1. INTRODUCTION

An important problem in control consists in considering a specific structure on the overall control scheme especially when dealing with complex or distributed systems (Šiljak et al., 1991).

This structure often depends on the own structure of the global system when subdivided in subsystems. It also depends on the accessible measured signals and the authority on actuation variables of each separated controller included in the global desired controller. In general, finding an $H_2/H_\infty$ controller under structural constraints is considered as a non-convex optimization problem (see (Yagoubi and Chevrel 2001) and references therein).

In this paper we consider a special class of structured controller design problems with sparsity structural constraints. The constraints on the controller, considered in this paper, satisfy the so called quadratic invariance property (Voulgaris 2001, Rotkowitz and Lall 2002) with respect to the considered system. In this case, a convex optimization of the Youla parameter (Youla et al., 1976) is possible to find an optimal $H_2/H_\infty$ controller under such constraints and a method is proposed to realize that. This new method can be considered as an alternative to the one proposed in (Qi et al., 2003). An application to a LBT system (Stanković et al., 2000, Claveau et al., 2003) shows the efficiency of the method.

The paper is organized as follows: The principal
notations and the position of the problem are first presented in section 2. Section 3 introduces the method based on an optimization of the Youla parameter in a special basis. In section 4, the platoon of vehicles control problem taken as a benchmark (Stanković et al., 2000, Claveau et al., 2003) is considered in order to test the method and compare it with other ones.

2. PROBLEM STATEMENT

2.1 Notations

The standard scheme of Fig. 1, where $P$ and $K$ are both linear time-invariant systems, is considered throughout this paper.

![Standard scheme](image)

Fig. 1 Standard scheme

$P$ is the standard model associated to the process model $G = P_{22}$ (1), and $K$ is the controller to be designed (2).

$P(s)$ is defined by

$$ P(s) = \begin{bmatrix} p_{11}(s) & p_{12}(s) \\ p_{21}(s) & p_{22}(s) \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} \tag{1} $$

and $K(s)$ by

$$ K(s) = \begin{bmatrix} A_K \quad B_K \\ C_K \quad D_K \end{bmatrix} \cong \bar{K} \tag{2} $$

where $x \in \mathbb{R}^n$, $w \in \mathbb{R}^{n_u}$, $u \in \mathbb{R}^{n_u}$, $z \in \mathbb{R}^{n_z}$, $y \in \mathbb{R}^{n_y}$. $T_{zw}$ denotes the closed-loop transfer matrix between the exogenous inputs $w$ and the weighted output $z$

$$ T_{zw} = F_1(P, K) = P_{11} + P_{12}K(I - GK)P_{21} \tag{3} $$

Let us consider, without loss of generality, that

$$ T_{zw} = \begin{bmatrix} A_{cl} & B_{cl} & B_{cl2} \\ C_{cl} & D_{cl1} & D_{cl2} \end{bmatrix} \tag{4} $$

where $A_{cl}$, $B_{cl1}$, $B_{cl2}$, $C_{cl}$, $D_{cl1}$, $D_{cl2}$ and $D_{2cl}$ depends affinely on $\bar{K}$.

Let $\mathbb{R}^{m \times p}_{\text{mp}}$ (resp. $\mathbb{R}^{m \times p}_{\text{mp}}$) be the set of matrix-valued real-rational proper (resp. strictly proper) transfer functions. The structure constraint on the controller is defined as

$$ \bar{K} = \Lambda_K \otimes \bar{R}, \quad \Lambda_K \in \{0, 1\}^{(n_k \times n_k), (n_u \times n_u)} \tag{5} $$

where $\Lambda_K \in \{0, 1\}^{(n_k \times n_k), (n_u \times n_u)}$ and $\otimes$ denotes the direct product of matrices ($n_k$ is the order of the controller).

Note that it is straightforward to prove the convexity of the set

$$ \Omega_{\Lambda_K} := \{ K \in \mathbb{R}^{n \times n}, \quad \bar{K} = \Lambda_K \otimes \bar{R}, \quad \Lambda_K \in \{0, 1\}^{(n_k \times n_k), (n_u \times n_u)} \} $$

2.2 Structured $H_2$ / $H_\infty$ control problems

In this paper, $H_2$ and $H_\infty$ control problems under the structure constraint $K \in \Omega_{\Lambda_K}$ are considered. A bilinear matrix inequality formulation is given in both cases.

Pb 1. Structured optimal $H_2$ control problem

It consists in finding the optimal controller $K^*$ such that

$$ K^* = \arg \min_{K \in \Omega_{\Lambda_K}} \| T_{zw} \|_2 $$

$K^*$ may be obtained by solving the BMI optimization problem (6)

$$ \begin{bmatrix} \min \text{trace}(Y) \\ A_{cl}^T X_{cl} + X_{cl} A_{cl} + C_{cl}^T C_{cl} \quad - I \quad 0 \\ 0 \quad - Y \quad B_{cl}^T X_{cl} \quad 0 \\ X_{cl} B_{cl} \quad - X_{cl} \end{bmatrix} < 0 \tag{6} $$

$$ K = \Lambda_K \otimes \bar{K} $$

Pb 2. Structured optimal $H_\infty$ control problem

It consists in finding the optimal controller $K^*$ such that

$$ K^* = \arg \min_{K \in \Omega_{\Lambda_K}} \| T_{zw} \|_{\infty} $$

$K^*$ may be obtained by solving the BMI optimization problem (7)

$$ \begin{bmatrix} \min \gamma \\ A_{cl}^T X_{cl} + X_{cl} A_{cl} \quad X_{cl} B_{cl} \quad C_{cl}^T \quad - I \\ B_{cl}^T X_{cl} \quad - \gamma I \quad 0 \quad - \gamma I \\ C_{cl} \quad 0 \quad - \gamma I \end{bmatrix} < 0 \tag{7} $$

Thus, in the case where a structure constraint is imposed on the controller the $H_2$ and $H_\infty$ control problems are both formulated as BMI optimization problems that can not be reduced to an LMI.
This paper shows, however, that for a large class of structured $H_2$ and $H_\infty$ control problems where the controller and the system verify a special \textit{quadratic invariance} property, it is possible to reduce problems (6) and (7) to some LMI optimization problems thanks to the use of a structured Youla parameter.

2.3 Problem formulation

Given $G \in \mathbb{R}_p^{n\times n}$, let us define the map 
\[ h : \mathbb{R}_p^{n\times n} \rightarrow \mathbb{R}_p^{n\times n} \] 
by $h(K) = K(I - GK)^{-1}$ for all $K \in \mathbb{R}_p^{n\times n}$. Note that, if $G \in \mathbb{R}_p^{n\times n}$ then $(I - GK)$ is invertible for all $K \in \mathbb{R}_p^{n\times n}$.

\textbf{Definition: Quadratic invariance (Rotkowitz and Lall 2002)}

The subspace $S \subset \mathbb{R}_p^{n\times n}$ is said quadratically invariant under $G \in \mathbb{R}_p^{n\times n}$ if $K G K \in S$ for all $K \in S$.

In order to introduce a characterization for the quadratic invariance property we need some more notations.

Let $H^{\bin} \in \{0,1\}^{m\times p}$ be a binary matrix. The subspace $\mathcal{S}(H^{\bin})$ is then defined as 
\[ \mathcal{S}(H^{\bin}) = \{ \widetilde{H} \in \mathbb{R}^{m\times p} ; \widetilde{H}_{ij} = 0 \ \forall (i,j) \text{ such that } H_{ij} = 0 \} \]

Also, if $\widetilde{H} \in \mathbb{R}^{m\times p}$ is a transfer matrix then $Pa(\widetilde{H})$ denotes the binary matrix 
\[ Pa(\widetilde{H}) = H^{\bin} \text{ with } H^{\bin}_{ij} = \begin{cases} 0 & \text{if } \widetilde{H}_{ij} = 0 \\ 1 & \text{otherwise} \end{cases} \]

Suppose that the subspace $S \subset \mathbb{R}_p^{n\times n}$ is quadratically invariant under $G = P_{22} \in \mathbb{R}_p^{n\times n}$, $K_{\bin} G_{\bin} K_{\bin}^T (I - K_{\bin} G_{\bin}) = 0$, for all $i,l = 1,\ldots,n_1$ and $j,k = 1,\ldots,n_2$.

\textbf{Theorem 1 : (Rotkowitz and Lall 2002)}

Let $K^{\bin} \in \{0,1\}^{n\times n}$, $S = \mathcal{S}(K^{\bin})$ and $G^{\bin} = Pa(G)$. Then the subspace $S$ is said quadratically invariant under $G$ if and only if $K^{\bin} G^{\bin} K^{\bin}^T (I - K^{\bin} G^{\bin}) = 0$, for all $i,l = 1,\ldots,n_1$ and $j,k = 1,\ldots,n_2$.

\textbf{Proof : (cf. (Rotkowitz and Lall 2002))}

\textbf{Remarks :}

A particular case of such subspace is obtained when $K^{\bin}$ and $G$ are both LBT (Lower Block Triangular). In this case $S = \mathcal{S}(K^{\bin})$ is quadratically invariant under $G$. It is important to notice that $G$ and $S$ having the same sparsity structure is not in general sufficient to imply that $S$ is quadratically invariant under $G$.

\textbf{Theorem 2 :}

Suppose that the subspace $S \subset \mathbb{R}_p^{n\times n}$ is quadratically invariant under $G = P_{22} \in \mathbb{R}_p^{n\times n}$. The optimal structured problem $\min_{K \in S} \|F_i(P,K)\|_{\infty}$ (named $H_2 / H_\infty$ control problems under sparsity constraints) admits a solution if and only if there exists $Q^* \in S$ such that 
\[ K^* = h(Q^*) = Q^*(I - GQ^*)^{-1} \] 
and 
\[ Q^* = \arg \min_{Q \in S} \|P_{11} + P_{12}QP_{21}\|_{\infty} \]

\textbf{Proof :}

This result is a direct consequence of the Youla parameterization (Youla et al., 1976) and the fact that if $S$ is quadratically invariant under $G$ then $h(S)$ is S. i.e. $h(Q) = Q(I - GQ)^{-1}$, $\forall Q \in S$.

3. AN LMI APPROACH BASED ON THE PROJECTION OF THE YOULA PARAMETER

Consider the general feedback system shown in Fig. 1. The set of achievable stable closed loop maps is given by $\Phi = \{ T_{zw} = P_{11} + P_{12}QP_{21} / Q \in \mathbb{R}_p^{n\times n} \}$ where $Q$ is a free parameter and the transfer matrices $P_{11}$, $P_{12}$, $Q$ and $P_{21}$ are all stable. If the system is not stable a stabilization step under structure constraint has to be considered first.

Let $Q = \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix}$. Assume that $D_{11}$ and $D_{22}$ are null matrices. It is then clear that a realization for $T_{zw} = F_i(P,K) = P_{11} + P_{12}K(I - GK)P_{21}$ can be given by 
\[ T_{zw} = \begin{bmatrix} A_0 & B_0 & C_0 & D_0 \\ 0 & A_1 & B_1 & C_1 & D_1 \end{bmatrix} \begin{bmatrix} \bar{A}_0 & \bar{B}_0 & \bar{C}_0 & \bar{D}_0 \\ \bar{A}_1 & \bar{B}_1 & \bar{C}_1 & \bar{D}_1 \end{bmatrix} \]

with
\[ \begin{align*}
\bar{A}_i &= \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \quad \bar{A}_2 &= \begin{bmatrix} A & B_0 C_2 \\ 0 & A \end{bmatrix}, \quad \bar{B}_i &= \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad \bar{B}_2 &= \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \\
\bar{B}_i &= \begin{bmatrix} B_0 D_{2i} \\ B_i \end{bmatrix}, \quad \bar{C}_1 = [C_1 \quad C_1], \quad \bar{C}_2 = \begin{bmatrix} I & 0 \\ 0 & C_2 \end{bmatrix} \\
\bar{D}_i &= D_{12}, \quad \bar{D}_2 = \begin{bmatrix} 0 \\ D_{21} \end{bmatrix}, \quad M_0 = [C_0 \quad D_0] 
\end{align*} \]

In the following, the \( Q \)-parameter will be chosen either static or defined in an orthonormal basis (e.g. the Niness orthonormal basis (Niness and Gustafsson 1997)):

\[
Q(s) = \sum_{i=1}^{N} \theta_i Q_i(s) \tag{11}
\]

with:

\[
Q_i(s) = \sqrt{2 \text{Re}(q_i)} \prod_{k=1}^{i-1} \frac{s - a_k}{s - \bar{a}_k} \tag{12}
\]

The poles of the elements of the basis have to be chosen accordingly to the problem considered. (This point will be discussed later).

Note that \( A_0 \) and \( B_0 \) are fixed from (12) (the poles of the \( Q \)-basis are chosen \textit{a priori}). \( n_q \) is the assumed order of \( A_0 \).

If \( Q \) is restricted to be a static parameter the following matrices have to be used in (10)

\[
\bar{A}_i = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \quad \bar{A}_2 = A, \quad \bar{B}_i = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \\
\bar{B}_2 = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \quad \bar{B}_3 = B_1, \quad \bar{C}_1 = [C_1 \quad C_1], \quad \bar{C}_2 = C_2 \\
\bar{D}_1 = D_{12}, \quad \bar{D}_2 = D_{21}, \quad M_0 = Q 
\]

The \( H_{2,\infty} \) control problems consist now in finding a (static or a dynamic) parameter \( Q \) solution of (9).

Using the state space formulation (10), the only free parameter is \( M_0 \).

The structure constraint on the \( Q \)-parameter can be defined as

\[
M_0 = \Lambda \otimes M_0, \quad \Lambda \in \{0,1\}^{n_x \times n_x} \tag{13}
\]

where \( M_0 \in \mathbb{R}^{n_x \times n_x} \), \( \Lambda \in \{0,1\}^{n_x \times n_x} \).

Note that \( \Lambda \in \{0,1\}^{n_x \times n_x} \) is chosen such that the \( Q \)-parameter has the same structure as \( K \). The convex structure constraint (13) reduces the number of decision variables.

\textbf{Theorem 3}:

For given matrices \( A_0 \) and \( B_0 \), the \( H_2 \) (resp. \( H_{\infty} \)) control problem under sparsity constraints is equivalent to the LMI problem (14) (resp. (15)).

\textbf{Proof}:

Let us consider the Lyapunov function \( X_{2,\infty} \)

\[
X_{2,\infty} = \begin{bmatrix} W & Z \\ Z^T & Y \end{bmatrix}
\]

partitioned into

\[
\begin{bmatrix} \bar{A}_1 & \bar{B}_2 M_0 \bar{C}_2 \\ 0 & \bar{A}_2 \end{bmatrix}
\]

The matrix inequalities (6) and (7) corresponding to \( H_{2,\infty} \) control problems applied to the closed loop system are non linear in the decision variables \( M_0 \) and \( X_{2,\infty} \). They can, however, be transformed by a change of variable and a congruence transformation.

\[
\begin{bmatrix}
\bar{A}_1 R + R \bar{A}_1^T & \bar{A}_1 S - S \bar{A}_2 + \bar{B}_2 M_0 \bar{C}_2 & R \bar{C}_1^T \\
(\cdot)^T & \bar{T} \bar{A}_1 + \bar{A}_1^T T & S^T \bar{C}_1 + \bar{C}_1^T M_0^T \bar{D}_1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
 \end{bmatrix} < 0 \tag{14}
\]

\[
\begin{bmatrix}
\bar{A}_1 R + R \bar{A}_1^T & \bar{A}_1 S - S \bar{A}_2 + \bar{B}_2 M_0 \bar{C}_2 & R \bar{C}_1^T \\
(\cdot)^T & \bar{T} \bar{A}_1 + \bar{A}_1^T T & S^T \bar{C}_1 + \bar{C}_1^T M_0^T \bar{D}_1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
 \end{bmatrix} < 0 \tag{15}
\]
Let us consider the change of variable (see Khargonekar and Rotea 1991):
\[
\begin{bmatrix}
W & Z
\end{bmatrix}
\rightarrow
\begin{bmatrix}
R & S \\
S^T & T
\end{bmatrix}
= 
\begin{bmatrix}
W^{-1} & -W^{-1}Z \\
-ZTW^{-1} & Y-ZTW^{-1}Z
\end{bmatrix}
\]
(16)
and define
\[
N = \begin{bmatrix}
R & 0 \\
S^T & I
\end{bmatrix}
\]
(17)

Let \(\Theta_1 = \text{diag}(N, I, I, N)\) (resp. \(\Theta_2 = \text{diag}(N, I, I)\)), then pre-multiplying and post-multiplying the matrix inequality involved in (5) (resp. in (6)) by \(\Theta_1^T\) and \(\Theta_1\) (resp. by \(\Theta_2^T\) and \(\Theta_2\)) yields to the LMI problem (14) (resp. (15)).

**Remark:**
The sub-optimal \(H_2\) (resp. \(H_{\alpha}\)) control problem under sparsity constraints can be formulated as the LMI optimization problems (14) (resp. (15)) which depends affinely on the variables \(R, S, T, \gamma\) and \(M_0\).

4. AN LMI-BASED METHOD

4.1 Control of a platoon of vehicles

In order to appreciate the efficiency of the proposed algorithm, the speed control of a platoon of vehicles is now considered. Many papers consider this problem (see e.g. (Ozgüner and Perkins, 1978, Stanković, et al., 2000, Levine and Athans, 1966, Rajamani and Shladover, 2001) and references therein). Briefly stated, the problem consists in keeping the platoon of vehicles with a constant velocity and constant intra platoon separations. A platoon of vehicles is indeed a potential large-scale system being composed of interconnected subsystems. A structured controller is a natural solution to meet the technological constraints (local embedded controllers). Moreover, limitations on information sharing between vehicles will restrict the admissible structures for the control.

![Fig. 2. Vehicles moving in string](image)

As in (Stanković, et al., 2000), the very simple model of platoon of vehicles given by (18) will be used. Consider a set of \(N\) vehicles moving in a straight line as illustrated in Figure 2. The variables parameterizing the model of the \(i\)-th vehicle are \(h_i(t), v_i(t), m_i, u_i(t), g_i v_i(t) = -\alpha_i v_i(t)\) which are respectively the position of the \(i\)-th vehicle at time \(t\), its velocity, its mass, the force applied, and lastly the drag force action, assumed to be locally linearly dependent of \(v_i(t)\). All the states are assumed to be measured. By considering the following deviations
\[
\begin{align*}
x_i &= v_i - v_d, \\
x_i' &= v_j - v_d, \\
x_j' &= \Delta h_j - \Delta h_i, \quad i = 2, \ldots, N
\end{align*}
\]
where \(v_d\) and \(\Delta h_j\) are respectively the desired velocity and intra platoon separation, a state-space representation of a platoon of \(N\) vehicles can be defined as (19).

![State-space representation](image)

To focus on the methodology, it is assumed from now (as in (Ozgüner and Perkins, 1978)) that \(\alpha_i = 1, m_i = 1,\) and \(N = 3\).

4.2 The \(H_2\) criterion

The standard model \(P(s)\) (1) for \(H_2\) control is defined accordingly to (Stanković, et al., 2000) with \(B_2 = [10000000],\)
\[
C_2 = \begin{bmatrix}
\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}, D_{11} = 0, D_{21} = 0
\]
and \(D_{22} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

**Remark:**
The choice for \(B_2\) comes from the initial value \(x_i(t = 0) = 10\) considered in (Stanković, et al., 2000).

4.3 Results

The following structured controllers have been compared: the static controller \(K_1\) obtained by the local / sequential \(H_2\) procedure in (Ozgüner and Perkins, 1978), the static controller \(K_2\) optimized thanks to a BMI algorithm (the heuristic proposed in
(Yagoubi and Chevrely 2001) for solving $H_2$ structured problems was used, a dynamic controller $K_3$ obtained thanks to the method proposed in section 3 with a static $Q$ parameter and a dynamic controller $K_4$ obtained thanks to the method proposed in section 3 with a dynamic $Q$ parameter. The global criterions $\|P_\infty\|$ obtained for each of them are compared. The results are summed up in Tab. 1. The differences observed between $K_2$ and $K_3$ or $K_4$ comes from the fact that if $K_2$ is probably the best static controller, $K_3$ and $K_4$ are dynamic controllers.

Remarks:

Note that the system of a platoon of $N$ vehicles (19) is unstable. Before applying the proposed method based on the optimization of a static or a dynamic $Q$-parameter, a stabilization step was performed by a classical sequential stabilization procedure (see for example (Stanković et al., 2000, Özgüner and Perkins 1978)).

Tab. 1. Criteria for several structured controllers

<table>
<thead>
<tr>
<th>Controllers</th>
<th>$K_1$</th>
<th>$K_2$</th>
<th>$K_3$</th>
<th>$K_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$|P_\infty|$</td>
<td>12,135</td>
<td>11,676</td>
<td>11,602</td>
<td>11,450</td>
</tr>
<tr>
<td>Order</td>
<td>0</td>
<td>0</td>
<td>5</td>
<td>17</td>
</tr>
</tbody>
</table>

5. CONCLUSION

In general, finding an $H_2/H_\infty$-norm-minimizing controller under structural constraints is considered as a non-convex optimization problem (see (Yagoubi and Chevrely 2001) and references therein). In this paper we consider a special class of structured control design problems with sparsity structural constraints.

In that case, a simple condition is recalled to test the quadratic invariance property (Voulgaris 2001, Rotkowitz and Lall 2002). The proposed LMI-based method finds an $H_2/H_\infty$-norm-minimizing controller under sparsity constraints by optimizing a $Q$-parameter. This parameter is either static or defined on an orthonormal base.

The comparison between the BMI, sequential $H_2$ and the proposed method has shown the interest of this last one when applied to a LBT system. In fact, the LMI-based proposed method has some advantages when applied to such systems.

First, this method optimizes a global criterion on the contrary of some sequential methods (see (Stanković et al., 2000, Özgüner and Perkins 1978)). Secondly, the proposed method makes it possible to deal with a multi-objective or multi-criterion structured design. Finally, the static $Q$-parameter gives a satisfactory controller with the same order of the system in an relatively short time.

Model reduction can be included as a part of the method to prevent a rapid increase of the controller degree when a dynamic $Q$-parameter is used.

REFERENCES


