_1) - f_i(\xi_2)| \leq L_i|\xi_1 - \xi_2|\]
for all \(\xi_1, \xi_2 \in \mathbb{R}\).

Consider the following boundary condition:
\[
\frac{\partial u(x,t)}{\partial N} = ... (v_j(t)) + \sum_{j=1}^{n} h_{ij}(t) g_j((v_j(t - \tau_j(t)))). \quad (14)
\]

Now we define a function
\[
V(t) = \sum_{i=1}^{n} \lambda_i |v_i(t)|^p. \quad (15)
\]

Theorem 1. Suppose (H1)-(H3) hold. If there exist real constants \(\zeta_{ij}, \eta_{ij}\) and positive constants \(\lambda_i > 0, p \geq 1, i = 1, 2, \cdots, n\) such that
\[
\min_{1 \leq i \leq n} \left\{ p \alpha_i(t) - \sum_{j=1}^{n} \left( \frac{\lambda_j}{\lambda_i} \right) L_j |w_{ji}(t)|^{p(1-\zeta_{ij})} + (p-1) L_j |h_{ij}(t)|^{\frac{p}{\eta_{ij}}} \right\} > \max_{1 \leq i \leq n} \left\{ \frac{\lambda_j}{\lambda_i} L_j \frac{1}{p} |h_{ij}(t)|^{p(1-\eta_{ij})} \right\}, \quad (9)
\]
then the equilibrium point \(u^*\) of system (1) is globally exponentially stable.

Proof. Integrating (1) with respect to \(x\) over the domain \(\Omega\), we have
\[
\frac{\partial}{\partial t} \left[ \int_{\Omega} u_i(x,t) dx \right]
= a_i(t) \int_{\Omega} \Delta u_i(x,t) dx + \sum_{k=1}^{n} h_{ik}(t) \int_{\Omega} f_i(u_i(x,t)) dx
- c_i(t) \int_{\Omega} u_i(x,t) dx + \sum_{j=1}^{n} w_{ij}(t) \int_{\Omega} f_j(u_j(x,t)) dx
+ \sum_{j=1}^{n} h_{ij}(t) \int_{\Omega} f_j(u_j(x,t - \tau_j(t))) dx + I_i. \quad (10)
\]
From Green’s formula and boundary condition (2), it follows that
\[
\int_{\Omega} \Delta u_i(x,t) dx = \int_{\partial \Omega} \frac{\partial u_i(x,t)}{\partial N} dS = 0, \quad t \geq 0 \quad (11)
\]
and
\[
\int_{\Omega} u_i(x,t - \rho_k(t)) dx = \int_{\partial \Omega} \frac{\partial u_i(x,t - \rho_k(t))}{\partial N} dS = 0,
\]
t \geq 0, k = 1, 2, \cdots, s, \quad (12)
where \(dS\) is the surface element on \(\partial \Omega\).
Combining (10)-(12), we have
\[
\frac{\partial}{\partial t} \left[ \int_{\Omega} u_i(x,t) dx \right]
= -c_i(t) \int_{\Omega} u_i(x,t) dx + \sum_{j=1}^{n} w_{ij}(t) \int_{\Omega} f_j(u_j(x,t)) dx
+ \sum_{j=1}^{n} h_{ij}(t) \int_{\partial \Omega} f_j(u_j(x,t - \tau_j(t))) dx + I_i. \quad (13)
\]
Let \(v_i(t) = \int_{\Omega} (u_i(x,t) - u_i^*) dx\), it follows from (13) that
\[
\frac{dv_i(t)}{dt} = -c_i(t)v_i(t) + \sum_{j=1}^{n} w_{ij}(t)g_j(v_j(t))
+ \sum_{j=1}^{n} h_{ij}(t)g_j(v_j(t - \tau_j(t))). \quad (14)
\]
Now we define a function
\[
V(t) = \sum_{i=1}^{n} \lambda_i |v_i(t)|^p. \quad (15)
\]
Calculating and estimating the upper right derivative $D^+V$ of $V$ along the solution of (14) as follows:

$$D^+V(t) = \sum_{i=1}^{n} \lambda_i p |v_i(t)|^{p-1} \text{sign}(v_i(t)) \dot{v}_i(t)$$

$$= \sum_{i=1}^{n} \lambda_i p |v_i(t)|^{p-1} \text{sign}(v_i(t)) \left[ -c_i(t) v_i(t) + \sum_{j=1}^{n} w_{ij}(t) g_j(v_j(t)) + \sum_{j=1}^{n} h_{ij}(t) g_j(v_j - \tau_j(t)) \right]$$

$$\leq \sum_{i=1}^{n} \lambda_i p \left[ -c_i(t) |v_i(t)|^p + \sum_{j=1}^{n} |w_{ij}(t)| L_j |v_j(t)|^{p-1} |v_j(t)| + \sum_{j=1}^{n} |h_{ij}(t)| L_j |v_j(t)|^{p-1} |v_j(t) - \tau_j(t)| \right]$$

$$= \sum_{i=1}^{n} \lambda_i p \left[ -c_i(t) |v_i(t)|^p + \sum_{j=1}^{n} L_j \left( |w_{ij}(t)|^{1-\xi_i} |v_j(t)| \right) \left( |w_{ij}(t)|^{\frac{\xi_i}{p}} |v_i(t)| \right)^{p-1} \times \left( |v_j(t)|^{\frac{\xi_i}{p}} |v_i(t)| \right)^{p-1} \right.$$  

$$\left. + \sum_{j=1}^{n} L_j \left( |h_{ij}(t)|^{1-\eta_i} |v_j(t) - \tau_j(t)| \right) \left( |h_{ij}(t)|^{\frac{\eta_i}{p}} |v_i(t)| \right)^{p-1} \times \left( |v_j(t) - \tau_j(t)|^{\frac{\eta_i}{p}} |v_i(t)| \right)^{p-1} \right].$$  

(16)

Let $a = |w_{ij}(t)|^{1-\xi_i} |v_j(t)|$, $b = \left(|w_{ij}(t)|^{\frac{\xi_i}{p}} |v_i(t)| \right)^{p-1}$, by Lemma 1, we have

$$\left( |w_{ij}(t)|^{1-\xi_i} |v_j(t)| \right) \left( |w_{ij}(t)|^{\frac{\xi_i}{p}} |v_i(t)| \right)^{p-1} \leq \frac{1}{p} |w_{ij}(t)|^{p(1-\xi_i)} |v_j(t)|^p$$

$$+ \frac{p-1}{p} |w_{ij}(t)|^{\frac{p\eta_i}{p}} |v_i(t)|^p.$$  

(17)

Similarly, let $a = |h_{ij}(t)|^{1-\eta_i} |v_j(t) - \tau_j(t)|$, $b = \left(|h_{ij}(t)|^{\frac{\eta_i}{p}} |v_i(t)| \right)^{p-1}$, by Lemma 1, we get

$$\left( |h_{ij}(t)|^{1-\eta_i} |v_j(t) - \tau_j(t)| \right) \left( |h_{ij}(t)|^{\frac{\eta_i}{p}} |v_i(t)| \right)^{p-1} \leq \frac{1}{p} |h_{ij}(t)|^{p(1-\eta_i)} |v_j(t) - \tau_j(t)|^p$$

$$+ \frac{p-1}{p} |h_{ij}(t)|^{\frac{p\eta_i}{p}} |v_i(t)|^p.$$  

(18)

Substituting (17) and (18) into (16), we obtain

$$D^+V(t) \leq \sum_{i=1}^{n} \lambda_i p \left[ -c_i(t) |v_i(t)|^p + \sum_{j=1}^{n} L_j \frac{1}{p} |w_{ij}(t)|^{p(1-\xi_i)} |v_j(t)|^p + \sum_{j=1}^{n} L_j \frac{1}{p} |h_{ij}(t)|^{p(1-\eta_i)} |v_j(t) - \tau_j(t)|^p \right]$$

$$= \sum_{i=1}^{n} \lambda_i \left[ pc_i(t) + \sum_{j=1}^{n} \frac{\lambda_j}{\lambda_i} L_j |w_{ij}(t)|^{p(1-\xi_i)} \right]$$

$$+ \sum_{j=1}^{n} (p-1) L_j |w_{ij}(t)|^{\frac{p\eta_i}{p}} |v_i(t)|^p$$

$$+ \sum_{j=1}^{n} (p-1) L_j |h_{ij}(t)|^{\frac{p\eta_i}{p}} \right] V(t)$$

$$+ \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^{n} \frac{\lambda_j}{\lambda_i} L_j \frac{1}{p} |h_{ij}(t)|^{p(1-\eta_i)} \right\} V(t).$$  

(19)

Applying Lemma 2, then it follows (9) and (15) that

$$\lambda_{\min} \int_{\Omega} |u(x, t) - u^*|^p dx \leq V(t) \leq V(t_0) e^{-\varepsilon(t-t_0)}. \quad (20)$$

So, we have

$$\int_{\Omega} |u(x, t) - u^*|^p dx \leq \frac{\lambda_{\max}}{\lambda_{\min}^{\frac{1}{p}}} e^{-\frac{\varepsilon}{p} t} \|\phi(x, t) - u^*\|^p. \quad (21)$$

Therefore, the proof is completed.

Corollary 1. Suppose (H1)-(H3) hold. If there exist constants $\lambda_i > 0$ such that

$$\min_{1 \leq i \leq n} \left\{ c_i(t) - \sum_{j=1}^{n} \left( \frac{\lambda_j}{\lambda_i} L_j |w_{ij}(t)| \right) \right\}$$

$$> \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^{n} \frac{\lambda_j}{\lambda_i} L_j |h_{ij}(t)| \right\}, \quad (22)$$

6047
then the equilibrium point \( u^* \) of system (1) is globally exponentially stable.

**Proof.** Taking \( p = 1, \zeta_{ij} = \eta_{ij} = 0 \) in theorem 1 above, then we can easily obtain Corollary 1.

**Corollary 2.** Suppose (H1)-(H3) hold. If there exist constants \( \lambda_i > 0 \) such that

\[
\min_{1 \leq i \leq n} \left\{ 2c_i(t) - \sum_{j=1}^{n} \left( \frac{L_i}{\lambda_i} |h_{ji}(t)| \right) + L_j (|w_{ij}(t)| + |h_{ij}(t)|) \right\} > \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^{n} \frac{\lambda_j}{\lambda_i} L_i \frac{1}{p} |h_{ji}(t)| p(1-\eta_{ji}) \right\},
\]

then the equilibrium point \( u^* \) of system (1) is globally exponentially stable.

**Proof.** It is easy to check that the inequality (8) is satisfied by taking \( p = 2, \zeta_{ij} = \eta_{ij} = 0.5 \) and hence the theorem 1 implies Corollary 2.

**Remark 1.** If the parameters \( \lambda_i, p, \zeta_{ij}, \eta_{ij} \) are properly chosen, we can easily obtain a series of corollaries.

**Remark 2.** In many papers, the delay function \( \tau_j(t) \) is needed to be differentiable, for example (Joy M. 1999; Cao and Wang 2003). However, the restriction is neglected. Moreover, we introduce the parabolic models to describe the delayed neural networks. Thus, the conditions given in this paper are less restrictive, general and conservative.

### 4. Numerical Examples

In this section, we will give two numerical examples to show the validity of our results.

**Example 1.** Consider a delayed parabolic neural network with variable coefficients

\[
\begin{aligned}
\frac{\partial u_i(x, t)}{\partial t} &= a_i(t) \Delta u_i(x, t) + \sum_{k=1}^{s} b_{ik}(t) \Delta u_i(x, t - \rho_k(t)) \\
&- c_i(t) u_i(x, t) + \sum_{j=1}^{n} w_{ij}(t) f_j(u_j(x, t)) \\
&+ \sum_{j=1}^{n} h_{ij}(t) f_j(u_j(x, t - \tau_j(t))) + I_i,
\end{aligned}
\]

\((x, t) \in (0, \pi) \times [0, \infty), \quad (24)\)

with boundary condition

\[
\frac{\partial u_i(0, t)}{\partial x} = \frac{\partial u_i(\pi, t)}{\partial x} = 0, \quad t \geq 0, \quad i = 1, 2.
\]

The system parameters are chosen as follows:

\[
\begin{aligned}
a_1(t) &= a_2(t) = 1, \\
b_{ik}(t) &= \epsilon^t \quad (i = 1, 2; \ k = 1, 2), \\
\rho_1(t) &= \frac{\pi}{2}, \quad \rho_2(t) = \frac{3\pi}{2}, \\
c_1(t) &= c_2(t) = 1, \quad I_1 = 1, \quad I_2 = 2, \\
W &= (w_{ij})_{n \times n} = \begin{bmatrix} 0.1 & -0.1 \\ 0.1 & 0.1 \end{bmatrix}.
\end{aligned}
\]

The activation function is \( f_j(y) = \begin{cases} y & \text{if } |y| < 1 \\ 1 & \text{otherwise} \end{cases} \). Clearly, \( f_j \) satisfies the assumption (H3) with \( L_1 = L_2 = 1 \). Furthermore, let \( \lambda_1 = \lambda_2 = 1 \), then one can easily check that

\[
\min_{1 \leq i \leq n} \left\{ 2c_i(t) - \sum_{j=1}^{n} \left( \frac{L_i}{\lambda_i} |h_{ji}(t)| \right) \right\} = 0.8
\]

\[
> \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^{n} \frac{\lambda_j}{\lambda_i} L_i |h_{ji}(t)| \right\} = 0.5. \quad (25)
\]

Therefore, by Corollary 1, the equilibrium point of (24) is globally exponentially stable. By a simple computation, we can easily seen that the matrix \( C + C^T = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \) is not negative semidefinite. Thus the condition in [Li et al., 2003] does not hold. For this example, our result is less restrictive than that given in (Li et al., 2003).

**Example 2.** If we do not consider the parabolic term, the system (24) reduce to a normal delayed neural network as follows

\[
\begin{aligned}
\frac{du_i(t)}{dt} &= -c_i(t)u_i(t) + \sum_{j=1}^{n} w_{ij}(t) f_j(u_j(t)) \\
&+ \sum_{j=1}^{n} h_{ij}(t) f_j(u_j(t - \tau_j(t))) + I_i,
\end{aligned}
\]

\( (26) \)

We choose the same parameters as (24) and let \( \tau(t) = 0.5 \). Then, by Corollary 1, the system (26) is exponentially stable. Fig. 1 indicates that \([x_1(t), x_2(t)]^T\) converge to \([1.3000, 2.7000]^T\) with the initial values \([-1.0, -3.0]^T\).

### 5. Conclusions

The global exponential stability of parabolic neural networks with variable coefficients and time-varying delays has been studied. Some stability criteria, which are independent of the delay parameter, have been derived by employing delay differential inequality and Young inequality. The conditions given in this paper are less restrictive,
general and conservative. Moreover, the approaches presented in this paper can be applied to some other neural networks, such as neural networks with reaction-diffusion terms, robust neural networks, and Cohen-Grossberg neural networks.

REFERENCES


