On the Integral Sliding-Mode Control for Sample-data Systems with State Time-Delay

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Abstract: A discrete-time integral sliding mode control (DISMC) scheme is proposed for sample-data systems with state time-delay and disturbance. A steady-state error about the magnitude of $O(T^2)$ is achieved for some delay systems. A variable replacement is applied to transform the original system to a new delay-free system; then a DISMC scheme is designed for the new system. Comparing with the existing methods dealing with time-delay systems, the new scheme is different and simple for some systems. The illustrative example demonstrates the validity of the proposed scheme.

1. INTRODUCTION

With the development of computer, research in discrete-time control has been intensified in recent years. This also necessitated a rework in the sliding-mode control (SMC) strategy for discrete-time delay-free systems (see Gao et al. [1995], Hui et al. [1999], Koshkouei et al. [2000], Goloa et al. [2000], Gao et al. [1995]). However, the reaching law method is not very good wether in theory or practice; and the equivalent control method based on the equivalent control and disturbance estimation (see Cheng et al. [2000]) just drives the sliding-mode into a region of $O(T^2)$. The integral sliding-mode control method (see Abidi et al. [2007]) eliminates the reaching phase and drives both sliding-mode and state to the region magnitude of $O(T^2)$.

In fact, discrete-time systems with state-delay have strong background in engineering applications. Though a great number of research results concerning time-delay systems have existed (see Xu et al. [2001], Richard [2003], Gao et al. [2007]), little progress has been reported for the sliding-mode control strategy in discrete-time delay-free systems (see Abidi et al. [2007]).

2. PROBLEM FORMULATION

2.1 Sample-data System

Consider the continuous-time system with state-delay and matched disturbance:

\[ x(t) = Ax(t) + A_1 x(t-\tau) + B[u(t) + f(t)] \]
\[ x(0) = x_0 \]

where the state $x \in \mathbb{R}^n$, the control $u \in \mathbb{R}^m$, and the disturbance $f \in \mathbb{R}^m$ is assumed smooth and bounded, $\tau$ is the constant time-delay in state. If $T$ is the sampling period and $\tau = hT + \tau_1, 0 \leq \tau_1 < T$, then the discretized counterpart of (1) can be given by

\[ x_{k+1} = \Phi x_k + \Phi_0 x_{\tau} + \Phi_1 x_{\tau-1} + \Gamma u_k + d_k \]
\[ x_i = 0 \quad \text{for} \quad i = -1, -2, \cdots \]
\[ x_0 = x_0 \]

where

\[ \Phi = e^{AT}, \Phi_0 = \int_{\tau_1}^{T} e^{As} ds \cdot A_1, \]
\[ \Phi_1 = \int_{0}^{\tau_1} e^{As} ds \cdot A_1, \Gamma = \int_{0}^{T} e^{As} ds \cdot B, \]

the disturbance $d_k = \int_{0}^{T} e^{As} Bf((k+1)T-s) ds$ represents the influence accumulated from $kT$ to $(k+1)T$. From the definition of $\Gamma$, it can be shown that

\[ \Gamma = BT + \frac{1}{2!} ABT^2 + \cdots + BT + MT^2 + O(T^3) \]

where $M = \frac{1}{2!} AB$ is a constant matrix, and it can be concluded that the magnitude of $\Gamma$ is $O(T)$.
Lemma 1. If the disturbance \( f(t) \) in (1) is bounded and smooth, then
\[
d_k = \Gamma f_k + \frac{1}{2} \Gamma g_k T + O(T^3) \tag{3}
\]
\[
d_k - d_{k-1} = O(T^2) \tag{4}
\]
\[
d_k - 2d_{k-1} + d_{k-2} = O(T^3) \tag{5}
\]
where \( f_k = f(kT), \ g_k = g(kT), \ g(t) = (d/dt)f(t) \).

See Abidi et al. [2007].

Note that the magnitude of the mismatched part \( d_k \) in the disturbance is of the order \( O(T^3) \).

2.2 Transformation

Consider the linear nominal model
\[
x_{k+1} = \Phi x_k + \Gamma u_k \tag{6}
\]

Lemma 2. With a feedback control law
\[
u_k^* = -K x_k \tag{7}
\]
the closed-loop system of (6) is asymptotically stable. \( K \) is chosen such that \( \Phi - \Gamma K = \Gamma \) has all different poles inside the unit disk in the complex \( z \)-plane.

For \( G \) has different poles in unit disk, it can be written as diagonal form
\[
G = PJ P^{-1}
\]
where \( P \) is a nonsingular transformation matrix and \( J \) is the diagonal matrix of \( \{\lambda_i; i = 1, \cdots, n\} \) and\( \{\lambda_i\} \) are the poles of \( G \), where \( J \) is that
\[
J = \begin{bmatrix}
\lambda_1 \\
\vdots \\
\lambda_n
\end{bmatrix}
\]
Obviously, \( \|J\| = \lambda, \lambda = \max_{1 \leq i \leq n} |\lambda_i| \).

Consider a controller with the following structure
\[
u_k = u_k^* + v_k
\]
then, the discretized system (2) is changed to
\[
x_{k+1} = Gx_k + \Phi_0 x_{k-h} + \Phi_1 x_{k-h-1} + \Gamma v_k + d_k \tag{8}
\]

Lemma 3. The transformation
\[
z_k = x_k - \sum_{j=h}^{k-1} G^{k-j-1} \Phi_0 x_{j-h} - \sum_{j=h+1}^{k-1} G^{k-j-1} \Phi_1 x_{j-h-1} \tag{9}
\]
can transfer the state time-delay system (8) to delay-free system
\[
z_k = Gz_k + \Gamma v_k + d_k \tag{10}
\]

Proof. Using (9) and (2), we obtain
\[
z_{k+1} = x_{k+1} - \sum_{j=h}^{k} G^{k-j} \Phi_0 x_{j-h} - \sum_{j=h+1}^{k} G^{k-j} \Phi_1 x_{j-h-1} = Gx_k + \Phi_0 x_{k-h} + \Phi_1 x_{k-h-1} + \Gamma v_k + d_k
\]
\[
\sum_{j=h}^{k} G^{k-j} \Phi_0 x_{j-h} - \sum_{j=h+1}^{k} G^{k-j} \Phi_1 x_{j-h-1} = Gz_k + \Gamma v_k + d_k
\]

Assumption 1. If \( G \) is an asymptotically stable matrix and \( x_k \) is bounded, then \( x_k \) has the same magnitude with \( z_k \).

Remark: This assumption is necessary for the control design of the system (8) and system (10); fortunately, there are some systems satisfy it. In fact
\[
\|z_k\| \geq 1 - \sum_{j=h}^{k} \left\| G^{k-j-1} \Phi_0 \frac{x_{j-h}}{x_k} \right\|
\]
\[
+ \sum_{i=h}^{k-2} \left\| G^{k-i-2} \Phi_1 \frac{x_{i-h}}{x_k} \right\|
\]
\[
\geq 1 + \prod_{j=h}^{k} \left\| J^{-1} \right\| \left\| P^{-1} \right\| \left\| \Phi_1 \right\| \left\| \frac{x_{k-1-h}}{x_k} \right\|
\]
\[
- \|P\| \left\| \sum_{j=h}^{k-1} \left( P^{-1} \Phi_0 + J^{-1} P^{-1} \Phi_1 \right) \frac{x_{j-h}}{x_k} \right\|
\]

So, by selecting proper \( G \), \( \|z_k\| \geq O(T) \) can be obtained.

3. STATE REGULATION WITH ISMC

From assumption 1, it is reasonable to consider the delay-free system (10). Define the integral sliding manifold as follows
\[
s_k = Dz_k + \varepsilon_k - Dz_0
\]
\[
\varepsilon_k = \varepsilon_{k-1} + \hat{E}z_{k-1} \tag{11}
\]
where \( \varepsilon_k \in \mathbb{R}^m, \varepsilon_k \in \mathbb{R}^m, \ k = 0, 1, \cdots, D, E \in \mathbb{R}^{m \times n} \) are constant matrices of rank \( m \) and \( D \) is invertible. It can be concluded that \( \sigma_0 = 0, \varepsilon_0 = 0 \); this means that the reaching phase is eliminated. In order to force the state trajectory to stay on the sliding manifold, the equivalent control is calculated by \( \sigma_{k+1} = 0 \). This leads to
\[
\varepsilon_k = (D^T)^{-1} z_0 - (D^T)^{-1}[ \left( D + E \right) z_k + Dd_k + \varepsilon_k]
\]

Lemma 4. With the integral sliding manifold and disturbance estimation
\[
\hat{d}_k = d_{k-1} = z_k - Gz_{k-1} - \Gamma v_{k-1}
\]
the practical control law
\[
v_k = (D^T)^{-1} z_0 - (D^T)^{-1}[ \left( D + E \right) z_k + D\hat{d}_k + \varepsilon_k]
\]
can make the sliding manifold \( \sigma_k \) go into a neighborhood of \( O(T^2) \).

Proof. Using (11), (12) and (4), we obtain
\[
s_{k+1} = Dz_{k+1} + \varepsilon_{k+1} - Dz_0
\]
\[
= Dz_k + D^T v_k + Dd_k + \varepsilon_k + E z_k - Dz_0
\]
\[
= Dz_k + Dd_k + \varepsilon_k + E z_k - Dz_0 + D \hat{d}_k
\]
\[
\left( (D^T)^{-1} D^T - (D^T)^{-1} \right) \left[ (D + E) z_k + D\hat{d}_k + \varepsilon_k \right]
\]
\[
= Dd_k - \hat{d}_k = D(d_k - d_{k-1}) = O(T^2)
\]

Assumption 2. \( \|I - \Gamma(D^T)^{-1} D\| = O(1) \) and \( \|P\|\|P^{-1}\| = O(T^{-1}) \).

Remark: This assumption is easy to be satisfied by selecting proper matrix \( D \), and enough small \( T \).

Theorem 1. If assumption 1 and 2 are satisfied, then with the control law (12), the closed-loop dynamic system
\[
z_{k+1} = Gz_k + \xi_k \tag{13}
\]
is set by the magnitude \( O(T^3) \), the ultimate bound of \( z_k \) is set by the magnitude \( O(T^2) \) with \( E = D - DG \) and \( \xi_k \in \mathbb{R}^m \). So, the original state vector \( x_k \) goes into the boundary layer with the magnitude \( O(T^2) \).
Proof. Substituting (12) into (10), we obtain
\[
z_{k+1} = [G - \Gamma(D^T)^{-1}(DG + E)]z_k - \Gamma(D^T)^{-1}\varepsilon_k \\
+ \Gamma(D^T)^{-1}Dz_0 + d_k - \Gamma(D^T)^{-1}Dd_k
\]  
(14)
using the integral sliding manifold (11), we obtain
\[
\varepsilon_k = \sigma_k - Dz_k + Dz_0
\]  
(15)
Substituting (15), \(E = D - DG, \sigma_{k+1} = D(d_k - d_{k-1})\) into (14), we obtain
\[
z_{k+1} = Gz_k + d_k - \Gamma(D^T)^{-1}Dz_{k-1} \\
- \Gamma(D^T)^{-1}D(d_{k-1} - d_{k-2})
\]  
(16)
Let
\[
\xi_k = d_k - \Gamma(D^T)^{-1}Dz_{k-1} - \Gamma(D^T)^{-1}D(d_{k-1} - d_{k-2})
\]  
(17)
then using (3), (5), \(|I - \Gamma(D^T)^{-1}D| = O(1)|\), we obtain
\[
\xi_k = d_k - 2d_{k-1} + d_{k-2} \\
+ (I - \Gamma(D^T)^{-1}D)(2d_{k-1} - d_{k-2}) \\
= O(T^3) + (I - \Gamma(D^T)^{-1}D)(2f_k - f_{k-2}) \\
+ \Gamma(g_{k-1} - 1/2g_{k-2})T + O(T^3)
\]  
Then
\[
O(T^3) + (I - \Gamma(D^T)^{-1}D)O(T^3) = O(T^3)
\]  
The solution of (13) is
\[
z_k = Gz_0 + \sum_{i=0}^{k-1} G^{k-i-1}\xi_i
\]  
\[
= PJ^kP^{-1}z_0 + P\sum_{i=0}^{k-1} P^{-1}J^{k-i-1}R^{-1}\xi_i
\]  
So
\[
\|z_k\| \leq \|P\|\|J^k\||P^{-1}\|z_0\| + \|P\|\sum_{i=0}^{k-1} \|J^{k-i-1}R^{-1}\xi_i\|
\leq \|P\|\|P^{-1}\|\|\lambda_k\|z_0\| + \sum_{i=0}^{k-1} \|\lambda^{k-i-1}\|O(T^3)]
\leq \|P\|\|P^{-1}\|\|\lambda_k\|z_0\| + \|P\|\|P^{-1}\|\|\frac{1}{1-\lambda}\|O(T^3)
\]
For \(\lambda\) is predict given, so it is easy to select \(\lambda \leq 0.9\) to make \(\frac{1}{1-\lambda} = O(1)\), so \(\|P\|\|P^{-1}\|\|\frac{1}{1-\lambda}\| = O(T^{-1})\). So, when \(k \to \infty\), \(\|z_k\| = O(T^2)\) and from assumption 1, the original states vector \(x_k\) go into the boundary layer of \(O(T^2)\).

4. NUMERICAL EXAMPLES
Consider (1) with the following parameters:
\[
A = \begin{bmatrix} 8 & 12 \\ 10 & 6 \end{bmatrix}, A_1 = \begin{bmatrix} 2 & 8 \\ 4 & 7 \end{bmatrix}, B = \begin{bmatrix} 10 \\ 15 \end{bmatrix}
\]
\[f(t) = 0.1(\cos(t) - \sin(t)), \tau = 0.265\]
The initial states are \(x_0 = [1 - 0.5]^T\). Take the sample time \(T = 0.01s\), then \(h = 26\), and \(\tau_1 = 0.005\). The discretized counterpart is given by
\[
\Phi = \begin{bmatrix} 1.0898 & 0.12896 \\ 0.10747 & 1.0683 \end{bmatrix}, \Phi_0 = \begin{bmatrix} 0.0108 & 0.4191 \\ 0.0206 & 0.0366 \end{bmatrix}
\]
\[
\Phi_1 = \begin{bmatrix} 0.0108 & 0.4191 \\ 0.0206 & 0.0366 \end{bmatrix}, \Gamma = \begin{bmatrix} 0.11376 \\ 0.16015 \end{bmatrix}, \|d(k)\| < 0.03
\]
The poles are selected as \(\lambda_1 = 0.7, \lambda_2 = 0.65\), then the gain matrices can be obtained
\[
K = \begin{bmatrix} 8.5728 \\ -1.0443 \end{bmatrix}
\]
Then
\[
G = \begin{bmatrix} 0.1145 & 0.2478 \\ -1.2655 & 1.2355 \end{bmatrix}, P = \begin{bmatrix} -0.4199 & -0.3897 \\ -1.9076 & -0.9209 \end{bmatrix}
\]
So, \(\|P\|\|P^{-1}\| = 73.5 < O(T^{-1})\). Select any matrix \(\Gamma\) which make \(\Gamma \Gamma^{-1}\) is invertible, here \(\Gamma = \begin{bmatrix} 1 & 4 \end{bmatrix}\), then \(\|I - \Gamma \Gamma^{-1}\| = 1.1346 = O(1)\). According to \(E = D - DG\)
\[
E = \begin{bmatrix} 2.6572 & -0.5775 \end{bmatrix}
\]
The delayed disturbance is used. Fig.1 shows that the state systems are ultimately bounded and are set by the magnitude \(O(T^2)\) from Fig.2. The ISMC goes into a boundary after the second step and is set by the magnitude \(O(T^2)\) from Fig.4; this is because that for the first step it is equal to \(Dd_0\). The control input is bounded and avoids chattering from Fig.5 and Fig.6.

5. CONCLUSION
This work proposes a method to deal with some sampled-data systems with time-delay. It is effective for some systems which satisfy assumption 1 and 2. So, it is meaningful to be after the necessary and sufficient conditions to select proper systems which satisfy the assumptions.
Fig. 4. magnified ISM s

Fig. 5. control output u

Fig. 6. magnified control output u

REFERENCES


