PARAMETRIZING ALL SOLUTIONS OF UNCONTROLLABLE MULTIDIMENSIONAL LINEAR SYSTEMS

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Abstract: Using an algebraic analysis approach, we derive a necessary and sufficient condition so that we can parametrize all solutions of a multidimensional linear system by glueing the controllable sub-behaviour with the autonomous elements. Effective algorithms checking this condition are obtained. This result generalizes a result of 1-D linear systems for a class of multidimensional linear systems. Copyright © 2005 IFAC

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1. INTRODUCTION

Let us first show how to parametrize all solutions of a time-invariant linear control system defined by ordinary differential equations.

We consider the commutative polynomial ring $D = \mathbb{R}\left[\frac{d}{dt}\right]$ of differential operators in $\frac{d}{dt}$ with coefficients in the field $\mathbb{R}$. An element of $D$ has the form $\sum_{i=0}^{n} a_i \frac{d^i}{dt^i}$ where $a_i \in \mathbb{R}$. Let us consider a full row rank matrix $R \in D^{q \times p}$, i.e., the rows of $R$ are $D$-linearly independent. Then, computing a Smith form for $R$, we obtain

$$R = U \text{diag}(d_1, \ldots, d_q) 0 V,$$

where the matrices $U \in D^{q \times q}$ and $V \in D^{p \times p}$ are unimodular, i.e., det $U$ and det $V$ are non-zero constants, diag denotes the diagonal matrix and $0 \neq d_i \in D$. Hence, $R$ can be written as:

$$\begin{cases}
R = R'' R', \\
R'' = U \text{diag}(d_1, \ldots, d_q) \in D^{q \times q}, \\
R' = (I_q 0) V \in D^{p \times p}.
\end{cases}$$

If we denote by $r = p - q$ and $V = (V_1^T V_2^T)^T$, $V_1 \in D^{q \times p}$, $V_2 \in D^{r \times p}$, then, from the latter of the previous equations, we obtain $R' = V_1$. Using the fact that $V$ is unimodular, then the entries of $V^{-1}$ belong to $D$. If we denote by $V^{-1} = (S \quad Q)$, where $S \in D^{p \times q}$ and $Q \in D^{p \times r}$, then we obtain:

$$\begin{pmatrix} R' \\ V_2 \\
\end{pmatrix} (S \quad Q) = I_p, \quad (1)$$

$$\begin{pmatrix} S \quad Q \\ R' \\
\end{pmatrix} V_2 = I_p. \quad (2)$$

Now, solving the system $R \eta = 0$ is equivalent to solve the following system:

$$\begin{cases}
R' \tau = 0, \\
\tau = R' \eta.
\end{cases}$$

The first system $R'' \tau = 0$ is equivalent to

$$d_1 \tau_1 = 0, \ldots, d_q \tau_q = 0, \quad (3)$$

where $\tau = (\tau_1 \ldots \tau_q)^T$. We denote by $\tau$ a fundamental solution of $R'' \tau = 0$ in a signal space $\mathcal{F}$ which has a $D$-module structure (e.g., $C^\infty, D'$).
Then, we need to solve the inhomogeneous system $R' \eta = \tau$. But, using (1), we obtain $R' S = I_q$, and thus, a particular solution for $R' \eta = \tau$ is given by $\eta = S \tau \in \mathcal{F}$. Moreover, (2) is equivalent to $S R' + Q V_2 = I_q$, and thus, if $R' \eta = 0$, then we have $\eta = S (R' \eta) + Q (V_2 \eta) = Q (V_2 \eta)$, showing that a general solution of the homogeneous system $R' \eta = 0$ is given by $\eta = Q \xi$ for a certain $\xi \in \mathcal{F}$. Therefore, $\text{Sol}_F(R) = \{ \eta \in \mathcal{F} \mid R \eta = 0 \}$ has the following explicit parametrization:

$$\eta = S \tau + Q \xi = (S \tau) \left( \begin{array}{c} \tau \\ \xi \end{array} \right), \quad \forall \xi \in \mathcal{F}. \quad (4)$$

We note that $\text{Sol}_F(R') = \text{Sol}_F(\text{diag}(d_1, \ldots, d_q))$

is a finite-dimensional $\mathbb{R}$-vector space. Let us call $l$ its dimension and let us denote by $\{ \theta_i \}_{1 \leq i \leq l}$ one of its bases. Then, the general solution $\tau$ can be written as $\tau = \sum_{i=1}^l c_i \theta_i$, where $c_i \in \mathbb{R}$. Therefore, we obtain the parametrization $\text{Sol}_F(R') = \Phi(\mathbb{R}^l \times \mathcal{F})$ where $\Phi$ is defined by:

$$\Phi : \mathbb{R}^l \times \mathcal{F} \longrightarrow \mathcal{F}, \quad (c_1 \ldots c_l, \xi)^T \longmapsto \sum_{i=1}^l c_i (S \theta_i) + Q \xi. \quad (5)$$

Finally, if the set of initial conditions of the system $R \eta = 0$ is known, then the corresponding constants $c_i$ can be explicitly computed.

We point out that the existence of non-trivial $d_i$ in the Smith form of $R$ (i.e., existence of $d_i \in D[R]$) is equivalent to the lack of controllability of the system $R \eta = 0$. The $\mathbb{R}$-vector space

$$\text{Sol}_F(R') = \{ \eta \in \mathcal{F} \mid R' \eta = 0 \} \quad (6)$$

called the controllable sub-behaviour of the behaviour $\text{Sol}_F(R) = \{ \eta \in \mathcal{F} \mid R \eta = 0 \}$, whereas

$$\text{Sol}_F(R') = \{ \tau \in \mathcal{F} \mid R' \tau = 0, \quad \tau = R' \eta \}$$

is called the autonomous behaviour. For time-invariant ordinary differential equations, it is well-known that $\text{Sol}_F(R')$ can be interpreted as a subbehaviour of $\text{Sol}_F(R)$ and we have:

$$\text{Sol}_F(R) = \text{Sol}_F(R') \oplus \text{Sol}_F(R''). \quad (7)$$

The controllable sub-behaviour $\text{Sol}_F(R')$ can be parametrized. See (Polderman and Willems, 1998; Pommaret and Quadrat, 1998) for more details.

The main interest of (4) is to parametrize the behaviour $\text{Sol}_F(R)$ and not simply the controllable sub-behaviour $\text{Sol}_F(R')$. Parametrizations (4) and (5) show how to glue elements of $\text{Sol}_F(R')$ with those of $\text{Sol}_F(R)$ in order to obtain all trajectories of the system $\text{Sol}_F(R)$.

The purpose of this paper is to show how and it is possible to extend the previous construction to multidimensional linear systems defined over Ore algebras (e.g., differential time-delay systems, partial differential equations, discrete systems) with constant or variable coefficients (Chyzak et al., 2005).

2. MODULE-THEORETIC APPROACH

For multidimensional linear systems defined over (non-commutative) multivariate polynomial rings, no Smith form exists. Therefore, we cannot copy the results obtained in the introduction in order to parametrize all solutions of such systems.

In order to cope with this problem, we introduce concepts of module theory. In what follows, $D$ denotes a (non-commutative) Ore algebra which is a left and right noetherian domain (Chyzak et al., 2005). Then, $D$ satisfies the left and right Ore properties, namely

$$\forall d_1, d_2 \in D, \exists (u_1, u_2), (v_1, v_2) \in D^2 \setminus (0, 0) : u_1 d_1 = u_2 d_2, \quad d_1 v_1 = d_2 v_2,$$

and

$$K = \{ d^{-1} n = \bar{n} \bar{d}^{-1} \mid 0 \neq d, n, 0 \neq \bar{d}, \bar{n} \in D \}$$

is the (left and right) quotient division ring of $D$.

Let $R \in D^{q \times p}$ and $M = D^{1 \times q}/(D^{1 \times q} R)$ be the finitely presented left $D$-module defined as the cokernel of the $D$-morphism:

$$R : D^{1 \times q} \longrightarrow D^{1 \times p}, \quad \lambda \longmapsto \lambda R. \quad (8)$$

The left $D$-module $M = D^{1 \times p}/(D^{1 \times q} R)$ is associated with the system $R \eta = 0$ in the sense that

$$\text{hom}_D(M, \mathcal{F}) = \{ \eta \in \mathcal{F} \mid R \eta = 0 \}, \quad (9)$$

where $\text{hom}_D(M, \mathcal{F})$ denotes the abelian group formed by the $D$-morphisms from $M$ to the left $D$-module $\mathcal{F}$. See (Pommaret and Quadrat, 2003) for more details. Moreover, $M$ is defined by the $D$-linear combinations of the equations $R y = 0$, where the components of $y = (y_1 \ldots y_p)^T$ are the generators of $M$, i.e., $y_i$ is the class in $M$ of the $i$th vector $e_i$ of the standard basis of $D^{1 \times p}$. See (Chyzak et al., 2005) for more details.

Definition 1. The left $D$-module $M$ is said to be:

- **Free** if there exists $r \in \mathbb{Z}_+$ such that $M$ is isomorphic to $D^{1 \times r}$ (denoted by $M \cong D^{1 \times r}$).
- **Projective** if there exist a left $D$-module $N$ and $r \in \mathbb{Z}_+$ such that $M \oplus N \cong D^{1 \times r}$.
- **Torsion-free** if the torsion-submodule $t(M) = \{ m \in M \mid \exists 0 \neq d \in D : d m = 0 \}$ of $M$ is trivial, i.e., $t(M) = 0$. Elements of $t(M)$ are called torsion elements of $M$.
- **Torsion** if $t(M) = M$. 


We give characterizations of the previous properties. We refer to (Pommaret and Quadrat, 1998; Pommaret and Quadrat, 2003) for the proofs.

**Theorem 1.** Let us consider $R \in D^{p \times p}$ and the left $D$-module $M = D^{1 \times p}/(D^{1 \times p} R)$. Then, we have:

1. $M$ is a free left $D$-module iff there exist $Q \in D^{p \times m}$ and $T \in D^{m \times p}$ such that:
   \[
   \begin{align*}
   \ker(Q) & \triangleq \{ \lambda \in D^{1 \times p} \mid \lambda Q = 0 \} = D^{1 \times q} R, \\
   QT & = I_m.
   \end{align*}
   \]
2. $M$ is a projective left $D$-module iff there exists $S \in D^{p \times m}$ such that $R S R = R$.
3. $t(M) = \left(\left(\left(K^{1 \times q} R\right) \cap D^{1 \times p}\right)/(D^{1 \times q} R)\right)$, and thus, $M$ is a torsion-free left $D$-module iff:
   \[
   (K^{1 \times q} R) \cap D^{1 \times p} = D^{1 \times q} R.
   \]

We recall some results (Rotman, 1979).

**Theorem 2.** (1) The following implications hold:

- $\text{free} \Rightarrow \text{projective} \Rightarrow \text{torsion-free}.$
- If $D = k[x_1, \ldots, x_n]$ is a commutative polynomial ring over a field $k$, then every projective $D$-module is free.
- If $D$ is a (left) principal ideal domain, i.e., every (left) ideal of $D$ can be generated by means of one element, (e.g., $D = \mathbb{R} \left[\frac{d}{dt}\right], \mathbb{R}(t) \left[\frac{d}{dt}\right]$), then every torsion-free (left) $D$-module $M$ is free.

There exists an algorithm which computes a matrix $R' \in D^{q \times p}$ such that:

\[
(K^{1 \times q} R) \cap D^{1 \times p} = D^{1 \times q} R'.
\]

See (Chyzak et al., 2005; Pommaret and Quadrat, 1998) for more details. This algorithm is implemented in the Maple package OreModules (Chyzak et al., 2003) (see procedure Extri) for rings of differential operators with polynomial or rational coefficients (i.e., $D = \mathbb{A}[\partial_1, \ldots, \partial_n]$, where $\mathbb{A} = \mathbb{R}, \mathbb{R}(x_1, \ldots, x_n)$, $\mathbb{R}(x_1, \ldots, x_n)$, for time-invariant time-delay systems (i.e., $D = \mathbb{R}[\frac{d}{dt}], \mathbb{R}[\frac{d}{dt}], \mathbb{R}[\frac{d}{dt}], \mathbb{R}[\frac{d}{dt}]$) with incommensurable delays $\delta_i$, $f(t) = f(t - h_i)$) and multidimensional discrete systems (i.e., $D = \mathbb{R}(\sigma_1, \ldots, \sigma_t)$, where $\sigma_i \mathbb{R}(n_1, \ldots, n_t) = \mathbb{R}(n_1, \ldots, n_{i+1}, \ldots, n_t)$ and $A = \mathbb{R}, \mathbb{R}(\delta_1, \ldots, \delta_n), \mathbb{R}(n_1, \ldots, n_t)$). Using OreModules, we can effectively test whether or not a finitely presented left $D$-module has non-trivial torsion elements, is torsion-free or projective.

**Definition 2.** A sequence of $D$-modules $P_i$ and $D$-morphisms $d_i : P_i \to P_{i-1}$ satisfying $d_i \circ d_{i+1} = 0$, i.e., $\text{im} d_{i+1} \subseteq \ker d_i$, is said to be exact at $P_i$ if $\ker d_i = \text{im} d_{i+1}$ and exact if it is exact at all $P_i$.

**Example 1.** The following sequence of morphisms

\[
0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0
\]

is exact iff $f$ is injective, $\text{im} f = \ker g$ and $g$ is surjective. (10) is called a short exact sequence.

Using the embedding $i$ of $t(M)$ into $M$ and the canonical projection $\rho$ of $M$ onto $M/t(M)$, we obtain the following short exact sequence:

\[
0 \to t(M) \xrightarrow{i} M \xrightarrow{\rho} M/t(M) \to 0.
\]

If $M = D^{1 \times p}/(D^{1 \times q} R)$, then, from the $D$-morphism (8), we have the exact sequence

\[
0 \to \ker R \to D^{1 \times q} \xrightarrow{R} D^{1 \times p} \xrightarrow{\pi} M \to 0,
\]

where $\pi(\lambda)$ is the class of $\lambda \in D^{1 \times p}$ in $M$.

**Definition 3.** A left $D$-module $\mathcal{F}$ is said to be injective if, for every short exact sequence (10), we have the following short exact sequence

\[
0 \to \text{hom}_D(M', \mathcal{F}) \xrightarrow{f^* \circ \text{hom}_D(M, \mathcal{F})} \text{hom}_D(M, \mathcal{F})
\]

where $f^*(\phi) \triangleq \phi \circ f$ for all $\phi \in \text{hom}_D(M, \mathcal{F})$.

**Theorem 3.** (Malgrange, 1966) If $\Omega$ is a convex open subset of $\mathbb{R}^n$, then $C^\infty(\Omega)$ and $D'f(\Omega)$ are injective $D = \mathbb{R}[\partial_1, \ldots, \partial_n]$-modules ($\partial_i = f/\partial x_i$).

**Definition 4.** A short exact sequence (10) splits if one of the following equivalent conditions holds:

1. There exists a $D$-morphism $h : M'' \to M$ such that $g \circ h = \text{id}_{M''}$.
2. There exists a $D$-morphism $k : M \to M'$ such that $k \circ f = \text{id}_{M'}$.
3. We have an isomorphism $M \cong M' \oplus M''$.

**Proposition 1.** (Rotman, 1979) If $M''$ is a projective $D$-module, then the short exact sequence (10) splits and we have $M \cong M' \oplus M''$.

3. A NECESSARY AND SUFFICIENT CONDITION

Let us investigate when the exact sequence (11) splits. A first case is when $M/t(M)$ is a projective left $D$-module. Indeed, by Proposition 1, the exact sequence (11) splits and we obtain:

\[
M \cong t(M) \oplus M/t(M).
\]

In particular, if $D = \mathbb{R}[\frac{d}{dt}]$ or $\mathbb{R}(t)[\frac{d}{dt}]$, then, using 3 of Theorem 2, we obtain that $M/t(M)$ is free, and thus, projective by 1 of Theorem 2. The same result holds over the ring $\mathbb{R}[t][\frac{d}{dt}]$ as every
torsion-free left \( D \)-module is projective (Chyzak et al., 2005). We shall show in Section 5 how the direct sum (13) of left \( D \)-modules implies the direct sum (7) between the controllable and autonomous sub-behaviours.

**Lemma 1.** Let us consider \( M = D^{1 \times p}/(D^{1 \times q} R) \) and \( M/t(M) = D^{1 \times p}/(D^{1 \times q} R') \). Then, there exists \( R'' \in D^{1 \times q} \) such that \( R = R'' R' \) and we have the following commutative exact diagram:

\[
\begin{array}{cccc}
D^{1 \times q} \overset{R'}{\longrightarrow} & D^{1 \times p} \overset{\pi}{\longrightarrow} & M & \longrightarrow 0 \\
R'' \downarrow & & \rho \downarrow & \\
D^{1 \times q'} \overset{R'''}{\longrightarrow} & D^{1 \times p} \overset{\pi'}{\longrightarrow} & M/t(M) & \longrightarrow 0,
\end{array}
\]  
(14)

where \( \pi' \) is defined by \( \pi' = \pi \circ \rho \).

**PROOF.** By 4 of Theorem 1, we have

\[
(K^{1 \times q} R) \cap D^{1 \times p} = D^{1 \times q} R'
\]

and we check that \( D^{1 \times q} R \subseteq (K^{1 \times q} R) \cap D^{1 \times p} \), which proves that \( D^{1 \times q} R \subseteq D^{1 \times q} R' \). Then, every row \( R_i \) of \( R \) belongs to \( D^{1 \times q} R' \), and thus, there exists \( R'' \in D^{1 \times q} \) such that \( R_i = R'' R' \). If we denote by \( R'' = ((R''_1)^T \ldots (R''_n)^T)^T \), then we obtain \( R = R'' R' \). The commutative diagram (14) directly follows from (11), (12) and \( R = R'' R' \).

The matrix \( R'' \in D^{q \times q} \) can be computed using the procedure FACTORIZOR OF OREMODULES.

**Theorem 4.** Let \( R \in D^{q \times p} \), \( M = D^{1 \times p}/(D^{1 \times q} R) \) be a left \( D \)-module and \( R' \in D^{p \times q} \) a matrix such that \( M/t(M) = D^{1 \times p}/(D^{1 \times q} R') \). Then, the short exact sequence (11) splits, i.e., we have (13), iff there exist \( S \in D^{p \times q} \) and \( V \in D^{q \times q} \) such that:

\[
R' - R' S R'' = V R.
\]

**PROOF.** Let us suppose that there exist matrices \( S \in D^{p \times q} \) and \( V \in D^{q \times q} \) satisfying (15) and let us denote by \( U = I_p - S R'' \). Then, we have:

\[
U = I_p - S R' \quad \text{and} \quad R' U = V R.
\]

From the last equality, we obtain the following commutative exact diagram:

\[
\begin{array}{cccc}
D^{1 \times q} \overset{R'}{\longrightarrow} & D^{1 \times p} \overset{\pi'}{\longrightarrow} & M/t(M) & \longrightarrow 0 \\
M \downarrow & & \rho \downarrow & \\
D^{1 \times q} \overset{R''}{\longrightarrow} & D^{1 \times p} \overset{\pi}{\longrightarrow} & M & \longrightarrow 0.
\end{array}
\]  
(17)

Then, the \( D \)-morphism \( h : M/t(M) \longrightarrow M \), defined by \( h(m') = \pi(\lambda U) \), where \( \lambda \in D^{1 \times p} \), is any element satisfying \( m' = \pi'(\lambda) \), is well-defined. Then, using \( \pi' = \rho \circ \pi \), for \( m' \in M/t(M) \), we have

\[
(\rho \circ h)(m') = \rho(\pi(\lambda U)) = \pi'(\lambda U),
\]

and thus,

\[
(\rho \circ h - id_{M/t(M)})(m') = \pi'(\lambda U) - \pi'(\lambda) = \pi'(\lambda(U - I_p)) = -\pi'(\lambda(S R')) = 0,
\]
because \( (\lambda S) R' \in D^{1 \times q} R' \) and \( \pi' \) is the canonical projection onto \( M/t(M) = D^{1 \times p}/(D^{1 \times q} R') \). Therefore, we have \( \rho \circ h = id_{M/t(M)} \) showing that (11) splits by 1 of Definition 4.

Conversely, let us suppose that there exists a \( D \)-morphism \( h \) satisfying \( \rho h = id_{M/t(M)} \). We denote by \( e_i \in D^{1 \times p} \) the vector with 1 in the \( i \)-th position and 0 elsewhere. Then, we have \( (h \circ \pi')(e_i) \in M \), and thus, there exists \( U_i \in D^{1 \times p} \) such that

\[
(h \circ \pi')(e_i) = \pi(U_i), \quad i = 1, \ldots, p,
\]
as \( \pi \) is surjective. If we define \( U = (U_1^T \ldots U_p^T)^T \in D^{p \times p} \), then we have \( \pi \circ (U) = h \circ \pi' \) and:

\[
\pi \circ (U \circ (R')) = h \circ \pi' \circ (R') = 0,
\]

\[
\implies D^{1 \times q} \circ (R' U') \subseteq \ker \pi = D^{1 \times q} R.
\]

In particular, if \( R'_i \) denotes the \( j \)-th row of \( R' \), then we have \( R'_j U \in D^{1 \times q} \), and thus, there exists \( V_j \in D^{1 \times q} \) such that \( R'_j U = V_j R \). If we denote by \( V = (V_1^T \ldots V_p^T)^T \in D^{p \times q} \), then we obtain \( R' U = V R \) and the commutative exact diagram (17). Composing (17) and (14), we obtain the following commutative diagram:

\[
\begin{array}{cccc}
D^{1 \times q} \overset{R'}{\longrightarrow} & D^{1 \times p} \overset{\pi'}{\longrightarrow} & M/t(M) & \longrightarrow 0 \\
(V R') \downarrow & & \rho \downarrow & \\
D^{1 \times q} \overset{R''}{\longrightarrow} & D^{1 \times p} \overset{\pi}{\longrightarrow} & M/t(M) & \longrightarrow 0.
\end{array}
\]  
(18)

We have \( (\rho \circ h)(\pi'(e_i)) = \pi'(e_i(U_i)) \), and using the fact that \( \rho \circ h = id_{M/t(M)} \), we obtain:

\[
\pi'(e_i) = \pi'(e_i(U_i)) \quad \Rightarrow \quad \pi'(e_i(I_p - U_i)) = 0,
\]

\[
\Rightarrow \exists S_i \in D^{1 \times q} : \quad e_i(I_p - U_i) = S_i R', \quad 1 \leq i \leq p,
\]

\[
\Rightarrow S = S_1^T \ldots S_p^T \in D^{p \times q} : \quad U = I_p - S R'.
\]

Therefore, we have just proved the existence of \( U \in D^{p \times p} \), \( V \in D^{q \times q} \) and \( S \in D^{p \times q} \) satisfying (16), or equivalently, (15) by eliminating \( U \).

We note that we have \( h(M/t(M)) \oplus t(M) = M \), where \( h \) is defined in the beginning of the proof.

We want to acknowledge this priority. However, the purposes of the last paper are different and we also study here the non-commutative case.

4. **ALGORITHMS**

We first consider the case where \( D \) is a commutative polynomial ring. Then, we use the fact that the product \( U \cdot V \cdot W \) of matrices \( U \in D^{a \times b} \), \( V \in D^{b \times c} \), \( W \in D^{c \times d} \) can be written as a row

\[
(V_1 \ldots V_b) \cdot (U^T \otimes W),
\]  
(19)
where $V_1, \ldots, V_q$ are the rows of $V$ and $\otimes$ denotes the tensor product of matrices. We continue to use single subscripts to denote the rows of a matrix. Then, it is easy to see that (15) can be written as:

\[
(R'_1 \ldots R'_{q'}) = (S_1 \ldots S_p) (R^T \otimes R') + (V_1 \ldots V_{q'}) (I_{q'} \otimes R).
\]

We obtain the inhomogeneous system $f T = g$:

\[
T = \begin{pmatrix} R^T \otimes R' \\ I_{q'} \otimes R \end{pmatrix} \in D^{(p+q') \times p'}, \quad (20)
\]

\[
g = (R'_1 \ldots R'_{q'}) \in D^{1 \times p'}. \quad (21)
\]

Algorithm 1. Input: $R \in D^{q \times p}$, $R' \in D^{d' \times p'}$.
Output: $U \in D^{p \times p}$ such that the $D$-module $h(M/t(M)) \cong \pi(D^{(1 \times p')})$ is a direct complement of $t(M)$ in $M$, or $\emptyset$ if no such complement exists.

**ComplementCoeff** $(R, R')$

Define $T$ and $g$ as in (20), (21).

$G \leftarrow$ Gröbner basis of the rows of $T$.

$r \leftarrow$ Normal form of $g$ modulo $G$.

if $r = 0$ then

From the reduction of $g$ modulo $G$,

find $f \in D^{(1 \times (p+q'))}$ s.t. $f T = g$.

Construct the matrix $S$ from $f$:

$S_{ij} \leftarrow f_{1(i-1)q'+j}, 1 \leq i \leq p, 1 \leq j \leq q'$.

return $U = I_p - S R'$

else

There exists no solution of (15); return $\emptyset$
end if

If now we consider the non-commutative Weyl algebra $D = K[x_1, \ldots, x_n][\partial_1, \ldots, \partial_n]$, then the product $U \cdot V \cdot W$ of matrices $U$, $V$, $W$ can no longer be written as in (19). If $S \in D^{q \times q'}$ were given, (15) could be viewed as the problem to factorize the matrix $R' - R' S R'$ as a product $V R$ with a suitable $V \in D^{q' \times q}$. Assuming $S_{ij} = \sum_{k,l \in \{0, \ldots, d\}} a_{k(i)} x^k \partial^l$ with indeterminates $a_{k(i)}$, we derive a system of equations in $a_{k(i)}$ that characterizes the above factorizability.

Algorithm 2. Input: $d \in \mathbb{N}, R \in D^{q \times p}$, $R' \in D^{d' \times p'}$.
Output: $U \in D^{d \times p}$ of order $d$ such that the left $D$-module $h(M/t(M)) \cong \pi(D^{(1 \times p')})$ is a direct complement of $t(M)$ in $M$, or $\emptyset$ if no such complement exists.

**Complement** $(R, R', d)$

Introduce the indeterminates $\lambda_j$, $j = 1, \ldots, p$,

$\mu_i, i = 1, \ldots, q$, and $k(i)$ over $D$.

$P \leftarrow \{\sum_{j=1}^p \lambda_j = \mu_i \mid i = 1, \ldots, q\}$.

Compute the Gröbner basis $G$ of $P$ in

$D \lambda_i \oplus \bigoplus_{j=1}^q D \mu_i \oplus \bigoplus_{k} D_k(i)$ w.r.t. an order which eliminates the $\lambda_i$'s.

Let $S_{ij} = \sum_{k,l \in \{0, \ldots, d\}} a_{k(i)} x^k \partial^l$.

$X \leftarrow R' - R' S R' \in D^{d' \times p}$; $H \leftarrow \emptyset$.

for $i = 1, \ldots, q$ do

$F_i \leftarrow$ normal form of $\sum_{j=1}^p X_{ij} \lambda_j$ modulo $G$.

Augment $H$ with all non-zero coefficients of $x^k \partial^l \lambda_j$ in $F_i$ for all $1 \leq k, l, \leq n, 1 \leq j \leq p$.

end for

Solve the linear system given by $H$ for $a_{k(i)}$.

if the linear system has a solution $(a_{k(i)})$ then

Plug all $a_{k(i)}$ into $S$.

return $U = I_p - S R'$.

else

No complement of order $d$; return $\emptyset$.
end if

5. PARAMETRIZING ALL SOLUTIONS

Now, we only investigate the case where condition (15) of Theorem 4 is fulfilled. Then, we have (13).

By applying the functor $\text{hom}_D(\cdot, \mathcal{F})$ to (14) and (17), we obtain the commutative exact diagrams

\[
\begin{align*}
0 \xrightarrow{\text{hom}_D(\pi)} & \text{hom}_D(t(M), \mathcal{F}) \\
\uparrow & \text{id} \\
\mathcal{F}'' \xrightarrow{\rho} \mathcal{F} & \text{Sol}_\mathcal{F}(R''') \xrightarrow{\text{Sol}_\mathcal{F}} 0, \\
\uparrow & \text{id} \\
\mathcal{F}' \xrightarrow{\rho} \mathcal{F} & \text{Sol}_\mathcal{F}(R') \xrightarrow{\text{Sol}_\mathcal{F}} 0, \\
\uparrow & \text{id} \\
\mathcal{F} \xrightarrow{\rho} \mathcal{F} & \text{Sol}_\mathcal{F}(R) \xrightarrow{\text{Sol}_\mathcal{F}} 0, \\
\uparrow & \text{id} \\
\mathcal{F} \xrightarrow{\rho} \mathcal{F} & \text{Sol}_\mathcal{F}(R) \xrightarrow{\text{Sol}_\mathcal{F}} 0.
\end{align*}
\]

where $k : M \rightarrow t(M)$ denotes the $D$-morphism satisfying $k \circ i = \text{id}_{t(M)}$. Then, we have:

\[
\text{Sol}_\mathcal{F}(R) = \rho^* (\text{Sol}_\mathcal{F}(R')) \oplus k^* (\text{hom}_D((D^{1 \times q'}/(D^{1 \times q} R), \mathcal{F})).
\]

$\text{Sol}_\mathcal{F}(R')$ is called the controllable sub-behaviour of $\text{Sol}_\mathcal{F}(R)$, whereas $\text{hom}_D(t(M), \mathcal{F})$ cannot generally be interpreted as a sub-behaviour of $\text{Sol}_\mathcal{F}(R)$.

However, in the previous case, $k^* (\text{hom}_D(t(M), \mathcal{F}))$ is a sub-behaviour of $\text{Sol}_\mathcal{F}(R)$ which we call the non-controllable sub-behaviour.

From (22), it follows that computing $\text{Sol}_\mathcal{F}(R)$ can be decomposed into two problems:
Theorem 5. Let $M' = D^1 \times p / (D^1 \times q \ R')$ be a torsion-free left $D$-module. Then, there exists a matrix $Q \in D^{p \times m}$ such that we have the following exact sequence $D^1 \times q' \to Q \to D^1 \times m$. We refer to (Chyzak et al., 2005) for a constructive proof and an implementation in OreModules.

Corollary 1. Let $F$ be an injective left $D$-module. With the hypothesis and the notations of Theorem 5, we obtain the following exact sequence $F' \xrightarrow{R'} F \xrightarrow{Q} F^m$, i.e., we have $\text{Sol}_F(R') = Q \ F^m$. This result holds for the $D = \mathbb{R} [\theta_1, \ldots, \theta_p]$-modules $F = C^\infty(\Omega)$ and $D'(\Omega)$ and an open convex subset $\Omega$ of $\mathbb{R}^n$.

If we denote by $\theta_i$ the class of the $i$th row of $R'$ in $t(M) = (D^1 \times q' / (D^1 \times q \ R))$, then $[\theta_i]_{1 \leq i \leq q'}$ is a family of generators of the torsion submodule $M$ of $M$ (Chyzak et al., 2005). Then, for every torsion element $\theta_i \neq 0$, there exists a family $\text{ann}_D(\theta_i)$ of non-zero elements of $D$ satisfying $\forall d \in \text{ann}_D(\theta_i), \ d \theta_i = 0$. We refer to (Chyzak et al., 2005; Pommaret and Quadrat, 1998) for a description of the algorithm computing $\text{ann}_D(\theta_i)$.

Lemma 2. Let us consider the left $D$-modules $M = D^1 \times p / (D^1 \times q \ R), M/t(M) = D^1 \times p / (D^1 \times q' \ R')$ and $R'' \in D^{q \times q'}$ the matrix defined by $R = R'' R'$ (see Lemma 1). Then, we have:

1. There exist $L \in D^{r \times q'}$ and $L' \in D^{r \times q}$ s.t.:

$$F \triangleq \ker \left( \begin{bmatrix} R' \\ R \end{bmatrix} \right) = D^{1 \times r} (L \ L')$$

2. If $\ker(R') = D^{1 \times r} T$, then we have:

$$F = D^{1 \times r} T \ (T \ 0) + D^{1 \times q} (R'' - I_q).$$

If $F$ is an injective left $D$-module, then $\tau_i$ defined in (24) satisfy the following equivalent systems:

$$L \tau = 0 \iff \begin{cases} R'' \tau = 0, \\ T \tau = 0. \end{cases}$$

Proof. 1 is satisfied as $D$ is a noetherian ring.

2. Let us consider $\lambda = (\lambda_1, \lambda_2) \in F$. Then, we have $\lambda_1 R' + \lambda_2 R = 0$ and, using $R = R'' R'$, we obtain $(\lambda_1 + \lambda_2 R'') R' = 0$, and thus, we have $\lambda_1 \lambda_2 R'' \in \ker(R') = D^{1 \times r'} T$. Then, there exists $\mu \in D^{1 \times r'}$ satisfying $\lambda_1 = \mu T - \lambda_2 R''$, implying $(\lambda_1, \lambda_2) = \mu (T - 0) - \lambda_2 (R'' - I_q) \Rightarrow \lambda = (D^{1 \times r'}(T - 0) + D^{1 \times q}(R'' - I_q))$. The converse inclusion trivially holds proving 2.

Now, applying $(L \ L')$ to the left of the system $R'' \eta = \tau, \ R \eta = 0$, we obtain $L \tau = 0$. Applying the matrix $\begin{bmatrix} T & 0 \\ R'' - I_q \end{bmatrix}$ to the left of (26), we obtain $T \tau = 0$ and $R'' \tau = 0$. The equivalences follow from the injectivity of $F$.

Using $I_p = (I_p - S R') + S R'$, i.e., $I_p = U + S R'$, for all $\eta \in \text{Sol}_F(R)$, we finally obtain:

$$\begin{cases} \eta = U \eta + S (R' \eta) = U \eta + S \tau, \\ U \eta \in \text{Sol}_F(R), \ \tau = (\tau_1, \ldots, \tau_q)^T = R' \eta, \\ \tau_i \in \text{hom}_F(t(M), F), \ 1 \leq i \leq q'. \end{cases}$$

Theorem 6. Let $F$ be an injective left $D$-module, $R \in D^q \times p$, $M = D^1 \times p / (D^1 \times q R)$ and $R' \in D^{q \times p}$ a matrix satisfying $M/t(M) = D^1 \times p / (D^1 \times q' \ R')$. If there exist $S \in D^{p \times q}$ and $V \in D^{q \times q'}$ satisfying (15), then every element $\eta \in \text{Sol}_F(R)$ has the form $\eta = S \tau + Q \xi, \ \forall \xi \in F^m$, where $Q \in D^{p \times m}$ is a matrix as in Theorem 5 and $\tau$ is a fundamental solution of (25) in $F'$. This result holds for the $D = \mathbb{R} [\theta_1, \ldots, \theta_p]$-modules $F = C^\infty(\Omega)$ and $D'(\Omega)$ and an open convex subset $\Omega$ of $\mathbb{R}^n$.

REFERENCES