QUADRATIC PERFORMANCE ANALYSIS FOR FINITE-HORIZON SYSTEMS

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Abstract: A finite dimensional condition is derived to test whether an integral quadratic constraint holds or not for a finite-horizon system with boundary conditions. A related parameter search problem is also considered and a cutting hyperplane generated by an infeasible parameter is derived. Copyright ©2005 IFAC

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1. PROBLEM FORMULATION

Consider a state-space equation

\[ \dot{x}(t) = Ax(t) + Bu(t) \]  

(1)

with a boundary condition

\[ \Omega x(0) + \Upsilon x(1) = 0 \]  

(2)

satisfying that

\[ \Sigma := \Omega + \Upsilon e^{A} \]  

(3)

is nonsingular, where \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, \Omega \in \mathbb{R}^{n \times n}, \) and \( \Upsilon \in \mathbb{R}^{n \times n}. \) The regularity of \( \Sigma \) is required for the well-posedness of (1) and (2). In fact (1) and (2) has a unique solution \( x = 0 \) for \( u = 0 \) if and only if \( \Sigma \) is nonsingular (Mirkin and Palmor, 1999).

The following is the first problem we study in this paper:

Problem 1. Let a real symmetric matrix \( \Pi = \Pi^* \in \mathbb{R}^{(n+m) \times (n+m)} \) be given. Determine whether

\[ \int_{0}^{1} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \Pi \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt < 0 \]  

(4)

holds for all \( u \in L_2[0, 1], u \neq 0 \) or not.

We remark that (4) is a finite-horizon IQC (integral quadratic constraint), and Problem 1 is motivated by the important role of (infinite-horizon) IQCs in recent robust control theory (Megretski and Rantzer, 1997; Rantzer and Megretski, 1997). The norm computation of finite-horizon systems, which is a special case of Problem 1, is required in \( H_\infty \) analysis and synthesis of delay systems (e.g. (Zhou and Khargonekar, 1987)) and sampled-data systems (e.g. (Chen and Francis, 1995)), and in the computation of the spatio-temporal frequency response of a class of spatially invariant systems (e.g. (Jovanović and Bamieh, 2003)). Hence it is expected that Problem 1 is required to be solved in order to develop a robust control theory based on IQCs for the systems mentioned above.

There are several analysis tools for infinite-horizon IQCs including the Kalman-Yakubovich-Popov lemma (Rantzer, 1996). This paper intends to provide a counterpart for finite-horizon IQCs based on the approach in (Dullerud, 1999; Fujioka, 2004), where norm computation of finite horizon systems is considered.

We also remark that a special case \( (\Omega = -\Gamma) \) of Problem 1 arises in robustness analysis of periodic systems (Kao et al., 2001; Jönsson et al., 2003). As in the infinite-horizon case, we also consider the following parameter search problem, which will be important for reduction of conservativeness of robustness analysis:
The proof is found in Appendix A.

2. QUADRATIC PERFORMANCE TEST

In this section, we provide a solution to Problem 1 as a condition on a matrix.

We introduce a partition of \( \Pi \):

\[
\Pi = \Pi' = \begin{bmatrix} \Pi_1 & \Pi_3 \\ \Pi_2 & \Pi_2 \end{bmatrix}
\]

where \( \Pi_1 \in \mathbb{R}^{n \times n}, \Pi_2 \in \mathbb{R}^{m \times m}, \) and \( \Pi_3 \in \mathbb{R}^{n \times m}. \) Then we have a condition so that the answer to Problem 1 is negative.

Proposition 3. There exists a \( u \in L_2[0, 1], u \neq 0 \) which violates (4) if \( \Pi_2 \) is not strictly negative-definite.

The proof is found in Appendix A.

Hence in the sequel we consider the case of \( \Pi_2 < 0 \) where the following Hamiltonian matrix \( H \) is well-defined:

\[
H := \begin{bmatrix} -A' & -\Pi_1 \\ 0 & A \end{bmatrix} - \begin{bmatrix} -\Pi_1 & \Pi_2 \end{bmatrix} \begin{bmatrix} B \\ \Pi_3 \end{bmatrix}.
\]

The following theorem provides a solution to Problem 1:

Theorem 4. Suppose that \( \Pi_2 < 0. \) The following two statements are equivalent:

(i) (4) holds for all \( u \in L_2[0, 1], u \neq 0. \)

(ii) \( \Phi < 0 \) where the matrix \( \Phi \) is defined as follows:

Step 1: Fix \( \theta \in (-\pi, \pi] \) such that

\[
e^{i\theta} \not\in \text{eig}(e^A), \quad e^{i\theta} \not\in \text{eig}(e^H).
\]

Step 2: Define \( M \) by

\[
M := R^* \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix} R,
\]

where

\[
Q := \int_0^1 e^{A't} \Pi_1 e^{At} \, dt,
\]

\[
R := \begin{bmatrix} \Xi^{-1}(\Omega e^{-j\theta} + \Upsilon) & 0 \\ 0 & I \end{bmatrix}.
\]

Define also \( W_m \) as in the bottom of this page where

\[
J := \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}.
\]

Step 3: Case 1) \( \text{eig}(H) \cap j\mathbb{R} \neq \emptyset: \) In this case

\[
\eta := \max \{ |\alpha| : \alpha \in \text{eig}(H) \cap j\mathbb{R} \} \geq 0
\]

is well-defined. Fix \( N \) as a nonnegative integer satisfying

\[ |\alpha_{N+1}| > \eta, \quad |\alpha_{N+2}| > \eta \]

where \( \{ \alpha_k \}_{k=0}^\infty \) is defined by

\[
\alpha_k := 2\pi v_k + \theta, \quad \{ v_k \}_{k=0}^\infty := \{ 0, 1, -1, 2, -2, \ldots \}.
\]

Then \( \Phi \) is defined by

\[
\Phi := \begin{bmatrix} K & 0 \\ 0 & -I \end{bmatrix} + \begin{bmatrix} L^* & 0 \\ V_{N+1} & M \end{bmatrix} \begin{bmatrix} L & V_{N+1} \end{bmatrix}
\]

where

\[
K := \begin{bmatrix} P_0^* \Pi P_0 & 0 \\ 0 & P_N^* \Pi P_N \end{bmatrix}, \quad L := \begin{bmatrix} S_0 & \cdots & S_N \end{bmatrix},
\]

\[
P_i := \begin{bmatrix} (j\alpha_i I - A)^{-1} B \\ I \end{bmatrix}, \quad S_i := \begin{bmatrix} -j\alpha_i I - A)^{-1} B \\ (j\alpha_i I - A)^{-1} \end{bmatrix} \begin{bmatrix} \Pi_1 & \Pi_3 \end{bmatrix} P_i.
\]

\( V_{N+1} \) is a column full rank matrix defined by a factorization:

\[
V_{N+1} W_{N+1} = W_m - \sum_{i=0}^N \bar{W}_i.
\]

where \( \bar{W}_i \) is given at the bottom of this page.
Case 2) $\text{eig}(H) \cap j\mathbb{R} = \emptyset$: In this case $\Phi$ is defined by
\[
\Phi := V_0'MV_0 - I, \quad V_0V_0' = W_w.
\]
where $V_0$ has its column full rank.

The proof is found in Appendix B.

3. SPECIAL CASES

Notice formally that $M$ in Theorem 4 is equal to 0 if
\[
\Omega e^{-j\theta} + \Upsilon = 0. \quad (11)
\]
Hence Theorem 4 is further simplified when (11) holds. Since both $\Omega$ and $\Upsilon$ are real matrices, (11) implies either (a) $\Omega = -\Upsilon$ and $\theta = 0$, or (b) $\Omega = \Upsilon$ and $\theta = \pi$.

In fact we can take $\theta = 0$ when $\Omega = -\Upsilon$, and $\theta = \pi$ when $\Omega = \Upsilon$. In the proof of Theorem 4, $M$ is constructed for $\theta$ satisfying $e^{j\theta} \not\in \text{eig}(e^{j\theta})$. In addition, once we find $M = 0$, we do not need additional conditions on $\theta$ like $e^{j\theta} \not\in \text{eig}(e^{j\theta})$. On the other hand, the regularity of $\Xi$ requires that $1 \not\in \text{eig}(e^{j\theta})$ when $\Omega = -\Upsilon$, and $-1 \not\in \text{eig}(e^{j\theta})$ when $\Omega = \Upsilon$, respectively.

In this section we will show reduced versions of Theorem 4 for the cases of $\Omega = -\Upsilon$ and $\Omega = \Upsilon$. We will also point out that both cases are related to periodic solutions of infinite horizon systems.

3.1 Case of $\Omega = -\Upsilon$

Noting the regularity of $\Xi$, the boundary condition in this case is
\[
x(0) = x(1).
\]
Then we can study periodic solutions (with period 1) of infinite horizon systems governed by (1). The reduced version of Theorem 4 for this case is given as follows:

Corollary 5. Suppose that $\Pi_2 < 0$ and $\Omega = -\Upsilon$. Then the following two statements are equivalent:

(i) (4) holds for all $u \in L_2[0, 1], u \neq 0$.
(ii) $\text{eig}(H) \cap j\mathbb{R} = \emptyset$, otherwise $P_i\Pi_iP_i < 0$ for all $i \in \{0, 1, \ldots, N\}$ where $N$ is defined as in Theorem 4 for $\theta = \pi$.

This case is closely related to (Kao et al., 2001; Jönsson et al., 2003). Moreover the approach in this paper is also closely related to the Fourier domain analysis in (Jönsson et al., 2003), where they derive a finite dimensional condition for time-varying $A, B$, and $\Pi$ under a certain assumption.

3.2 Case of $\Omega = \Upsilon$

In this case, the boundary condition is
\[
x(0) = -x(1),
\]
which is related to periodic signals $f$ with period 2 satisfying
\[
f(t) = -f(t + 1), \quad f(t) = f(t + 2).
\]

The reduced version of Theorem 4 for this case is given as follows:

Corollary 6. Suppose that $\Pi_2 < 0$ and $\Omega = \Upsilon$. Then the following two statements are equivalent:

(i) (4) holds for all $u \in L_2[0, 1], u \neq 0$.
(ii) $\text{eig}(H) \cap j\mathbb{R} = \emptyset$, otherwise $P_i\Pi_iP_i < 0$ for all $i \in \{0, 1, \ldots, N\}$ where $N$ is defined as in Theorem 4 for $\theta = \pi$.

4. RELATED FEASIBILITY PROBLEM

In this section we consider Problem 2. We here derive a cutting hyperplane generated from an infeasible parameter, with which one can easily construct a concrete cutting plane algorithm to solve Problem 2, as in (Kao et al., 2001; Jönsson et al., 2003).

Let us introduce a partition of $\hat{\Pi}_k (k = 0, 1, \ldots, q)$:
\[
\hat{\Pi}_k = \hat{\Pi}_k^e = \begin{bmatrix} \hat{\Pi}_{k1} & \hat{\Pi}_{k3} \\ \hat{\Pi}_{k2} & \hat{\Pi}_{k4} \end{bmatrix}
\]
where $\hat{\Pi}_{k1} \in \mathbb{R}^{n \times n}, \hat{\Pi}_{k2} \in \mathbb{R}^{n \times m},$ and $\hat{\Pi}_{k3} \in \mathbb{R}^{n \times m}$. The following theorem provides a cutting hyperplane:

Theorem 7. Given $\lambda \in \Lambda$ such that
- (5) is violated by $\lambda = \hat{\lambda}$ and some $u \in L_2[0, 1], u \neq 0$, and
- $\Pi_2 < 0$ where $\Pi_2 \in \mathbb{R}^{m \times m}$ is given in (7) for $\Pi$ defined by
\[
\Pi = \hat{\Pi}(\hat{\lambda}).
\]
The following two statements are equivalent:

(i) There exists a $\lambda \in \Lambda$ such that (5) holds for all $u \in L_2[0, 1], u \neq 0$.
(ii) There exists a $\lambda \in \Lambda \cap \{\lambda : \alpha + \beta^* \lambda \leq 0\}$ such that (5) holds for all $u \in L_2[0, 1], u \neq 0$, where $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}^q$ are defined as follows:

Step 1: Fix $\theta \in (-\pi, \pi]$ as in Theorem 4.

Step 2: Define $\hat{M}_k$ by
\[
\hat{M}_k := R^t \begin{bmatrix} \hat{Q}_k & (e^{i\theta}I - e^A)^* \end{bmatrix} R
\]
where
\[
\hat{Q}_k := \int_0^1 e^{A^*t}\hat{\Pi}_{k1}e^{Bt}dt.
\]
Define also $\hat{\Omega}_{k_0}$ and $\hat{\Gamma}_{k_0}$ by

$$
\hat{\Omega}_{k_0} := \hat{\Omega}_{k_0} + \hat{\Omega}_{k_0}^* + \hat{\Omega}_{k_0},
$$
$$
\hat{\Gamma}_{k_0} := W_m + \hat{\Omega}_{k_0} + \hat{\Omega}_{k_0}^*,
$$
respectively, where $W_m$ is defined in Theorem 4 and

$$
\hat{\Omega}_{k_0} := \begin{bmatrix} 0_n & 0 \\ -\frac{e^{i\theta}}{1 - e^{i\theta}} (e^{i\theta} - e^{-i\theta})^{-1} \end{bmatrix},
$$
$$
\hat{\Gamma}_{k_0} := \frac{1}{2} \left[ \begin{array}{c} \hat{C} \left( (e^{i\theta} - e^{-i\theta})^{-1} (e^{i\theta} + e^{-i\theta}) \right) \hat{B}, \\ \hat{C} \end{array} \right],
$$
$$
\hat{\Omega}_{k_0} := \frac{1}{2} \left[ \begin{array}{c} \hat{A} \hat{B} \hat{C} \hat{*} \end{array} \right] \left[ \begin{array}{c} -A^* \hat{F}_k \hat{I}_{2n} \\ 0 \hat{H} \hat{I}_{2n} \\ \hat{I}_{2n} \hat{I}_{2n} \end{array} \right],
$$
$$
\hat{\Gamma}_{k_0} := \hat{\Gamma}_{k_0}^* \hat{F}_k \hat{I}_{2n}^T \hat{I}_{2n}^T
$$

where $\hat{A}$, $\hat{B}$, and $\hat{C}$ are defined in Theorem 4, and

$$
\hat{\Omega}_{k_0} := (j\omega I - H)^{-1} \hat{E} (\hat{\Omega} - \Pi) \hat{E} (j\omega I - H)^{-1}.
$$

Case 2) $\text{eig}(H) \cap |\Re| = \emptyset$: Define $\hat{\Phi}_k$ by

$$
\hat{\Phi}_k := \hat{\Omega}_{k_0} - I + \hat{\Gamma}_{k_0} \hat{\Theta}_k \hat{\Gamma}_{k_0},
$$
$$
\hat{\Omega}_{k_0} := V_0^T \hat{\Omega}_{k_0} (V_0^T)^T, \quad \hat{\Gamma}_{k_0} := \hat{\Gamma}_{m} (V_0^T)^T.
$$

where $V_0$ is defined in Theorem 4.

Step 4: $\alpha$ and $\beta$ are given by

$$
\alpha := p^* (\hat{\Phi}_0 - \hat{\Phi}), \quad \beta_k := p^* \hat{\Phi}_k p
$$

where $\hat{\Phi}$ is defined in Theorem 4, and $p$ is a vector satisfying

$$
p^* \hat{\Phi} p \geq 0.
$$

The proof is omitted for the paper brevity.

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Appendix A. Proof of Proposition 3

Define an operator $G$ on $L_2[0, 1]$ by

$$
G : u \mapsto \begin{bmatrix} x \\ u \end{bmatrix}
$$

where $x$ is governed by (1) and (2). Consider the unitary operator $\Psi : L_2[0, 1] \rightarrow L_2$ mapping $f \mapsto \{ \phi_i \}_{i=0}^{\infty}$ defined by

$$
\phi_i := \int_0^1 e^{-j\omega f(t)} \, dt
$$

which is a key tool in (Dullerud, 1999). Identifying the matrix $\Pi$ and the corresponding multiplication operator on $L_2[0, 1]$, we have the following lemma (Fujikata, 2004):

**Lemma 8.** Assume that $e^{i\theta} \notin \text{eig}(e^{i\theta})$. The $(k, l)$-th block of the matrix expression of $\Psi G^* \Pi \Psi^*$ is given by

$$
\delta_{k_0} P_k^T \Pi P_l + S_k MS_l,
$$

where $P_k$, $S_l$, and $M$ are defined in (9), (10), and (8), respectively.

The proof completes by noting that

$$
\lim_{\delta \rightarrow \infty} (P_k^T \Pi P_l + S_k MS_l) = \Pi_2.
$$
Appendix B. PROOF OF THEOREM 4

By using $G$ and $\Psi$ defined in Appendix A, the purpose of Problem 1 is to check whether

$$G^*\Pi G < 0$$

holds or not.

Suppose that we have a unitary operator $U: L_2[0, 1] \to \mathbb{R}^n \boxtimes X$ for a Hilbert space $X$ such that $U G^* \Pi U^*$ is expressed as the sum of a block-diagonal and a finite rank operators:

$$
\begin{bmatrix}
K_0 & 0 \\
0 & \mathcal{K}
\end{bmatrix} +
\begin{bmatrix}
L_0 & 0 \\
0 & \mathcal{L}
\end{bmatrix} M_0 \begin{bmatrix}
L_0 & 0 \\
0 & \mathcal{L}
\end{bmatrix}
$$

where $K_0: \mathbb{R}^n \to \mathbb{R}^n$, $\mathcal{K}: X \to X$, $M_0: \mathbb{R}^n \to \mathbb{R}^n$, $L_0: \mathbb{R}^n \to \mathbb{R}^n$, $\mathcal{L}: X \to \mathbb{R}^n$, and furthermore $\mathcal{K} < 0$ holds. Then $G^* \Pi G < 0$ is equivalent to

$$
\begin{bmatrix}
K_0 & 0 \\
0 & -I
\end{bmatrix} +
\begin{bmatrix}
L_0 & 0 \\
0 & \mathcal{L}
\end{bmatrix} M_0 \begin{bmatrix}
L_0 & 0 \\
0 & \mathcal{L}
\end{bmatrix} < 0
$$

where $\mathcal{L} = \mathcal{L}(\mathcal{K})^{-\frac{1}{2}}$. This turns to

$$
I - \begin{bmatrix}
I & 0 \\
0 & \mathcal{L}^*
\end{bmatrix} \Theta \begin{bmatrix}
I & 0 \\
0 & \mathcal{L}^*
\end{bmatrix} > 0,
$$

$$
\Theta := \begin{bmatrix}
I + K_0 & 0 \\
0 & 0
\end{bmatrix} + \begin{bmatrix}
L_0^* & 0 \\
0 & I
\end{bmatrix} M_0 \begin{bmatrix}
L_0 & I
\end{bmatrix}.
$$

We then have an equivalent condition:

$$
\rho \left( \Theta \begin{bmatrix}
I & 0 \\
0 & W
\end{bmatrix} \right) < 1 \quad (B.1)
$$

where $W := \mathcal{L}^* \mathcal{K}^{-1} \mathcal{L}^*$. (B.1) is a finite dimensional condition since $W: \mathbb{R}^n \to \mathbb{R}^n$. With a (matrix) factorization of $W = VV^*$, (B.1) turns to

$$
I - \begin{bmatrix}
I + K_0 & 0 \\
0 & 0
\end{bmatrix} \Theta \begin{bmatrix}
I & 0 \\
0 & 0
\end{bmatrix} > 0,
$$

and hence

$$
\begin{bmatrix}
K_0 & 0 \\
0 & -I
\end{bmatrix} + \begin{bmatrix}
L_0^* & 0 \\
0 & V
\end{bmatrix} M_0 \begin{bmatrix}
L_0 & V
\end{bmatrix} < 0.
$$

The rest of the proof is a derivation of concrete formulas for $K_0, L_0, M_0$ and $V$, which is similar to that in (Fujioka, 2004), so it is omitted.

Appendix C. PROOF OF THEOREM 7

Let (5) be violated by $u = u_0$ when $\lambda = \hat{\lambda}$. Then $\alpha$ and $\beta$ are given by

$$
\alpha = \sigma_{\Pi_0}(u_0) - \sigma_{\Pi}(u_0), \quad \beta_k = \sigma_{\Pi_k}(u_0)
$$

since (5) is affine in $\lambda$ and

$$
\sigma_{\Pi_0}(u_0) + \beta^* \lambda = \sigma_{\Pi}(u_0) \geq 0
$$

where $\sigma_{\Pi}: L_2[0, 1] \to \mathbb{R}$ is defined by

$$
\sigma_{\Pi}(u) := \int_0^1 \begin{bmatrix} x(t) \end{bmatrix} \Pi \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt
$$

and $x$ is determined by (1) and (2). Hence our task here is to characterize $u_0$ and to derive formulas for $\sigma_{\Pi_k}(u_0)$ and $\sigma_{\Pi}(u_0)$.

With symbols used in Appendix B, we have

$$
\sigma_{\Pi} \left( \Psi^{-1} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} p \right) = p^* \Phi p
$$

for any compatible vector $p$ by using the following facts:

$$
-I = \mathcal{C}^* \mathcal{K} \mathcal{C}, \quad V = \mathcal{L} \mathcal{C}
$$

where

$$
\mathcal{C} := -\mathcal{K}^{-1} \mathcal{L}^* (\mathcal{L}^*)^\dagger.
$$

Hence we can characterize $u_0$ by

$$
u_0 = \Psi^{-1} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} p
$$

by taking $p$ as a vector satisfying $p^* \Phi p \geq 0$. Note that such a vector $p$ exists due to Theorem 4.

We have already seen that $\sigma_{\Pi}(u_0)$ is given by $p^* \Phi p$. Hence we derive a computational formula for $\sigma_{\Pi_k}(u_0)$ in the sequel. For the purpose we compute

$$
\hat{\alpha}_k := \mathcal{L} \mathcal{K}^{-1} \hat{\alpha}_k \mathcal{K}^{-1} \mathcal{L}^*
$$

and

$$
\hat{\Gamma}_k := -\mathcal{L}_k \mathcal{K}^{-1} \mathcal{L}^*.
$$

since

$$
\sigma_{\Pi_k} \left( \Psi^{-1} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} p \right) = p^* \left( \begin{bmatrix} \hat{\alpha}_k & 0 \\ 0 & \hat{\Gamma}_k \hat{\alpha}_k (\hat{\Gamma}_k)^\dagger \end{bmatrix} + \begin{bmatrix} \hat{\alpha}_k & 0 \\ 0 & \hat{\Gamma}_k \hat{\alpha}_k (\hat{\Gamma}_k)^\dagger \end{bmatrix} M_k \begin{bmatrix} \hat{\alpha}_k (\hat{\Gamma}_k)^\dagger \\ \hat{\Gamma}_k (\hat{\Gamma}_k)^\dagger \end{bmatrix} \right) p
$$

where $\mathcal{K}_k$ and $\mathcal{L}_k$ are respectively defined similarly to $\mathcal{K}$ and $\mathcal{L}$ but replacing $\Pi$ by $\hat{\Pi}_k$.

We get

$$
\tilde{\alpha}_k = \sum_{i=0}^\infty S_i (P_i^* \Pi P_i)^{-1} (P_i^* \hat{\Pi}_k P_i) (P_i^* \Pi P_i)^{-1} S_i,
$$

$$
\hat{\alpha}_k = -\sum_{i=0}^\infty \hat{S}_i (P_i^* \Pi P_i)^{-1} S_i,
$$

where $i_0$ is determined as in Appendix B.
We compute \( \tilde{\Omega}_k \) first. It is readily to see that
\[
\tilde{\Omega}_k = \sum_{i=0}^{\infty} S_i (P_i^* \Pi P_i)^{-1} (P_i^* (\tilde{\Pi}_k - \Pi) P_i) (P_i^* \Pi P_i)^{-1} S_i^* - \bar{W}.
\]

Noting that
\[
(P_i^* \Pi P_i)^{-1} S_i^* = \left( -\Pi_2^{-1} \tilde{C} (j \omega I - H)^{-1} B \Pi_2^{-1} + \Pi_2^{-1} \tilde{C} (j \omega I - \tilde{A})^{-1} \right) \Pi_2^{-1} \tilde{C} (j \omega I - H)^{-1}
\]
we have
\[
P_i (P_i^* \Pi P_i)^{-1} S_i^* = \left[ \begin{array}{c} (j \omega I - A)^{-1} B \\
\Pi_2^{-1} \tilde{C} (j \omega I - H)^{-1}
\end{array} \right] \Pi_2^{-1} \tilde{C} (j \omega I - H)^{-1}
\]
Substituting (C.1) we get
\[
S_i (P_i^* \Pi P_i)^{-1} (P_i^* (\tilde{\Pi}_k - \Pi) P_i) (P_i^* \Pi P_i)^{-1} S_i^* = \tilde{\Omega}_{ki}
\]
and hence
\[
\tilde{\Omega}_k = \sum_{i=0}^{\infty} \tilde{\Omega}_{ki} - \sum_{i=0}^{n-1} \tilde{\Omega}_{ki} - \bar{W}
\]
We also get
\[
\sum_{i=0}^{\infty} \tilde{\Omega}_{ki} = \tilde{\Omega}_{k\infty}, \quad \sum_{i=0}^{n-1} \tilde{\Omega}_{ki} = \tilde{\Omega}_{k\infty}, \quad \sum_{i=0}^{n-1} \tilde{\Omega}_{ki} = \tilde{\Omega}_{k\infty}.
\]
Consequently we have
\[
\sum_{i=0}^{\infty} \tilde{\Omega}_{ki} = \tilde{\Omega}_{k\infty}.
\]
Next we move to computation of \( \hat{I}_k \): We have
\[
\hat{S}_k = S_1 + \left[ \begin{array}{c} (j \omega I - A)^{-1} B (\tilde{\Pi}_k - \Pi_1 \tilde{\Pi}_3 - \Pi_3) P_i \end{array} \right].
\]
Noting (C.1), we have
\[
\left[ \begin{array}{c} (j \omega I - A)^{-1} B (\tilde{\Pi}_k - \Pi_1 \tilde{\Pi}_3 - \Pi_3) P_i \end{array} \right] \times (P_i^* \Pi P_i)^{-1} S_i^* = -\tilde{\Omega}_{ki} - \tilde{\Omega}_{ki}.
\]
Hence we get
\[
\tilde{I}_k = \sum_{i=0}^{\infty} \tilde{I}_k i
\]
and
\[
\tilde{I}_{k\infty} = \sum_{i=0}^{\infty} \tilde{I}_ki.
\]
This completes the proof of Theorem 7.

REFERENCES


