Abstract: This article discusses a problem of stabilization for an inverted pendulum on a cart. In this work the cart does not travel along an horizontal line, which is the case commonly considered in the literature, but along an arbitrary path in the horizontal-vertical plane. The main obstacle encountered lies in the fact that the feedback controlled system becomes unstable in some "critical regions" along the cart path curve (close to the points where the path is vertical). The overall closed loop system is characterized by an alternation of periods of stability and instability and its state is proved to be stable under some conditions, which involve the geometry of the path. Simulations results are provided for the case in which the path is a circle. 

Keywords: Nonlinear control, time-varying systems, feedback stabilization, non-minimum phase systems, mechanical systems.

1. INTRODUCTION

The inverted pendulum on a cart is an important benchmark for nonlinear control techniques and has been widely analyzed under different points of view, see for instance (Astrom and Furuta, 2000) for an energy based approach, (Bloch et al., 2000) for a controlled Lagrangian one, (Shiriaev et al., 2000) for a passivity based approach, (Holzhuter, 2004) for an Euler-Lagrange backward integration approach, (Angeli, 2001) for an approach based on a continuous feedback, finally a tracking problem is presented in (Mazenc and Bowong, 2003).

In the most common configuration, the cart travels along an horizontal line and, to the best of our knowledge, the general case in which the cart travels on a generic curve on the vertical plane has not been considered in detail in the literature. This article presents an analysis of this more general case, presenting a control strategy that, under some geometrical hypotheses, allows the inverted pendulum to be practically stabilized at the upright position while the cart travels with a given velocity on an arbitrary path. The control function is generated on the basis of the current curve direction and curvature, i.e. the controller uses the same strategy that would stabilize the system when the path is a circle. It is proved that, unless of choosing the reference speed sufficiently low, it is possible to follow any given curve, provided that this curve has a globally bounded curvature and a non null curvature at the points where the tangent is vertical. The main obstacle encountered in this problem lies in the fact that the internal dynamics of the constrained system become unstable in some "critical regions" along the cart path (namely those close to the points where the tangent to the reference curve is vertical). This article is an improvement of (Consolini and Tosques, 2004) that focuses on the VTOL aircraft, in fact as it is shown in Section 3 the path following problem for the VTOL aircraft is strictly connected to the one considered in this article (see also (Fliess et al., 1999) for an explanation of the relationship between the VTOL and the inverted pendulum). Simulations results are provided for the case where the curve is a circle. The main limitation of the work is that the basin of attraction of the equilibrium is rather small.
2. THE MODEL

Let \( \gamma = (\gamma_1, \gamma_2) \in C^\infty([0, +\infty), \mathbb{R}^2) \), with \( \|\gamma'(\lambda)\| = 1, \forall \lambda \geq 0 \) and \( \Gamma = \{\gamma(\lambda) | \lambda \geq 0\} \) be the image of the curve \( \gamma \). Consider an inverted pendulum of mass \( m \) linked to a moving cart of mass \( M \) through a massless rod of length \( l \), in Figure 1 the pendulum is represented as the smaller sphere and the cart as the bigger one. It is supposed that, during the motion, the cart center of mass \( P \) is constrained to stay on the curve \( \Gamma \) and a force \( f(t) \) is applied on it in the direction tangent to \( \Gamma \). Remark that a pendulum model similar to this one can be found in (Fliess et al., 1999), with the only difference that in that case the pendulum is not constrained and has actually two degrees of freedom.

The end is to show that starting from the initial point \( \gamma_0 = \gamma(0) \) with an initial angle \( \theta_0 \) it is possible to find a control force \( f(t) \), which has to be applied to the center \( P \) of mass of the cart, such that the resulting motion \( (\lambda(t), \theta(t)) \) of that system satisfies that \( \lim_{t \to +\infty} \lambda(t) = +\infty \) (in other words all the curve \( \Gamma \) will be covered) and \( \theta(t) \) remains close to the upright position. To determine the dynamic equations on motion, take \( q = (\frac{\lambda}{\theta}) \) as coordinates vector, where \( \lambda \) is the arc-length coordinate representing the position of \( P \) along the curve \( \Gamma \) and \( \theta \) is the angle between the rod and the vertical axis. The kinetic energy \( T \) of this two-masses system is

\[
T = \dot{q}^T H \dot{q},
\]

where the inertia matrix \( H \) is given by

\[
H(q) = \begin{pmatrix}
M + m & -ml(\gamma'(\lambda))^T v(\theta) \\
-lm(\gamma'(\lambda))^T v(\theta) & ml^2
\end{pmatrix},
\]

the potential energy \( U \) is given by

\[
U(q) = (M + m)g\gamma(\lambda) + mgl\cos \theta.
\]

Setting \( L = T - U \) the dynamic equations are derived through the Euler-Lagrange equation

\[
\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = \tau,
\]

which implies that

\[
H(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = \tau
\]

where

\[
C(q, \dot{q}) = \begin{pmatrix}
0 \\
-ml(\gamma''(\lambda))^T v(\theta)\dot{\lambda}
\end{pmatrix},
\]

\[
G(q) = \begin{pmatrix}
-mg\gamma'(\lambda) \\
-mg\lambda\sin \theta
\end{pmatrix}
\]

is the Coriolis term, \( G = \begin{pmatrix}
(m + M)g\gamma'(\lambda) \\
-mgl\sin \theta
\end{pmatrix} \) is the gravity term and \( \tau = \begin{pmatrix} f \ 0 \end{pmatrix} \) is the vector of forces.

By left-multiplying equation (1) by vectors \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \),

\[
\begin{pmatrix} ml^2 \\ ml(\gamma'(\lambda))^T v(\theta) \end{pmatrix},
\]

we obtain respectively the following equations:

\[
i\dot{\lambda} = \dot{\gamma}'' + \begin{pmatrix} 0 \\ g \end{pmatrix}, \quad \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} > + < \gamma', \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} > \dot{\lambda},
\]

\[
\dot{\lambda} = \frac{ml^2}{\det H} (ml(\gamma')^T v\dot{\theta}^2 - (M + m)g\gamma' + f + mg(\gamma')^T v\sin \theta).
\]

where the notation \( <a, b> \) with \( a, b \in \mathbb{R}^n \) denotes the inner product.

Assuming for simplicity that \( l = m = 1 \), we get that the following system has to be verified

\[
\begin{cases}
\dot{\theta} = \dot{\gamma}'' + \begin{pmatrix} 0 \\ g \end{pmatrix}, \quad \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} > + < \gamma', \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} > u \\
\dot{\lambda} = u \\
\theta(0) = \theta_0, \quad \theta'(0) = \dot{\theta}_0 \\
\lambda(0) = 0, \quad \lambda'(0) = 0
\end{cases}
\]

which represents the pendulum dynamics and the required force is given by

\[
f = [(M + 1)g\gamma' - (\gamma'')^T v\lambda^2 + g(\gamma')^T v\sin \theta - (\gamma')^T v\dot{\theta}^2] + \det Hu.
\]

3. EQUIVALENCE WITH THE VTOL AIRCRAFT PATH-FOLLOWING PROBLEM

In this section it is shown that a path-following problem formulation for the VTOL aircraft leads to the same equation as the inverted pendulum, this is related to the fact that these two systems are equivalent by static feedback (as stated in (Fliess et al., 1999)).

Consider the simplified VTOL aircraft model introduced in (Sastry et al., 1992):

\[
\begin{cases}
\dot{x} = u_1 \left( -\sin \theta \cos \phi \right) + \varepsilon u_2 \left( \cos \theta \sin \phi \right) + \begin{pmatrix} 0 \\ g \end{pmatrix} \\
\dot{\theta} = \mu u_2 \left( -\sin \phi \right) + \varepsilon u_2 \left( \cos \phi \right) + \begin{pmatrix} 0 \\ g \end{pmatrix} \\
\dot{(x(0), y(0))} = \gamma(0) \\
(x(0), y(0)) = \gamma(0)v_0 \\
\theta(0) = \theta_0, \quad \dot{\theta}(0) = \dot{\theta}_0
\end{cases}
\]

where \((x, y)^T\), the output of the system, are the coordinates of the center of mass \( P \) of the aircraft on a fixed inertial frame, \( v_0 \) the initial scalar velocity, \( \theta \) is...
the angle between the aircraft symmetry axis and the vertical y-axis (see figure 2).

The goal is to show that starting from the initial point \( y_0 = \gamma(0) \) with an initial angle \( \theta_0 \) it is possible to find a control \((u_1, u_2)\) such that the resulting motion \((\lambda(t), \theta(t))\) of the VTOL verifies that \( \lim_{t \to +\infty} \lambda(t) = +\infty \) (in other words all the curve \( \Gamma \) is covered) and \( \theta(t) \) remains close to 0, that is the VTOL symmetry axis remains close to the vertical axis. To this end, if \( \lambda \in C^\infty([0, +\infty), \mathbb{R}) \) is such that \((x(t),y(t)) = \gamma(\lambda(t))\), it must be \((x \ y) = \gamma' \lambda, (\dot{x} \ y) = \gamma'' \lambda^2 + \gamma' \dot{\lambda}\). Therefore it follows from (4) that

\[
\begin{align*}
\begin{cases}
\begin{aligned}
\dot{u}_1 &= -\frac{\sin \theta}{\cos \theta} + \varepsilon u_2 \left( \frac{\cos \theta}{\sin \theta} \right) = \gamma'' \lambda^2 + \\
\dot{u}_2 &= \frac{0}{g}
\end{aligned}
\end{cases}
\end{align*}
\]
which implies, setting for simplicity \( \varepsilon = \mu = 1 \), that the following system holds:

\[
\begin{align*}
\begin{cases}
\dot{\theta} &= < \lambda^2 \gamma''(\lambda) + \left( \frac{0}{g} \right), \left( \frac{\cos \theta}{\sin \theta} \right) > + \\
&\quad + < \gamma'(\lambda), \left( \frac{\cos \theta}{\sin \theta} \right) > u \\
\dot{\lambda} &= u \\
\theta(0) &= \theta_0, \quad \dot{\theta}(0) = \dot{\theta}_0 \\
\lambda(0) &= 0, \quad \dot{\lambda}(0) = v_0,
\end{cases}
\end{align*}
\]

which represents the equation of the internal constrained dynamics for the VTOL and the control \((u_1, u_2)\) is given by the following equations:

\[
\begin{align*}
u_1 &= < \lambda^2 \gamma'' + \left( \frac{0}{g} \right), \left( -\frac{\sin \theta}{\cos \theta} \right) > + \\
&\quad + < \gamma', \left( -\frac{\sin \theta}{\cos \theta} \right) > u \\
u_2 &= < \lambda^2 \gamma'' + \left( \frac{0}{g} \right), \left( \frac{\cos \theta}{\sin \theta} \right) > + \\
&\quad + < \gamma', \left( \frac{\cos \theta}{\sin \theta} \right) > u
\end{align*}
\]

then the solution of system (4) will verify (5), that is the center of mass of the VTOL will stay on \( \gamma \).
A solution to (9) is given by
\[
\begin{align*}
\theta &= l \sin(\omega t + \phi) \\
v &= v_r \\
u &= 0,
\end{align*}
\]
where \(l = \frac{\tau \theta}{r^2 + \omega^2}\). Denote this solution as as \(r^{\omega, \phi} = (\theta, \dot{\theta}, v_r)^T\). Note that this solution has the remarkable property of being obtained with a null control.

Set \(x_e = x - r^{\omega, \phi} = (e, \dot{e}, w)^T\), as the difference between the solution to (9) and (10), then
\[
\begin{align*}
\dot{e} &= ge + \cos(\omega t + \phi)u \\
w &= u,
\end{align*}
\]
which can be written as
\[
x_e = f^{\omega, \phi}(x_e, u) = A_e(t)x_e + B_e(t)u,
\]
where \(\omega, \phi\) indicate the dependence of the vector field \(f\) from these two parameters, the subscript \(e\) denotes the “error equation” and \(A_e(t), B_e(t)\) are matrix functions.

The following Proposition shows that the time varying periodical system (11) can be globally asymptotically stabilized with a static feedback control.

**Proposition 1.** If \((\omega, \phi) \notin \{(0, \frac{\pi}{2}), (0, -\frac{\pi}{2})\}\), then system (11) is globally asymptotically stabilized by state feedback.

**Proof.** Consider the coordinate transformation \(z = T(t)x_e\), where \(T(t)\) is the matrix
\[
\begin{pmatrix}
-\omega \sin(\omega t + \phi) & \cos(\omega t + \phi) & -\cos(\omega t + \phi) \\
(\omega^2 + 4\omega \sin(\omega t + \phi) - 2\omega^2 \cos(\omega t + \phi)) & (-\omega \sin(\omega t + \phi) - 2\omega^2 \cos(\omega t + \phi)) & 2\omega \sin(\omega t + \phi) \\
(\omega^2 + 4\omega \sin(\omega t + \phi) + 2\omega^2 \cos(\omega t + \phi)) & (-\omega \sin(\omega t + \phi) + 2\omega^2 \cos(\omega t + \phi)) & -2\omega \sin(\omega t + \phi)
\end{pmatrix}
\]
in z-coordinates equation (11) takes the form
\[
\dot{z} = (-a(t), z) + (b_1 + b_2(t))u
\]
where \(a(t)\) is a zero mean periodic function of \(t\), \(b_1 = \frac{-2\omega^2}{\omega^2 + 4\omega \sin(\omega t + \phi) - 2\omega^2 \cos(\omega t + \phi)}\) and \(b_2(t) = \frac{2\omega^2}{\omega^2 + 4\omega \sin(\omega t + \phi) + 2\omega^2 \cos(\omega t + \phi)}\). Moreover
det\(T(t)\) = \((-b_1 + b_2(t))^2\). Notice that system (11) has not a well defined relative degree, because \(b(t)\) zeroes periodically. Choose
\[
u = -\frac{\text{sign}(b_1 + b_2(t))}{k_a} < k, z >,
\]
where \(k_a\) is the mean value of function \((b_1 + b_2(t))\) and \(k = (k_1, k_2, k_3)^T\) is a gain vector such that \(p(s) = k_1 + k_2 s^2 + k_3 s^3\) is Herwitz. The closed loop system takes the form
\[
\dot{z} = A_2(t)z + k < k, z >,
\]
where \(A_2(t)\) is a bounded matrix in which every line is null but the last one and every component has zero mean. Let \(A_e\) be the companion form matrix associated to \(p(s)\), consider the solution \(P\) to the Lyapunov equation
\[
A_e^T P + PA_e = -I,
\]
and define the potential \(V(z) = z^T P z\). Then
\[
\dot{V} = \|z\|^2 + z^T (A_2(t)P + PA_2 z) \leq -V(z) + \frac{1}{\lambda_M} - (m(t))\]
where \(\lambda_M\) is the maximum eigenvalue of \(P\) and \(m(t)\) is a periodic and zero mean function. Therefore
\[
V(t) \leq V(0) e^{-\frac{1}{\lambda_M} \int_0^t m(t) dt},
\]
and
\[
\lim_{t \to +\infty} V(t) = 0.
\]
Remark that being \(T(t)\) singular for some \(t\), (13) does not in general imply that \(\lim_{t \to +\infty} x_e(t) = 0\). To prove this last part remark that because of the periodicity of (9), its solution can be expressed as
\[
x_e(t) = P(t)e^{F}x_e(0),
\]
where \(P(t)\) is periodic. Now \(z(t) = T(t)x(t) = T(t)P(t) e^{F}x(0)\) and, being the term \(T(t)P(t)\) periodic, \(F\) must be negative definite and
\[
\lim_{t \to +\infty} x_e(t) = 0,
\]
which ends the proof. \(\square\)

Consider system (11) when the reference speed is null, i.e. \(v_r = 0\). It reduces to the following time invariant system
\[
\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \cos \phi \\ 0 & 1 \end{pmatrix} u
\]

**Proposition 2.** For any \(\eta > 0\) there exists gain constants \(K_1, K_2, K_3\) for which

- system (11) is stable
- system (14) is stable for every \(\phi\) such that \(|\cos \phi| > \eta\).

**Proof omitted for sake of brevity.** The following definition describes the set of paths for which the control procedure presented here works; it consists of those in which the curvature and its derivatives are bounded, the curvature is non-null whenever the curve is vertical and the vertical points are separated by a distance which is always greater than a positive constant.

**Definition 1.** If \(M_1, M_2, M_3, M_4 \in \mathbb{R}^+\) and \(M_1 > M_3\), set
\[
\mathcal{A}(M_1, M_2, M_3, M_4) = \{ \gamma : \|\dot{\gamma}(\lambda)\| < M_1, \|\ddot{\gamma}(\lambda)\| < M_2, \forall \lambda \in [0, +\infty); \|\dddot{\gamma}(\lambda)\| > M_3 \}
\]
\[
\forall \lambda \text{ such that } \phi(\lambda) \triangleq \arg(\dot{\gamma}(\lambda)) \in \left[\frac{\pi}{2}, \frac{\pi}{2}\right]; |
\lambda_1 - \lambda_2| > M_5,
\]
\[
\forall \lambda_1 \neq \lambda_2 : \phi(\lambda_1), \phi(\lambda_2) \in \left[\frac{\pi}{2}, -\frac{\pi}{2}\right]\}
\]
\[
\mathcal{A}(M_1, M_2, M_3, M_4) \text{ will be the set of the “admissible” curves } \gamma \text{ which can be followed by the system.}
\]

**Definition 2.** When \((\omega, \phi) \notin \{(0, \frac{\pi}{2}), (0, -\frac{\pi}{2})\}\) the transition matrix \(\Phi_{\omega, \phi}(\tau, t)\) the solution to the following differential system
\[
\frac{d\Phi_{\omega, \phi}(\tau, t)}{d\tau} = A_e(\tau)\Phi_{\omega, \phi}(\tau, t) + B_e(\tau)u
\]
\[
\Phi(0, t) = I
\]
Lemma 1. If \( P(\omega, \phi) \notin \{(0, \frac{\pi}{2}), (0, -\frac{\pi}{2})\} \), the quadratic form \( P_{\omega, \phi}(\tau, t) \) is defined as follows
\[
P_{\omega, \phi}(\tau, t) = \int_0^{+\infty} \Phi_{\omega, \phi}(\tau, t) T \Phi_{\omega, \phi}(\tau, t) dt
\]

Remark that \( P(\omega, \phi) \) is the solution at time \( t = 0 \) to the Lyapunov differential equation associated to (11). From Proposition 1, \( P(\omega, \phi) \) is well defined whenever \( \omega, \phi \notin \{(0, \frac{\pi}{2}), (0, -\frac{\pi}{2})\} \).

Definition 4. Given \( \gamma \in \mathcal{P}(M_1, M_2, M_3, M_4) \) the \( \forall \lambda \geq 0 \) define the following potential
\[
V_{\gamma}(\lambda, x) = x^T P_{\gamma}(\lambda, x) x
\]
where
\[
\lambda = \frac{1}{\gamma}(\lambda)
\]
Remark that if \( \gamma \neq 0 \), \( P_{\gamma}(\omega, \phi) \) is well defined for every curve \( \gamma \in \mathcal{P}(M_1, M_2, M_3, M_4) \) by Proposition 1 because \( \gamma \notin \{(0, \frac{\pi}{2}), (0, -\frac{\pi}{2})\} \).

Definition 5. For any \( \eta > 0 \), the critical set \( \mathcal{C}_\eta \) is
\[
\mathcal{C}_\eta = \{ \lambda : \cos \phi(\lambda) \leq \eta \}
\]
Given a function \( V(x) \), the derivative of \( V \), with respect to a vector field \( f(x) \) is denoted as
\[
L_f V(x) = V'(x) f(x)
\]
where \( V'(x) \) is the Jacobian matrix of \( V \) at \( x \).

Lemma 1. If the path \( \gamma \) is a circle of curvature \( \kappa \) and such that \( \arg(\gamma(\lambda)) = \phi(\lambda) \), then the derivative of the potential \( V_{\gamma}(\lambda, x) \) is defined as the vector field \( f(\lambda, x, u) \) in \( x \), satisfies
\[
L_{f(\lambda, x, u)} V_{\gamma}(\lambda, x) = -\|x\|^2
\]
Proof. This comes directly from the definition of \( V_{\omega, \phi} \), being the solution to the Lyapunov differential equation associated to the time varying system (11). □

Proposition 3. If \( r_{\gamma}(\lambda, x) \) is the reference solution and \( x \) is the solution of the nominal system \( f(x, u) \), then
\[
\frac{d}{dt} (r_{\gamma}(\lambda, x) \phi(\lambda)) = r_{\gamma}(\lambda, x) \phi(\lambda) + \psi_1(x) + \psi_2(x) u
\]
where \( \psi_1(x) \) and \( \psi_2(x) \) are suitable functions such that there exists \( c_1, c_2 > 0 \):
\[
\psi_1(x) \leq v_1^2 c_1 \|x\|, \psi_2(x) \leq v_2^2 c_2 \|x\|
\]
for \( \|x\| \) and \( v \) sufficiently small.

Proof omitted for sake of brevity.

Lemma 2. The derivative of the potential \( V_{\gamma}(\lambda, x) \) with respect to system (15), computed along a generic curve \( \gamma(\lambda) \), if \( \lambda \in \mathcal{C}_\eta \) can be written as
\[
L_f V_{\gamma}(\lambda, x) \leq -\|x\| + \sigma(\gamma) \frac{d\kappa}{d\lambda}
\]
where \( \sigma(\gamma) \) is a suitable function such that \( \lim_{\gamma \to 0} \sigma(\gamma) = 0 \), uniformly on \( \mathbb{R}^3 \).

Proof omitted for sake of brevity.

Lemma 3. Given a curve \( \gamma \) if \( \lambda_1, \lambda_2 \notin \mathcal{C}_\eta \) are such that \( \cos \phi(\lambda_1) = \cos \phi(\lambda_2) \) then
\[
\lim_{\gamma \to 0} P_{\gamma}(\lambda_1, x) = \lim_{\gamma \to 0} P_{\gamma}(\lambda_2, x)
\]
Proof. It comes from the fact that, as \( \gamma \) approaches \( 0 \), \( P_{\gamma}(\lambda_1, x) \) and \( P_{\gamma}(\lambda_2, x) \) are the solution to the same differential equation. □

Proposition 4. Set \( M_1, M_2, M_3, M_4 \in \mathbb{R}^+ \), with \( M_1 > M_3 \), for any \( \gamma \in \mathcal{P}(M_1, M_2, M_3, M_4) \) and for any \( \varepsilon > 0 \) there exists a one dimensional smooth manifold \( \Sigma \in \mathcal{P}(\mathbb{R}, \mathbb{R}) \), in the state space and a sufficiently small \( r > 0 \) such that \( \forall (\theta, \phi, t) \in B(\Sigma(0), r) \) there exists a feedback \( u(\theta, \phi, \gamma(t)) \in \mathcal{P}(\mathbb{R}, \mathbb{R}) \) such that the solution to system (6) has the following properties:
\[
\begin{aligned}
\lambda \in \mathcal{P}(\mathbb{R}, \mathbb{R}) \text{ is a strictly monotone function such that} \\
\lim_{t \to \infty} \lambda(t) = +\infty \quad (16) \\
\theta(t), \phi(t), \lambda(t) \in B(\Sigma(\lambda(t)), \varepsilon), \forall t \geq 0 \quad (17)
\end{aligned}
\]

Sketch of the proof. Given \( \varepsilon > 0 \), set
\[
V_{\max}(P_{\gamma}(\lambda, x, u)) = \max_{\lambda} P_{\gamma}(\lambda, x, u)
\]
where \( \max_{\lambda} \) is the maximum eigenvalue of \( P \), then if \( V_{\max}(\varepsilon(t)) < V_{\max}(\varepsilon(t)) < \varepsilon \) for all \( t > 0 \) then \( \|x(t)\| < \varepsilon \) and the thesis is proved.
Given any \( \lambda \neq 0 \), choose \( \eta \) such that, for any \( \lambda \in \mathcal{C}_0 \), \( L_{\sigma}(\lambda) < 0 \).

Given \( \gamma \in \mathcal{P}(M_1, M_2, M_3, M_4) \), for any \( \lambda \neq \mathcal{C}_\eta \) it is from Lemma 2
\[
L_f V_{\gamma}(\lambda, x) \leq -\|x\| + \sigma(\gamma) \frac{d\kappa}{d\lambda}
\]
with \( \lim_{\gamma \to 0} \sigma(\gamma) = 0 \). When \( \lambda \in \mathcal{C}_\eta \), unless of reducing \( \eta \) there exists \( t_1, t_2 \) such that \( \lambda(t_1) = \lambda_1, \lambda(t_2) = \lambda_2 \), with \( \lambda_2 = \inf_{\eta \in \mathcal{C}_\eta} \lambda \geq \lambda_1, \lambda_1 = \sup_{\eta \in \mathcal{C}_\eta} \lambda \leq \lambda_1 \).

Consider the difference
\[
V_{\gamma}(\lambda_2(x(t_2)) - V_{\gamma}(\lambda_1, x(t_1)) = x(t_2) - x(t_1)
\]
from Lemma 3 it is
\[ \lim_{v \to 0} P_r(v, \kappa(\lambda_2), \phi(\lambda_2)) = \lim_{v \to 0} P_r(v, \kappa(\lambda_1), \phi(\lambda_1)). \]

By combining these equations it is possible to prove that by reducing \( \eta \) and \( v \), and choosing a sufficiently small radius \( r \), then for any initial condition \( e(0) \) with \( \|e(0)\| < R \) we have that \( V_r(\lambda(t), x(t)) < V_r \), for every \( t > 0 \). Now it is sufficient to define \( \Sigma(\lambda) = (r(0, \arg(y(k)))) \), \( \forall \lambda \geq 0. \]

6. SIMULATIONS

This section presents a simulation for the case where the cart lies on a circle. The circle has a radius \( r = 5 \) m, the reference speed is \( v = 0.4 \) m/s, \( (x(0), y(0)) = (0, 0) \), the initial speed is \( (\dot{x}(0), \dot{y}(0)) = (0.4, 0) \), and \( \theta_0 = 0, \phi_0 = 0 \). The assigned eigenvalues are \( \{-10, -12, -14\} \) and the threshold value for \( \det R \) is \(-0.1\).

The simulation results are reported in figure 3, where it can be noticed that the small perturbations of the speed along the trajectory happen when the determinant approaches zero and the tangent to the trajectory becomes parallel to the y-axis.

7. CONCLUSIONS

This article is about the control of an inverted pendulum on a cart that moves along an arbitrary curve. A control strategy which proves to be effective also in simulation, has been introduced and a theoretical explanation has been given. This particular problem seems very interesting and the approach may be generalized to more general non minimum phase systems.

REFERENCES


