A general adaptive robust nonlinear motion controller combined with disturbance observer

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Abstract: A general adaptive robust nonlinear motion controller combined with disturbance observer (DOB) for positioning control of a nonlinear single-input-single-output (SISO) mechanical system is proposed. Theoretical performance such as transient performance, ultimate tracking error bound and mean square tracking error bound are analyzed rigorously, and simulation results are provided to support the theoretical results.

Keywords: Disturbance observer; Input-to-state stability; Robust control; Adaptive control.

1. INTRODUCTION

In this report, we propose a general adaptive robust nonlinear motion controller combined with DOB. With the help of nonlinear damping terms, the input-to-state stability (ISS) property [3] of the overall nonlinear control system is ensured. Basically, the boundedness of the internal signals are ensured by the robustifying nonlinear damping terms. The DOB is employed to compensate the lumped uncertainties without the necessity of parameterization. The adaptive laws are employed to furthermore account for fast-changing uncertainties which the DOB cannot handle sufficiently.

Our major contribution is to incorporate the DOB technique and the adaptive technique which have been considered as two contrastively different approaches in the literature, into one controller under the framework of ISS property. Moreover, transient performance, ultimate tracking error bound and mean square tracking error bound are analyzed rigorously, with transparent physical meaning. Finally, simulation results are provided to support the theoretical results.

2. STATEMENT OF THE PROBLEM

Consider the following SISO nonlinear mechanical system.

\[ \begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= F(x) + d(x, t) + G(x)u
\end{align*} \]  

(1a) 

(1b)

where, \( x = [x_1, x_2]^T \), \( x_1 \) and \( x_2 \) are the position and velocity respectively, \( u \) is the control input; \( G(x) \) and \( F(x) \) are the modelable nonlinear functions; and \( d(x, t) \) is the completely unknown term composed of unmodelled nonlinearities and disturbances.

Denoting the nominal nonlinearities based on prior knowledge as \( F_0(x) \) and \( G_0(x) \), we can model \( F(x) \) and \( G(x) \) as

\[
F(x) = F_0(x) + \Delta_F(x), \quad G(x) = G_0(x) + \Delta_G(x) \tag{2}
\]

where the modelling errors \( \Delta_F(x) \) and \( \Delta_G(x) \) can be approximated by the linear-in-the-parameters networks. Then we have

\[
\begin{align*}
\hat{F}(x, w_F) &= F_0(x) + \Delta_F(x, w_F) \\
\hat{G}(x, w_G) &= G_0(x) + \Delta_G(x, w_G)
\end{align*}
\]

(3)

where

\[
\Delta_F(x, w_F) = \phi_F^T(x)w_F, \quad \Delta_G(x, w_G) = \phi_G^T(x)w_G
\]  

(4)

The regressor vectors \( \phi_F(x) \) and \( \phi_G(x) \) are defined as

\[
\begin{align*}
\phi_F(x) &= [\phi_{F1}(x), \ldots, \phi_{FN_F}(x)]^T \\
\phi_G(x) &= [\phi_{G1}(x), \ldots, \phi_{GN_G}(x)]^T \\
w_F &= [w_{F1}, \ldots, w_{FN_F}]^T \\
w_G &= [w_{G1}, \ldots, w_{GN_G}]^T
\end{align*}
\]  

(5)

(2)~(5) lead to the following relations.

\[
\begin{align*}
F(x) &= \hat{F}(x, w_{\hat{F}_1}) + \Delta_F(x, \hat{w}_{\hat{F}_1}) \\
&= \hat{F}(x, w_{\hat{F}_1}) + \eta_F(x, w_{\hat{F}_1}) \\
&= \hat{F}(x, w_{\hat{F}_1}) + \eta_F(x, w_{\hat{F}_1}) - \phi_F^T(x)w_{\hat{F}_1}
\end{align*}
\]

(6a)

\[
\begin{align*}
G(x) &= \hat{G}(x, w_{\hat{G}_1}) + \Delta_G(x, \hat{w}_{\hat{G}_1}) \\
&= \hat{G}(x, w_{\hat{G}_1}) + \eta_G(x, w_{\hat{G}_1}) \\
&= \hat{G}(x, w_{\hat{G}_1}) + \eta_G(x, w_{\hat{G}_1}) - \phi_G^T(x)w_{\hat{G}_1}
\end{align*}
\]

(6b)
where
\[
\eta_{F}(x, w_F) = \Delta_{F}(x) - \tilde{\Delta}_{F}(x, w_F)
\]
\[
\eta_{G}(x, w_G) = \Delta_{G}(x) - \tilde{\Delta}_{G}(x, w_G)
\]
\[
w_{F}^{*} = \arg\min_{w_{F}} \left\{ \sup_{x \in \Omega_{x}} \left| \eta_{F}(x, w_{F}) \right| \right\}
\]
\[
w_{G}^{*} = \arg\min_{w_{G}} \left\{ \sup_{x \in \Omega_{x}} \left| \eta_{G}(x, w_{G}) \right| \right\}
\]
\[
\hat{w}_{Ft} = w_{Ft} - w_{F}^{*}, \quad \hat{w}_{Gt} = w_{Gt} - w_{G}^{*}
\]

We impose the following standing assumptions for \(x\) on the desired domain of operation \(\Omega_{x}\).

**Assumption 1.** The lower and upper bounds of the parameter vectors are known a priori:

\[
w_{F} \leq w_{F}^{*} \leq \bar{w}_{F}, \quad w_{G} \leq w_{G}^{*} \leq \bar{w}_{G}
\]

**Assumption 2.** \(G(x) > 0\) and the parameter bounds satisfy

\[
\tilde{G}(x, w_G) = G_{0}(x) + \tilde{\Delta}_{G}(x, w_G) > 0
\]

where \(\bar{w}_{G} \leq \hat{w}_{Gt} \leq \bar{w}_{G}\).

**Assumption 3.** The adaptively updated \(\tilde{G}(x, w_{Gt})\) satisfies

\[
\frac{\tilde{G}(x, w_{Gt})}{G(x)} \leq 3M_{G} < \infty, \quad \frac{|\tilde{\eta}_{G}(x, w_{Gt})|}{G(x)} \leq 3M_{G} < \infty
\]

**Assumption 4.** There exist known continuous bounding functions \(\bar{F}(x) > 0\) and \(\bar{d}(x, t) > 0\) such that

\[
\frac{|\hat{\eta}_{F}(x, w_{Ft})|}{\bar{F}(x)} \leq 3M_{F} < \infty, \quad \frac{|d(x, t)|}{\bar{d}(x, t)} \leq 3M_{d} < \infty
\]

Assumption 5. The reference trajectory \(y_{r}(t)\) is appropriately chosen as a sufficiently smooth function such that \(\dot{y}_{r}\) and \(\ddot{y}_{r}\) are known and

\[
\mathcal{D}_{y_r} = \{y_r, \ddot{y}_r, \dot{y}_r \mid \{y_r, \dot{y}_r\}^{T} \in \Omega_{y} \subset \Omega_{x}, |\ddot{y}_r| \leq 3M_{r} < \infty\}
\]

3. CONTROLLER DESIGN

The proposed controller is designed in a backstepping procedure, composed of two steps.

**Step 1:** Define the position error and velocity error signals respectively as

\[
z_{1} = x_{1} - y_{r}, \quad z_{2} = x_{2} - \alpha_{1}
\]

where \(\alpha_{1}\) is the virtual input to stabilize \(z_{1}\).

Then from (1a) we have subsystem \(S1:\)

\[
S1: \quad \dot{z}_{1} = \alpha_{1} + z_{2} - \ddot{y}_{r}
\]

The virtual input \(\alpha_{1}\) is designed based on the common PI control technique:

\[
\alpha_{1} = -c_{1p} z_{1} - c_{1i} \int_{0}^{t} z_{1} dt + \ddot{y}_{r}
\]

where \(c_{1p}, c_{1i} > 0\).

The next step is to stabilize the velocity error \(z_{2}\).

**Step 2:** From (1b), (6) and (15) we have subsystem \(S2:\)

\[
S2: \quad \dot{z}_{2} = \tilde{F}(x, w_{Ft}) - \alpha_{1} + \tilde{G}(x, w_{Gt}) u + d(x, t) + \eta_{F}(x, w_{Ft}) - \bar{\phi}_{F}^{T}(x) w_{Ft} + \eta_{G}(x, w_{Gt}) + \bar{\phi}_{G}^{T}(x) w_{Gt} u - \bar{\phi}_{F}^{T}(x) w_{Ft} - \bar{\phi}_{G}^{T}(x) w_{Gt} u
\]

Denoting the uncertain terms as the lumped disturbance \(w\), we have

\[
\dot{w} = \dot{z}_{2} - \tilde{F}(x, w_{Ft}) + \tilde{G}(x, w_{Gt}) u - \alpha_{1}
\]

\[
= d(x, t) + \eta_{F}(x, w_{Ft}) + \eta_{G}(x, w_{Gt}) + \bar{\phi}_{F}^{T}(x) w_{Ft} + \bar{\phi}_{G}^{T}(x) w_{Gt} u
\]

Therefore, the lumped disturbance \(w\) can be obtained as \(w_{2}\).

\[
Q(s) = \frac{1}{1 + \lambda_{s}}, \quad \tilde{Q}(s) = 1 - Q(s) = \frac{\lambda_{s}}{1 + \lambda_{s}}
\]

This is the so called DOB in the literature [1, 2, 4]. The benefit of compensating the control input by \(\tilde{w}\) is obvious. Replacing \(u\) in (18) by \(u = (v - \bar{F}(x, w_{Ft}) + \tilde{\alpha}_{1} - \tilde{G})(x, w_{Gt})\) and assuming \(\tilde{w} \approx w\), we have

\[
\dot{z}_{2} \approx v
\]

where \(v\) is a nominal linear input. A simple controller can therefore be designed. The simplest design is, for example, to let \(v = -c_{2}z_{2}\).

However, actually we can only expect \(\tilde{w} \approx w\) at low-frequencies. If the disturbance and model mismatch are fast-changing, the estimation error \(\tilde{w} - \tilde{G}\) cannot be neglected and can destroy the stability of the closed-loop in the case of large model mismatch [4].

To stabilize the subsystem \(S2\), we design the following controller.

\[
u_{u} = \frac{\alpha_{20}}{5} u + u_{w} + u_{d1} + u_{d2} + u_{d3} + u_{d4} + u_{d5}
\]

\[
u_{e} = -\frac{\alpha_{24}}{G(x, w_{Gt})} e_{G} + e_{F}
\]

where

\[
\alpha_{20} = -c_{2}z_{2} + \tilde{\alpha}_{1} + \tilde{\bar{F}}(x, \bar{w}_{Ft})
\]

\[
u_{d1} = \kappa_{21} \tilde{F}(x)
\]

\[
u_{d2} = \kappa_{22} \tilde{\alpha}_{2d}
\]

\[
u_{d3} = \kappa_{23} \tilde{\alpha}_{2d}(x, t)
\]

\[
u_{d4} = \kappa_{24} \tilde{\bar{G}}(x, t)
\]

\[
u_{d5} = \kappa_{25} e_{G} \tilde{\alpha}_{2d}
\]

and \(c_{2}, \kappa_{21}, \kappa_{22}, \kappa_{23}, \kappa_{24}, \kappa_{25} > 0\).

\(u_{d1}z_{2}\) is a feedback controller with model compensation. \(u_{w}\) is a compensating term by the DOB's output. \(u_{d1}z_{2}\) and \(u_{d3}z_{2}\) are nonlinear damping terms to counteract
\[ \Delta F(x) - \hat{\Delta F}(x, \hat{w}_F t), \Delta G(x) - \hat{\Delta G}(x, \hat{w}_G t) \text{ and } d(x, t) \]
respectively. \( u_{d4} = \{i = 1, \cdots, 3\} \) are designed as time-varying control gains so that they grow at least as the same order as the corresponding uncertain terms to be counteracted.

\[ u_{d4} \] is a nonlinear damping term to ensure boundedness of \( z_2 \) when \( \hat{w} \) is used. As will be seen in (25) and (26), \( u_e \) is introduced to compensate the terms \( e_G \) and \( e_F \) defined in (26). Notice that \( e_G \) and \( e_F \) stem from the fact that adaptive laws are not applicable directly to the terms \( Q(s) [\phi_F^T(x)] \hat{w}_G t \) and \( Q(s) [\phi_F^T(x)] \hat{w}_F t \), but are applicable to the terms \( \{Q(s) \phi_F^T(x)\} \hat{w}_G t \) and \( \{Q(s) \phi_F^T(x)\} \hat{w}_F t \).

Finally, \( u_{d5} z_2 \) is a nonlinear damping term to ensure boundedness of \( z_2 \) when \( u_e \) is used. The roles of the nonlinear damping terms will be shown later through stability analysis. See (35) and (36).

Applying \( u \) to \( S2 \), we have
\begin{equation}
\dot{z}_2 = -c_2 z_2 + G(x, \hat{w}_G t) u_t + \eta_F(x, \hat{w}_F t) - \phi_F^T(x) \hat{w}_F t + d(x, t) + \eta_G(x, \hat{w}_G t) u - \phi_G^T(x) \hat{w}_G t u_t
\end{equation}
\begin{equation}
= -c_2 z_2 + G(x, \hat{w}_G t) u_t + \eta_G(x, \hat{w}_G t) u - \phi_F^T(x) \hat{w}_F t + d(x, t) + \eta_G(x, \hat{w}_G t) u - \phi_G^T(x) \hat{w}_G t u_t
\end{equation}

\begin{equation}
e_F = \{Q(s) \phi_F^T(x)\} \hat{w}_F t - \{Q(s) \phi_F^T(x)\} \hat{w}_G t
\end{equation}
\begin{equation}e_G = \{Q(s) \phi_G^T(x) u\} \hat{w}_G t - \{Q(s) \phi_G^T(x) u\} \hat{w}_G t
\end{equation}

To let the adaptive parameters stay in the prescribed range we adopt the following adaptive laws with projection.

\begin{align}
\dot{w}_F \text{ or } w_G = \begin{cases}
0 & \text{for } w_F \text{ or } w_G < 0 \\
\gamma_F [Q(s) \phi_F(x) u] & \text{for } w_F \text{ or } w_G \geq 0.
\end{cases}
\end{align}

where \( n = 1, \cdots, N_F, \gamma_F \geq 0 \).

\begin{align}
\dot{w}_G \text{ or } w_G = \begin{cases}
0 & \text{for } w_G \geq 0 \\
\gamma_G [Q(s) \phi_G(x) u] & \text{for } w_G < 0.
\end{cases}
\end{align}

where \( n = 1, \cdots, N_G, \gamma_G \geq 0 \).

It can be verified that the adaptive laws satisfy
\begin{align}
\hat{w}_F \leq \hat{w}_F t \leq \hat{w}_F, \quad \hat{w}_G \leq \hat{w}_G t \leq \hat{w}_G
\end{align}

Remark 1. Inspection of (27) and (28) reveals interesting physical features of the adaptive laws. Since the complementary filter \( Q(s) \) is high-pass, in the case of slowly-changing signals, the amplitudes of \( \{Q(s) \phi_F(x)\}^n \) and \( \{Q(s) \phi_G(x) u\} \) are trivial so that the adaptive laws become to be "lazy". Contrastively, the DOB in (20) is more efficient for low-frequency uncertainties.

4. STABILITY ANALYSIS

4.1 ISS property analysis of each subsystem

Applying \( \alpha_1 \) to the subsystem \( S1 \), we have
\begin{equation}
\dot{z}_1 = z_1 - c_1 p_1 z_1 - c_1 \int_0^t z_1 dt
\end{equation}

Denote the Laplace operator as \( s \). Then the subsystem \( S1 \) can be expressed as
\begin{equation}
\dot{z}_1 = \frac{s z_2}{s^2 + c_1 p_1 s + c_1}
\end{equation}

This can be rewritten into the state-space form:
\begin{equation}
z_{1a} = A z_{1a} + B z_2
\end{equation}

where \( z_{1a} = [\int_0^t z_1 dt, z_1]^T \),
\begin{equation}
A = \begin{bmatrix} 0 & 1 \\ -c_1 & -c_1 \end{bmatrix}, \quad B = [0 1]^T
\end{equation}

The ISS property of the subsystem \( S1 \) is described in the following lemma.

Lemma 1. If the virtual input \( \alpha_1 \) is applied to the subsystem \( S1 \), and if \( z_2 \) is made uniformly bounded at the next step, then the subsystem \( S1 \) is ISS, i.e., for \( \lambda_0, \lambda_0 > 0 \),
\begin{equation}
|z_{1a}| \leq \lambda_0 \|z_{1a}(0)\| + \lambda_0 \sup_{t \leq \tau \leq t} |z_2|)
\end{equation}

To establish the ISS property of the subsystem \( S2 \), we first rewrite (25):
\begin{equation}
\dot{z}_2 = -c_2 z_2 + G(x, \hat{w}_G t) u_t + \eta_G(x, \hat{w}_G t) u - \phi_F^T(x) \hat{w}_F t + d(x, t) + \eta_F(x, \hat{w}_F t) + \eta_G(x, \hat{w}_G t) u_t
\end{equation}

Then we have
\begin{equation}
d \frac{d}{dt} \left[ \frac{z_2}{2} \right] = -c_2 z_2 + D_2 z_2 + d z_2
\end{equation}

where
\begin{equation}
\mu_2(t) = \frac{d_2}{\tau_2 + D_2}
\end{equation}

According to Assumptions 1–5, it is obvious that each term in the numerator of \( \mu_2 \) is counteracted by a nonlinear damping term in the denominator which grows at least as the same order as the corresponding term in the numerator, so that \( \mu_2 \) is uniformly bounded. Furthermore, from (35) we have
\begin{equation}
|z_2| \geq \mu_2(t) \Rightarrow \frac{d}{dt} \left[ \frac{z_2}{2} \right] \leq -c_2 z_2^2
\end{equation}
and hence
\[ |z_2(t)| \leq |z_2(0)|e^{-c_2t/2} + \sup_{0 \leq \tau \leq t} \mu_2(\tau) \] (39)

Therefore the uniform boundedness of \( z_2 \) can be ensured by the nonlinear damping terms. Thus, Lemma 1 holds, which implies \( |z_{1a}| \) is bounded. Since the reference trajectory \( y_r, \dot{y}_r, \text{ and } \ddot{y}_r \) are uniformly bounded (Assumption 5), we, therefore, conclude that all the internal signals of the two subsystems are uniformly bounded.

In the above analysis, the main attention is to show the boundedness of the internal signals of the closed-loop. No analysis yet has been done for the attenuation effects of \( w - \hat{w} \). Without such an analysis, we cannot clearly see how the DOB’s output \( \hat{w} \) can bring an improvement. We now attempt to make such an effort.

Then, keeping that all the internal signals are bounded in mind, we rewrite (35) by using (34):
\[ \frac{d}{dt} \left( \frac{z_2^2}{2} \right) = -\frac{c_2}{2}z_2^2 - \left( c_2 + D_{2w} \right) z_2^2 + w z_2 - (\hat{w} + e_G + e_F) z_2 \]
\[ \leq -\frac{c_2}{2}z_2^2 - \left( c_2 + D_{2w} \right) z_2^2 |z_2| \] (40)

where
\[ \mu_2(t) = |w - \hat{w}| + |e_G + e_F| = \frac{D_{2w}^2 + 2c_2}{2} \]
\[ D_{2w} = \kappa_2 \bar{P}(x) + \kappa_2 a_2 \alpha_2 + \kappa_2 \bar{d}(x, t) \]
\[ + \kappa_2 \bar{d}(\hat{w} + e_G + e_F) \] (41)

Notice that we can write \( w - \hat{w} \) as:
\[ w - \hat{w} = \bar{Q}(s) \left( d(x, t) + \eta_F(x, \hat{w}_F) + \eta_G(x, \hat{w}_G) \right) u \]
\[ = \frac{\eta_G(x, \hat{w}_G)}{G(x, \hat{w}_G)} \left( -D_{2w} z_2 + \alpha_2 - e_F - e_G \right) \]
\[ + \eta_F(x, \hat{w}_F) + d(x, t) - \frac{G(x)}{G(x, \hat{w}_G)} \hat{w} \] (42)

It should be commented here that \( \mu_2 \) has very transparent physical meaning. At low-frequencies, we can expect \( w - \hat{w} \approx 0 \). And any nonzero \( w - \hat{w} \) at high-frequencies is counteracted by \( c_2 + D_{2w} \) so that \( z_2 \) is quite robust against \( w - \hat{w} \).

**Remark 2.** As mentioned previously, the terms \( e_G \) and \( e_F \) defined in (26) are inevitably due to the time-varying effects of the adaptively updated parameters. In the case of \( \gamma_F = \gamma_G = 0 \), i.e., the adaptive laws are switched off, we have \( e_G = e_F = 0 \).

**Remark 3.** In the case of \( \gamma_F = \gamma_G = 0 \), the controller is reduced to a fixed robust controller with DOB [4], and thus the boundedness of \( \mu_2 \) and \( \mu_{2w} \) still holds under Assumptions 1~5.

**Remark 4.** At the present stage, we are mainly concentrated on the boundedness of the DOB’s output \( u_c \). If we do not adopt the compensation term \( u_c \) in (23), \( e_G + e_F \) does not appear in the numerators of \( \mu_2 \) and \( \mu_{2w} \), and thus the boundedness of \( \mu_2 \) and \( \mu_{2w} \) is still ensured. Empirically, the amplitudes of \( e_G \) and \( e_F \) given in (26) are trivial in most cases since \( \hat{w}_F \) and \( \hat{w}_G \) do not change so fast compared to the high-passed signals, and thus the control performance do not degenerate significantly when \( u_c \) in (23) remains. However, for the convenience to show how the adaptive laws bring improved ultimate bound and mean square bound of \( z_2 \) theoretically, it is recommended to employ \( u_c \) so that the time-varying effects of the adaptively updated parameters are fully compensated. See (25).

**Remark 5.** When the DOB is not used, i.e., \( Q(s) = 0 \), we have \( \hat{w} = 0, e_F = 0, e_G = 0 \), and thus \( u_{d2}, u_{d5}, u_c \) in (23) can be removed. In this case, the boundedness of \( \mu_2 \) and \( \mu_{2w} \) is still ensured, owing to the nonlinear damping terms.

Finally, we have
\[ |z_2| \geq \mu_{2w}(t) = \frac{d}{dt} \left( \frac{z_2^2}{2} \right) \leq -c_2 z_2^2 \] (43)

and hence the result of Lemma 2.

**Lemma 2.** Let Assumptions 1~5 hold. If the control input \( u \) is applied to the subsystem \( S2 \), then the subsystem \( S2 \) is ISS and the error signal \( z_2(t) \) is uniformly bounded as
\[ |z_2(t)| \leq |z_2(0)|e^{-c_2t/2} + \sup_{0 \leq \tau \leq t} \mu_2 (\tau) \]

Notice that lemmas 1 and 2 imply that the internal signals of the two subsystems are bounded, i.e., the boundedness is ensured by the nonlinear damping terms no matter if the adaptive laws are activated.

**4.2 Error bounds achieved by DOB and adaptive laws**

We first consider the subsystem \( S1 \) in (32) and (33). Since there exists a positive definite symmetric matrix \( P \) satisfying \( A^T P + PA = -Q \) for any positive definite symmetric matrix \( Q \), we have
\[ \frac{d}{dt} \left( \frac{z_{1a}^2}{2} \right) = -\frac{1}{2} z_{1a}^T Q z_{1a} + z_{1a}^T P B z_2 \]
\[ \leq -\frac{\lambda_{Q_{min}}}{2} |z_{1a}|^2 + |z_{1a}| |P B| |z_2| \]
\[ \leq -\frac{\lambda_{Q_{min}}}{4} |z_{1a}|^2 + \frac{1}{\lambda_{Q_{min}}} |P B|^2 |z_2|^2 \] (44)

where \( \lambda_{Q_{min}} \) is the minimal eigenvalue of \( Q \). Then we have

**Lemma 3.** If \( a_1 \) is applied to the subsystem \( S1 \), and if \( z_2 \) is made uniformly ultimately bounded with ultimate bound \( \tau_2 \) at the next step, the error signal \( z_{1a}(t) \) is uniformly ultimately bounded such that
\[ |z_{1a}(t)| \leq C_1 \tau_2 \]
and the mean square of \( z_{1a}(t) \) satisfies
\[ \lim_{T \to \infty} \frac{1}{T} \int_0^T |z_{1a}|^2 dt \leq \frac{4 |P B|^2}{\lambda_{Q_{min}}} \lim_{T \to \infty} \frac{1}{T} \int_0^T |z_2|^2 dt \]

Furthermore, to analyze how the adaptive laws help to improve \( |z_2| \), we impose one more assumption:

**Assumption 6.** The networks are sufficiently complex such that the approximation errors are sufficiently small on the desired domain of operation \( \Omega_X \), i.e., there exist \( w_F^* \) and \( w_G^* \) satisfying
\[ \sup_{x \in D_X} |\eta_F(x, w_F^*)| \leq \varepsilon_F \sup_{x \in \Omega_X} |\eta_G(x, w_G^*)| \leq \varepsilon_G \] (45)
Then we define the following Lyapunov function for \( \gamma_F, \gamma_G > 0 \):
\[
V_2 = \frac{z_2^2}{2} + \frac{\overline{w}_F^T \overline{w}_F}{2\gamma_F} + \frac{\overline{w}_G^T \overline{w}_G}{2\gamma_G}
\]  
(46)
By using the results of (25) and (29), we have
\[
\dot{V}_2 \leq -c_F D_{2w} z_2^2 + | \overline{Q}(s) w^* | z_2
\]
\[
= -\left( c_F + D_{2w} \right) z_2^2 + \frac{c_F + D_{2w}}{\gamma_F} | z_2 |^2
\]
\[
+ | \overline{Q}(s) w^* | z_2 | \geq \delta^2_{2m} + \delta^2_{2m}
\]
\[
\leq -c_F D_{2w} | z_2 |^2 + \delta^2_{2m}
\]
\[
\leq -c_F D_{2w} \left( V_2 \right) \left( V_2 - \frac{\gamma_F}{2} \right)
\]
\[
= -c_F D_{2w} \left( V_2 - \frac{\gamma_F}{2} \right)
\]
\[
= -c_F D_{2w} \left( V_2 - \frac{\gamma_F}{2} \right)
\]
\[
(47)
\]
where
\[
M^2_F = (\overline{w}_F - w_F^*)^T (\overline{w}_F - w_F^*) \geq \overline{w}_F^T \overline{w}_F
\]
\[
M^2_G = (\overline{w}_G - w_G^*)^T (\overline{w}_G - w_G^*) \geq \overline{w}_G^T \overline{w}_G
\]
\[
\delta^2_{2m} = \sqrt{2 \lambda_F \gamma_G}
\]
\[
\delta^2_{2u} = \sqrt{\frac{M^2_F}{\gamma_F} + \frac{M^2_G}{\gamma_G} + \frac{| \overline{Q}(s) w^* |^2}{(c_F + D_{2w})^2}}
\]
\[
w^* = d(x, t) + \eta_F (x, w_F^*) + \eta_G (x, w_G^*) u
\]
\[
\leq d(x, t) + \varepsilon_F + \varepsilon_G u
\]
Then we have the following results.

**Lemma 4.** Let Assumption 6 and the assumptions and results of Lemma 2 hold. If the control input \( u \) and the adaptive laws (27) and (28) are applied to the subsystem \( S_2 \), then the following results hold:

1. The adaptive parameters satisfy
\[
\overline{w}_F \leq \overline{w}_F^* \leq w_F^* \leq \overline{w}_G \leq w_G^*, \text{ for all } t \geq 0
\]
2. The ultimate bound of \( z_2 \) is obtained as
\[
| z_2 | \leq \sup_{t \geq 5 T} \delta_{2u}(t) \geq \sup_{t \geq 5 T} \delta_{2u}(t)
\]

(3) The mean square error \( z_2 \) satisfies
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T | z_2 |^2 dt \leq \frac{\gamma_F}{c_F} \lim_{T \to \infty} \frac{1}{T} \int_0^T \delta_{2m}^2 dt
\]
**Remark 6.** Inspection of (47) and (48) reveals clearly the ultimate boundedness and mean square error bound achieved by the DOB and adaptive laws. \( \varepsilon_F \) and \( \varepsilon_G \) defined in Assumption 6 imply the best approximation error achieved by the employed networks. If the network complexities are limited, they cannot be made to be zero. Since \( d(x, t) \) is an unparameterizable disturbance term, we cannot handle it by parameter adaptation. Therefore, \( w^* \) is what we can achieve by the adaptive laws. However, \( \overline{Q}(s) w^* \) clearly implies that the low-frequency components of \( w^* \) can be removed owing to the DOB.

### 4.3 Stability of the overall error system

Lemmas 1 and 2 imply that the overall error system is a cascade of two ISS subsystems. Define the error signal vector
\[
z(t) = [z_{10}(t), z_2(t)]^T
\]
(49)
Then along the same lines of the proof of Lemma C.4 in the monograph [3], we can derive the following results.
\[
| z(t) | \leq \beta_1 e^{-\rho_1 t} | z(0) | + \beta_2 \sup_{0 \leq \tau \leq t} \mu_{2w}(\tau)
\]
(50)
where
\[
\beta_1 = \sqrt{2 \left( \lambda_F^2 + \frac{3}{3} \frac{\lambda_G^2}{\rho_0} + 3 \frac{\lambda_0}{\rho_0} + 3 \right)}
\]
\[
\rho_1 = \min \left( \rho_0 / 2, c_2 / 4 \right)
\]
\[
\beta_2 = 2 \frac{\lambda_0}{\rho_0} + 3 \frac{\lambda_0}{\rho_0} + 1
\]
Furthermore, from Lemmas 3 and 4, we have the mean square bound and ultimate bound of the position tracking error. Additionally, (31) implies that the zero-frequency component of \( z_1 \) converges to zero. The results discussed above are summarized as follows.

**Theorem 1.** The following results hold for the overall error system:

1. Let the assumptions and results of Lemmas 1 and 2 hold. The overall error system is ISS such that
\[
| z(t) | \leq \beta_1 e^{-\rho_1 t} | z(0) | + \beta_2 \sup_{0 \leq \tau \leq t} \mu_{2w}(\tau)
\]
(52)
2. Let the assumptions and results of Lemmas 3 and 4 hold. The ultimate bound of the position tracking error can be made sufficiently small such that
\[
| z_1(t) | \leq \frac{2 \rho_0}{\lambda_{Q min}} \sup_{0 \leq \tau \leq t} \delta_{2u}(\tau)
\]
as \( T \to \infty \)
3. Let the assumptions and results of Lemmas 3 and 4 hold. The mean square bound of the position tracking error can be made sufficiently small such that
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T | z_1 |^2 dt \leq \frac{8 \rho_0^2}{c_2 \lambda_{Q min}^2} \lim_{T \to \infty} \frac{1}{T} \int_0^T \delta_{2u}^2 dt
\]
(4) The zero-frequency component of \( z_1 \) converges to zero.

### 5. SIMULATION EXAMPLES

Extensive simulations have been performed for the motion control problem of a linear motor where the friction and periodic ripple disturbances, and unmodelable external disturbance affect the control performance simultaneously. The nonlinear functions in the system model (1) are described as follows.
\[
F(x) = F_p(x_1) + F_f(x), \quad G(x) = \frac{1}{M}
\]
\[
d(x, t) = d_f(x_2, \varepsilon) + d_e(t)
\]
(52)
\[
F_p(x_1) = -2.5 \sin(2 \pi x_1 / P) - 3.1 \sin(4 \pi x_1 / P + 0.05 \pi)
\]
(53)
It is assumed that the magnet pitch is known such that \( P = 0.03 [m] \).

\( F_f(x) \) and \( d_f(x_2, \varepsilon) \) are respectively the modelable and unmodelable effects of friction:
\[
\dot{z} = x_2 - \frac{|x_2|}{h(x_2)} z, \quad z_s = h(x_2) \text{sgn}(x_2)
\]
\[
h(x_2) = \frac{F_c + (F_s - F_c)e^{-(x_2/x_s)^2}}{\sigma_0}
\]

It is known that
\[
|d_f(x_2, \varepsilon)| \leq \Delta_{d1}|x_2| + \Delta_{d2}, \quad \exists \Delta_{d1}, \exists \Delta_{d2} > 0
\]

In the above models, the true but unknown values of the physical parameters are given as
\[
M = 1[\text{kg}], \quad \sigma_0 = 10^5[\text{N/m}]
\]
\[
\sigma_1 = \sqrt{10^5}[\text{Ns/m}], \quad \sigma_2 = 1[\text{Ns/m}]
\]
\[
F_c = 2[\text{N}], \quad F_s = 4[\text{N}], \quad \dot{x}_s = 0.01[\text{m/s}]
\]

Finally, the unmodelable external disturbance \(d_e(t)\) shown in Fig. 1 is generated by passing a stochastic signal through a low-pass filter.

Due to the limit of paper length, the details of the controller design are omitted here. It can be found in Fig. 2 that due to the presence of unmodelable external disturbance \(d_e(t)\), the adaptive laws do not bring satisfactory improvement. However, we can find from the results of Fig. 3 that the proposed adaptive robust nonlinear controller incorporating DOB brings significant improvement to suppress the error amplitudes. The results match the theoretical analysis quite well.

6. CONCLUSIONS

In this paper, a general adaptive robust nonlinear motion controller combined with DOB for positioning control of a nonlinear SISO mechanical system was proposed. Rigorous stability analysis was performed as well. Extensive simulation studies were carried out to support the theoretical analysis given in the paper. Our major academic contribution is to incorporate the DOB technique and adaptive technique which have been considered as two contrastively different approaches in the literature, under the framework of ISS property.

REFERENCES


