Abstract: In this paper, the purpose is to design a filter for a stochastic bilinear system which satisfies an $H_\infty$ prescribed norm constraint and such that the estimation error is mean-square stable. The system under consideration is bilinear in control input and is subjected to multiplicative noise in both the state and the measurement equations. The system is also corrupted by deterministic perturbations. The proposed approach is based on the resolution of LMI and the filter design requires to satisfy a rank condition.

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Keywords: Filtering, Stochastic systems, Itô equation, Lyapunov stochastic function, LMI.

1. INTRODUCTION

The bilinear system is sometimes a good mean to represent physical systems when the linear representation is not sufficiently significant. The stochastic systems get a great importance in the last decades as shown by numerous references (Has’minskii, 1980; Kozin, 1969; Florchinger, 1997; Mao, 1997; Carravetta et al., 2000; Germani et al., 2002; Xu and Chen, 2003).

Generally, the term of bilinear stochastic system designs a system with multiplicative noise instead of additive one (Carravetta et al., 2000; Germani et al., 2002). The $H_\infty$ filtering for systems with multiplicative noise has been treated in many papers (Gershon et al., 2001; Xu and Chen, 2002; Stoica, 2002). In (Stoica, 2002), the problem of reduced-order $H_\infty$ filtering for a class of stochastic systems is solved in terms of two LMIs conditions coupled by a rank condition. The considered system is deterministic while the measurements are subjected to a multiplicative noise. In (Xu and Chen, 2002), the reduced-order $H_\infty$ filtering for stochastic systems with multiplicative noise and corrupted by deterministic input disturbance is treated. Notice that the measurement equation in this paper is not corrupted by noise, the problem is resolved in term of two LMIs and a coupling non convex rank constraint set. The $H_\infty$ filtering with noisy measurement equation is considered in (Gershon et al., 2001). The dynamic output feedback for stochastic system subjected to both deterministic and stochastic perturbations is solved in (Hinrichsen and Pritchard, 1998). In this paper, the bilinearity is also between the state and the control input. The goal is to design a filter for this kind of stochastic systems such that the estimation error system is mean-square stable and satisfies an $H_\infty$ norm constraint. The approach is based on a change of variable on the control input to transform the problem into a robust stochastic filtering one. Then the Itô formula and the LMI method permit to obtain a condition to be verified to ensure the existence of the filter. Let $L_2(\Omega, \mathbb{R}^k)$ be the space of square-integrable $\mathbb{R}^k$-valued functions on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ where $\Omega$ is the sample space, $\mathcal{F}$ is a $\sigma$-algebra of subsets of the sample space called
Let us consider the following Itô stochastic bilinear system (Has’minskii, 1980; Florchinger, 1997)

\[
\begin{align*}
\frac{dx(t)}{dt} &= (Ax(t) + u(t)Bx(t))dt + Fx(t)dw(t) + Gw(t)dt \\
y(t) &= Cx(t)dt + Hx(t)dw(t) \\
z(t) &= Lx(t)
\end{align*}
\]

where \(x(t)\in \mathbb{R}^n\) is the state vector, \(y(t)\in \mathbb{R}^p\) is the output, \(u(t)\in \mathbb{R}\) is the control input, \(v(t)\in \mathbb{R}^q\) is the disturbance vector and \(z(t)\in \mathbb{R}^q\) is the state linear combination to be estimated. \(w(t)\) is a zero mean scalar Wiener process verifying (Has’minskii, 1980)

\[
\mathbf{E}(dw(t)) = 0 \quad \text{and} \quad \mathbf{E}(dw(t)^2) = dt.
\]

To simplify the notation and without loss of generality, we consider only the single control input case. As in the most cases for physical processes, assume that the stochastic bilinear system (1) has known bounded control input. Let \(u(t)\in \Gamma \subset \mathbb{R}\), where

\[
\Gamma := \{u(t)\in \mathbb{R} \mid u_{\min} \leq u(t) \leq u_{\max}\}.
\]

**Assumption 1.** We suppose that \(v(t)\in \hat{L}_2\) holds.

**Definition 2.** (Hinrichsen and Pritchard, 1998; Ryashko and Schurtz, 1996) The stochastic system (1) with \(v(t) \equiv 0\) is asymptotically mean-square stable if all initial states \(x(0)\), yield

\[
\lim_{t \to \infty} \mathbf{E}\|x(t)\|^2 = 0, \quad \forall u(t) \in \Gamma.
\]

**Assumption 3.** The stochastic bilinear system (1) with \(v(t) = 0\) is assumed to be asymptotically mean-square stable.

In this paper, the aim is to design a filter in the following form

\[
\begin{align*}
\frac{d\hat{x}(t)}{dt} &= (A\hat{x}(t) + u(t)B\hat{x}(t))dt + K(d\hat{y}(t) - C\hat{x}(t)dt) + u(t)\overline{K}(d\hat{y}(t) - C\hat{x}(t)dt) \\
\hat{z}(t) &= L\hat{x}(t)
\end{align*}
\]

with \(K\) and \(\overline{K}\) are the gains to design in order to ensure that the estimation error \(x - \hat{x}\) is mean-square stable. Notice that the estimation error \(x(t) - \hat{x}(t)\) has the following dynamics

\[
\begin{align*}
d\epsilon(t) &= d\hat{x}(t) - d\hat{\hat{x}}(t) \\
&= (A - KC + u(t)(B - \overline{K}C))\epsilon(t)dt + (F - KH - u(t)\overline{K}H)x(t)dw(t) + G\epsilon(t)dt.
\end{align*}
\]

Let us consider the following augmented state vector

\[
\xi(t) = \begin{bmatrix} x^T(t) \epsilon^T(t) \end{bmatrix}.
\]

Then the dynamics of the augmented system is given by

\[
\begin{align*}
d\xi(t) &= (A + u(t)B)\xi(t)dt + (A_0 + u(t)B_0)\xi(t)d\hat{w}(t) + \overline{G}v(t)dt \\
\hat{z}(t) &= z(t) - \hat{z}(t) = \overline{L}\xi
\end{align*}
\]

with

\[
\begin{align*}
A &= \begin{bmatrix} A & 0 \\
0 & A - KC \end{bmatrix}, & B &= \begin{bmatrix} B & 0 \\
0 & B - \overline{K}C \end{bmatrix}, & \overline{G} &= \begin{bmatrix} G \\
0 \end{bmatrix}, & A_0 &= \begin{bmatrix} F & 0 \\
0 & -KH \end{bmatrix}, & \overline{L} &= \begin{bmatrix} 0 \\
0 \end{bmatrix}.
\end{align*}
\]

Now, we introduce the two following definition

**Definition 4.** (Hinrichsen and Pritchard, 1998; Xu and Chen, 2003) The stochastic system (8) is said to be externally stable if, for every \(v(t)\in \hat{L}_2([0, \infty); \mathbb{R}^m)\), \(\exists \gamma > 0\) such that \(\hat{z}(t) = z(t) - \hat{z}(t)\) is mean-square stable and the following \(H_{\infty}\) performance

\[
\|\hat{z}(t)\|_{L_2}^2 \leq \gamma \|v(t)\|_{L_2}^2
\]

holds.

**Problem 5.** Given a real \(\gamma > 0\), the goal is to design an asymptotically stable filter of the form of (5) such that the augmented system (8) is asymptotically mean-square stable and the following \(H_{\infty}\) performance

\[
\|\hat{z}(t)\|_{L_2}^2 \leq \gamma \|v(t)\|_{L_2}^2
\]

is achieved.
using the Itô’s formula (Has’minskii, 1980),
(Florchinger, 1997; Kozin, 1969; Mao, 1997; Xu and Chen, 2003) we have
\[
d V(\xi(t)) = LV(\xi(t)) dt + 2\xi^T(t) P (A_{u} + u(t) B_{0}) \xi(t) \, dw(t) \tag{13}
\]
with
\[
LV(\xi(t)) = 2\xi^T(t) P (A + u(t) B) \xi(t) + \xi^T(t) (A_{0} + u(t) B_{0})^T P (A_{0} + u(t) B_{0}) \xi(t). \tag{14}
\]
Then relation (13) is rewritten as
\[
d V(\xi(t)) = \xi^T(t) \{ P A_{t} + A_{t}^T P + u(t) (B^T P + PB) \\
+ (A_{0} + u(t) B_{0})^T P (A_{0} + u(t) B_{0}) \} \xi(t) \, dt \\
+ 2\xi^T(t) P (A_{0} + u(t) B_{0}) \xi(t) \, dw(t). \tag{15}
\]
To study the stability of this system, we introduce a change of variable on the control input \( u(t) \) in order to reduce the conservatism introduced by the assumption that \( u(t) \) is bounded (see (3)). Let
\[
u(t) = \alpha + \sigma \epsilon(t) \tag{16}
\]
where \( \alpha \in \mathbb{R} \) and \( \sigma \in \mathbb{R} \) are given by
\[
\alpha = \frac{1}{2}(u_{\text{min}} + u_{\text{max}}), \quad \sigma = \frac{1}{2}(u_{\text{max}} - u_{\text{min}}). \tag{17}
\]
The new “uncertain” variable is \( \epsilon(t) \in \Gamma \subset \mathbb{R} \) where the polytope \( \Gamma \) is defined by
\[
\Gamma := \{ \epsilon(t) \in \mathbb{R} \mid \epsilon_{\text{min}} = -1 \leq \epsilon(t) \leq \epsilon_{\text{max}} = 1 \}. \tag{18}
\]
Equation (15) is rewritten as
\[
d V(\xi(t)) = \xi^T(t) \{ P A_{t} + A_{t}^T P + \sigma \Delta A(t) + \Delta A(t)^T P \\
+ (A_{0t} + \Delta A_{0t}(t))^T P (A_{0t} + \Delta A_{0t}(t)) \} \xi(t) \, dt \\
+ 2\xi^T(t) P (A_{0t} + \Delta A_{0t}(t)) \xi(t) \, dw(t) \tag{19}
\]
for the system (see (8)) :
\[
\begin{cases}
\dot{\xi}(t) = (A_{t} + \sigma \Delta A(t)) \xi(t) \, dt + \sigma \epsilon(t) \xi(t) \, dt \\
+ (A_{0t} + \Delta A_{0t}(t)) \xi(t) \, dw(t)
\end{cases} \tag{20}
\]
where
\[
A_{t} = (A + \alpha B), \quad \Delta A(t) = H_{1} \Delta \epsilon(t) H_{2}, \tag{21a}
\]
\[
A_{0t} = (A_{0} + \alpha B_{0}), \Delta A_{0t}(t) = H_{10} \Delta \epsilon(t) H_{2}(t) \tag{21b}
\]
with
\[
H_{1} = \sigma B, \quad H_{10} = \sigma B_{0}, \quad \Delta \epsilon(t) = \epsilon(t) \quad \text{and} \quad H_{2} = I_{2n}. \tag{22}
\]
From the majoration lemma (Xu and Chen, 2003), we have (with \( \epsilon_{1} > 0 \) and \( \epsilon_{2} > 0 \))
\[
2\xi^T(t) P \Delta A(t) \xi(t) \leq \xi^T(t) \left[ \epsilon_{1} + \epsilon_{1} P \sigma B B^T \sigma P \right] \xi(t) \tag{23}
\]
and
\[
(A_{0t} + \Delta A_{0t}(t))^T P (A_{0t} + \Delta A_{0t}(t)) \leq A_{0t}^T (P^{-1} - \epsilon_{2}^{-1} H_{10} H_{10}^T)^{-1} A_{0t} + \epsilon_{2} H_{2}^2 H_{2}. \tag{24}
\]
So
\[
d V(\xi(t)) \leq \xi^T(t) \{ P A_{t} + A_{t}^T P + \epsilon_{1} I_{2n} \\
+ \epsilon_{1} P \sigma B B^T \sigma P + A_{0t}^T (P^{-1} - \epsilon_{2}^{-1} H_{10} H_{10}^T)^{-1} A_{0t} \\
+ \epsilon_{2} H_{2}^2 H_{2} \} \xi(t) \, dt \\
+ 2\xi^T(t) P (A_{0t} + \Delta A_{0t}(t)) \xi(t) \, dw(t). \tag{25}
\]
Then the following theorem is given to ensure the asymptotically mean-square stability of the augmented system (20).

**Theorem 6.** The system (20) with \( v(t) = 0 \) is mean-square stable if there exist \( P = P^T > 0 \) and two real \( \epsilon_{1} > 0, \epsilon_{2} > 0 \) such that the following LMI
\[
\begin{bmatrix}
(1,1) & P H_{1} & A_{0t} \xi(t) = 0 \\
H_{1}^T \xi(t) & -\epsilon_{1} I_{2n} & 0 \\
A_{0t} & 0 & -P \\
0 & 0 & H_{10}^T \xi(t) - \epsilon_{2}^{-1} I_{2n}
\end{bmatrix} < 0 \tag{26}
\]
holds, with
\[
(1,1) = P A_{t} + A_{t}^T P + (\epsilon_{1} + \epsilon_{2}) H_{2}^2 H_{2}. \tag{27}
\]
\[
\text{Proof.} \text{ Using the Schur lemma, theorem 6 gives}
\]
\[
P A_{t} + A_{t}^T P + \epsilon_{1} H_{2}^2 H_{2} + \epsilon_{1} \frac{1}{2} P H_{1} H_{2}^2 P \\
+ \epsilon_{2} (P^{-1} - \epsilon_{2}^{-1} H_{10} H_{10}^T)^{-1} A_{0} + \epsilon_{2} H_{2}^2 H_{2} = -K < 0.
\]
Note that from (26) \( \lambda_{\text{min}}(K) > 0 \) where \( \lambda_{\text{min}} \) is the smallest eigenvalue of \( K \). This and (25) yield to
\[
d V(\xi(t)) \leq -\lambda_{\text{min}}(K) \| \xi(t) \|^2 \, dt \\
+ 2\xi^T(t) P (A_{0t} + \Delta A_{0t}(t)) \xi(t) \, dw(t). \tag{28}
\]
Let \( \beta > 0 \) be given, using the integration-by-part formula (Mao, 1997; Xu and Chen, 2003), we can derive that
\[
d \left[ e^{\beta t} V(\xi(t)) \right] = e^{\beta t} [\beta V(\xi(t)) \, dt + d V(\xi(t))] \tag{29}
\]
which can be bounded as
\[
d \left[ e^{\beta t} V(\xi(t)) \right] \leq e^{\beta t} \left[ -\beta \lambda_{\text{max}}(P) - \lambda_{\text{min}}(K) \right] \| \xi(t) \| \, dt \\
+ 2e^{\beta t} \xi^T(t) P (A_{0t} + \Delta A_{0t}(t)) \xi(t) \, dw(t). \tag{30}
\]
Since
\[
[-\beta \lambda_{\text{max}}(P) - \lambda_{\text{min}}(K)] \| \xi(t) \| \leq 0, \tag{31}
\]
then inequalities (30) and (31) imply that
\[
d \left[ e^{\beta t} V(\xi(t)) \right] \leq 2e^{\beta t} \xi^T(t) P (A_{0t} + \Delta A_{0t}(t)) \xi(t) \, dw(t). \tag{32}
\]
Integrating both sides from 0 to \( t > 0 \) and then taking expectation give
\[
e^{\beta t} \mathbb{E} \left[ \xi^T(t) P \xi(t) \right] = e^{\beta t} \mathbb{E} \left[ \xi^T(0) P \xi(0) \right] \leq \int_{0}^{t} 2e^{\beta s} \xi^T(s) P (A_{0t} + \Delta A_{0t}(s)) \xi(s) \, dw(s). \tag{33}
\]
Now using the properties (2), the right term of inequality (33) is given by

$$
\int_0^t 2e^{\beta t} \xi^T(s)P(\mathcal{A}w + \Delta \mathcal{A}(s))\xi(s)E\left[ d(w(s)) \right] = 0.
$$

(34)

Then (33) can be rewritten as

$$
e^{\beta t} E\left[ \xi^T(t)P\xi(t) \right] \leq c
$$

(35)
or equivalently

$$
\lambda_{\min}(P) E\left[ ||\xi||^2 \right] \leq E\left[ \xi^T(t)P\xi(t) \right] \leq ce^{-\beta t}
$$

(36)

where $c = E[\xi^T(0)P\xi(0)]$ is a positive constant.

Finally, from (36) the following inequality

$$
E\left[ ||\xi||^2 \right] \leq \frac{c}{\lambda_{\min}(P)} e^{-\beta t}
$$

(37)

ensures that the augmented system (7) is asymptotically mean-square stable.

\[
\square
\]

4. $\mathcal{H}_\infty$ PERFORMANCE

From section 3, the following theorem is then given for the filter synthesis.

Theorem 7. The filtering problem 5 is resolved for the system (1) with the filter (5) if there exist matrices $P_1 = P_1^T > 0$, $P_2 = P_2^T > 0$, $P_3 > 0$, $Y_1$, $Y_2$, $Y_3$ and $Y_4$ such that the following LMIs hold

$$
\begin{bmatrix}
(1.1) & (1.2) & (1.3) \\
(1.2)^T & (1.2) & (1.3) \\
\sigma^T(P_1 + P_2) & \sigma^T(\mathcal{A} + \Delta \mathcal{A}(s))\xi(s)E\left[ d(w(s)) \right] & 0 \\
\sigma^T(P_1 + P_2) & \sigma^T(\mathcal{A} + \Delta \mathcal{A}(s))\xi(s)E\left[ d(w(s)) \right] & 0 \\
\sigma^T(P_1 + P_2) & \sigma^T(\mathcal{A} + \Delta \mathcal{A}(s))\xi(s)E\left[ d(w(s)) \right] & 0 \\
\sigma^T(P_1 + P_2) & \sigma^T(\mathcal{A} + \Delta \mathcal{A}(s))\xi(s)E\left[ d(w(s)) \right] & 0 \\
\end{bmatrix} < 0,
$$

(38)

where

$$(1.1) = P_1A_{\mathcal{A}} + A_{\mathcal{A}}^TP_1 + \sigma \xi(s)E\left[ d(s) \right],$$

$$\sigma = \frac{\gamma^2}{\gamma^2} \lambda_{\min}(P)$$

$$P_2 = P_2^T$$

$$P_3 = P_3^T$$

$$P_4 = P_4^T$$

$$Y_1 = \frac{\xi^T(0)P\xi(0)}{\lambda_{\min}(P)}$$

$$Y_2 = \frac{\xi^T(0)P\xi(0)}{\lambda_{\min}(P)}$$

$$Y_3 = \frac{\xi^T(0)P\xi(0)}{\lambda_{\min}(P)}$$

$$Y_4 = \frac{\xi^T(0)P\xi(0)}{\lambda_{\min}(P)}$$

Proof. Consider the Lyapunov matrix $\mathcal{P}$ given by

$$\mathcal{P} = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}$$

with $P_j \in \mathbb{R}^{n \times n}$ $j = 1, \ldots, 3$.

We will use Itô formula again to the system (20) and we get

$$
dV(\xi(t)) = LV(\xi(t))dt + 2\xi^T(t)P(\mathcal{A}w + \Delta \mathcal{A}(s))\xi(s)E\left[ d(w(s)) \right]
$$

(43)

with

$$LV(\xi(t)) = 2\xi^T(t)P(\mathcal{A}w + \Delta \mathcal{A}(s))\xi(t) + \tilde{G}(v(t)) + \tilde{G}(v(t))$$

(44)

Similarly to the derivation of (28), we have

$$dV(\xi(t)) \leq \left[ \xi^T(t) v(t)^T \right] \Theta \left[ \xi(t) v(t) \right] dt + 2\xi^T(t)P(\mathcal{A}w + \Delta \mathcal{A}(s))\xi(s)E\left[ d(w(s)) \right]
$$

(45)

with

$$\Theta = \begin{bmatrix} P_\mathcal{A} + A_{\mathcal{A}}^TP + \xi^T \xi \mathcal{H}_1 \mathcal{H}_1^T \mathcal{P} \mathcal{P}^T \tilde{G} \tilde{G}^T \mathcal{P} & 0 \\ 0 & 0 \end{bmatrix}
$$

(46)

By applying Schur we have

$$\Theta = \begin{bmatrix} P_\mathcal{A} + A_{\mathcal{A}}^TP + \xi^T \xi \mathcal{H}_1 \mathcal{H}_1^T \mathcal{P} \mathcal{P}^T \tilde{G} \tilde{G}^T \mathcal{P} & 0 \\ 0 & 0 \end{bmatrix}
$$

(47)

then by applying Schur to the second term we have

$$\Theta = \begin{bmatrix} P_\mathcal{A} + A_{\mathcal{A}}^TP + \xi^T \xi \mathcal{H}_1 \mathcal{H}_1^T \mathcal{P} \mathcal{P}^T \tilde{G} \tilde{G}^T \mathcal{P} & 0 \\ 0 & 0 \end{bmatrix}
$$

(48)

Integrating both sides of (45) from 0 to $t > 0$ with $\Theta$ given by (46) and taking the expectation, we have

$$E[V(\xi(t))] \leq E[V(\xi(0))]
$$

(49)

Then, from this and by applying the Schur lemma (three times) to (38) and finally pre-multiplying...
by $[\xi^T \ v(t)^T]^{T}$ and post-multiplying by its transpose, we deduce the following inequality

$$
\begin{bmatrix}
[\xi(t)^T \ v(t)^T]
\end{bmatrix}
\Theta
\begin{bmatrix}
[\xi(t)]
\end{bmatrix}
+ 
\begin{bmatrix}
[\xi(t)^T \ v(t)^T]
\end{bmatrix}
\begin{bmatrix}
L^T \ L & 0 \\
0 & -\gamma^2 I
\end{bmatrix}
\begin{bmatrix}
[\xi(t)] \\
v(t)
\end{bmatrix}
< 0,
$$

(50)

notice that the LMI (38) is equivalent to

$$
\Theta + 
\begin{bmatrix}
L^T \ L & 0 \\
0 & -\gamma^2 I
\end{bmatrix}
< 0
$$

(51)

which ends the proof.

The gains $K$ and $\bar{K}$ are then obtained by solving the following equation

$$
\begin{bmatrix}
-P_2 \\
-P_3
\end{bmatrix}
\begin{bmatrix}
K \\
\bar{K}
\end{bmatrix}
= 
\begin{bmatrix}
Y_2 & Y_2 \\
Y_3 & Y_3
\end{bmatrix}.
$$

(52)

Note that $K$ and $\bar{K}$ exist if and only if the following rank condition is satisfied

$$
\text{rank} \ \begin{bmatrix}
P_2 \\
P_3
\end{bmatrix} = \text{rank} \ \begin{bmatrix}
Y_2 & Y_2 \\
Y_3 & Y_3
\end{bmatrix}.
$$

(53)

5. CONCLUSION

In this paper, a method has been proposed to resolve the problem of $\mathcal{H}_\infty$ filter design for bilinear stochastic system with multiplicative noise and bounded control output. An LMI approach and a rank condition are proposed for the design of the filter to ensure an $\mathcal{H}_\infty$ disturbance attenuation.

6. REFERENCES


