STABILIZATION OF LTI TIME-DELAYED PROCESSES
USING ANALYTICAL PID CONTROLLERS

Linlin Ou¹, Weidong Zhang¹,²*, Danying Gu¹

¹ Department of Automation, Shanghai Jiaotong University, Shanghai 200030, P. R. China
² Institute of System Theory in Engineering, University Stuttgart, Stuttgart 70550, Germany
Tel: +49.711.6856526; Fax: +49.711.6857735; Email: zhang@ist.uni-stuttgart.de

Abstract: The analytical proportional-integral-derivative (PID) controllers, the tuning of which depends on simple formulas with one adjustable parameter, have been developed for processes with time delay. In this paper, using a simple method called dual-locus diagram, the solutions to the problems of stabilizing stable, integrating and unstable processes with time delay using analytical PID controllers are presented, respectively. For the stable and integrating process, the stabilizing range of the adjustable parameter is only related to the delay of the plant, while, for the unstable process, it depends on both the time constant of the plant and the ratio of the delay to the time constant. With the available stabilizing solution, the time-consuming stability check can be avoided in the controller application. Copyright © 2005 IFAC

Keywords: stability, proportional-integral-derivative (PID) control, processes with time delay, analytical.

1. INTRODUCTION

Despite great development of advanced control strategies and continual improvement in control theory, the majority of control systems in the industrial process are operated by PID controllers (Åström, 1995). The popularity of PID controllers stems from their robust performance in a wide range of operating conditions and functional simplicity, which allow process engineers to operate them in a simple and straightforward manner (Ming, 2002). In terms of the fact that practical requirements on the design of control systems are usually specified in terms of time-domain or frequency-domain response, such as overshoot and stability margin, the analytical PID controllers that can meet these indexes have been developed based on optimal control theory for the stable, integrating and unstable processes with time delay, respectively (Zhang, 2002a). The analytical PID controller of this kind has only one adjustable parameter and can provide the quantitative time-domain and frequency-domain responses.

Although the analytical PID controllers can be operated more conveniently and provide good performance, they may fail to stabilize the original time-delayed plants in spite of their complete stabilization for the approximated plants since the rational approximation is employed in the controller design. Hence, certain constraint has to be imposed on the single adjustable parameter. However, it is not a trivial task to analyze the stability of the process with time delay since the corresponding closed-loop system has an infinite number of poles. Recently, by using the extended Hermite-Biehler Theorem, the complete set of the classical PID controllers for the first-order plants with time delay have been derived (Silva et al., 2002). Whereas, the method on the basis of the extended Hermite-Biehler Theorem is complex and there are great difficulties to achieve the stabilizing range of the adjustable parameter by taking this method. The D-partition technique has been used to derive the complete set of stabilizing PID controller parameters for first-order plus time-delayed unstable processes (Hwang, 2004), but it is inapplicable to the case of the analytical PID controller with single parameter since the nature of the D-partition technique is to determine the boundaries of the stability region by three equations. Though the method developed by Xu (2003) can be used to compute the stabilizing region of the PID controller parameter, the analytical results and the relation between the stabilizing region and the plant parameters will not be derived. Compared with these above-mentioned approaches, the analytical stabilizing criterion called dual-locus diagram offers remarkable simplicity and ease of mathematical
calculations. The dual-locus diagram was proposed by Satche (1949) and then developed by Smith (1958). Using the dual-locus, Olgac (1995) discussed the design of delayed resonators as tunable active vibration absorbers. Zhong (2003) applied it to the robust stability analysis of simple systems controlled over communication networks.

In this paper, the dual-locus diagram method is employed to analyze the problems of stabilizing processes with time delay using analytical PID controllers. The analytical ranges of the adjustable parameters for which the analytical PID controllers can stabilize stable, integrating and unstable processes with time delay are respectively presented. With the analytical stabilizing range of the adjustable parameter being available for a given process, it can avoid the time-consuming stability check in controller application and thereby to save the controller tuning time.

2. PRELIMINARY KNOWLEDGE FOR DUAL-LOCUS DIAGRAM METHOD

The dual-locus diagram method is based on the following argument principle.

Lemma 1 (Argument principle (Roger, 2000)): Let a function \( f \) be meromorphic in the domain interior if we denote as \( s \) is analytic and nonzero on \( \theta \), where \( \theta \) is the setpoint, and \( s \) is the time delay involved in the controller, \( sF \) corresponds to 1 () for \( s = 1 \), and \( sF \) is the number of zeros traverses the Nyquist contour. The argument of equation may be transformed into

\[
H(s) = -e^{\theta s}
\]

(3)

where \( H(s) \) is the delay-free open-loop transfer function and \( \theta \) is the time delay involved in the plants. Denote as \( m \) and \( n \) the degrees of the numerator and denominator of \( H(s) \), respectively. Here, \( H(s) \) and \( -e^{\theta s} \) correspond to \( F_1(s) \) and \( F_2(s) \) in Eq. (2), respectively. Hence, for the plants with time delay, the closed-loop system is stable if and only if the variation of the argument of \( H(s) + e^{\theta s} \) is zero. Equivalently, the criterion of the dual-locus diagram is simply illustrated as follows:

**Corollary 1**: If the following conditions are satisfied, the system is stable. Otherwise, it is unstable.

1) \( n > m \), or, \( |b_m/a_n| < 1 \) for \( n = m \), where \( a_n \) and \( b_m \) are leading coefficients of the numerator and denominator of \( H(s) \), respectively.

2) Either the loci of \( H(s) \) and \( -e^{\theta s} \) have no intersection or the locus of \( H(s) \) arrives at the point of intersection earlier than that of \( -e^{\theta s} \) if the two loci intersect.

It is seen that the dual-locus diagram method for the systems with time delay employs a counterclockwise unity circle prepared beforehand and only requires the plotting of a simple curve derived from the open-looped transfer function free of delay. Thus, the criterion presents remarkable simplicity and ease of mathematical calculations over other available conventional methods.

![Feedback control system equation](image)

**Fig. 1. Feedback control system equation**

3. STABILIZATION USING ANALYTICAL PID CONTROLLERS

For the purpose of controller design and system analysis, practical industrial processes are frequently expressed as first-order and second-order time-delayed models. Since the stability analysis process of the analytical PID controller for second-order plants with time delay is similar to the case of first-order, only the first-order plants with time delay are considered in this paper. The feedback control system is shown in Fig. 1 where \( G(s) \) is the plant to be controlled, \( C(s) \) is the analytical PID controller, \( r \) is the setpoint, and \( y \) is the output of the plant. The analytical PID controller is of the following type

\[
C(s) = K_c(1 + \frac{1}{T_i s} + T_p s) \frac{1}{T_p s + 1}
\]

(4)

where \( K_c, T_i, T_p \) and \( T_p \) are the functions with respect to \( \lambda \) if we denote as \( \lambda \) the adjustable positive parameter of the analytical PID controller.
The cases of stable, integrating and unstable processes with time delay are considered as follows.

### 3.1 Stable processes with time delay

For the first-order stable plant with the following transfer function

\[ G(s) = \frac{k}{\tau s + 1} e^{-\theta s} \]  

(5)

\[ K_c, \ T_p, \ T_d \] and \[ T_r \] of the analytical PID controller \( C(s) \) in (4) are respectively written as (Zhang, 1996):

\[ T_p = \frac{\lambda^2}{2\lambda + \theta / 2}, \quad T_i = \tau + \frac{\theta}{2} \]

\[ T_d = \frac{T_r}{2T_p}, \quad K_c = \frac{T_i}{K(2\lambda + \theta / 2)} \]

The characteristic equation of the system in Fig.1 is given by

\[ 1 + C(s)G(s) = 1 + \frac{(1 + \theta s / 2)}{\lambda^2 s^2 + (2\lambda + \theta / 2)s} e^{-\theta s} = 0 \]  

(6)

It is equivalent to

\[ H(s) = -e^{-\theta s} \]  

(7)

where

\[ H(s) = \frac{2 + \theta s}{2\lambda^2 s^2 + (4\lambda + \theta)s} \]  

(8)

The loci of \( H(s) \) and \( -e^{-\theta s} \) are shown in Fig.2. When \( \omega \) increases from 0 to \( +\infty \), the locus of \( -e^{-\theta s} \) is the counterclockwise unit circle starting at \((-1,0)\) and the locus of \( H(s) \) is a curve from the bottom up in Fig.2. The locus of \( H(s) \) corresponding to \( \omega = -\infty \) is symmetric with respect to the real axis and is thus not taken into account. Since the transfer function \( H(s) \) has one pole at the origin, its Nyquist plot shifts clockwise from \( \pi / 2 \) to \(-\pi / 2 \) with the infinite radius when \( s \) changes from \(-j0 \) to \( j0 \). Firstly, compute the critical frequency \( \omega_c \) at which the locus of \( H(s) \) intersects with the unity circle.

\[ \frac{\theta \omega_j + 2}{2\lambda^2 (\omega_j)^2 + (4\lambda + \theta)\omega_j} = 1 \]  

(9)

Simplifying (9) yields

\[ 4\lambda^2 \omega_c^4 + (16\lambda^2 + 8\lambda \theta)\omega_c^2 - 4 = 0 \]  

(10)

Eq.(10) has four analytical roots, of which the valid solution for \( \omega_c \) is

\[ \omega_c = \sqrt[5]{4\beta^2 + 4\beta^4 + \beta^2 - 2\beta^2 - \beta} \]  

(11)

where \( \beta = \lambda / \theta \). The other three roots are either negative or complex values. The phase angle of \( H(s) \) at \( \omega_c \) is

\[ \phi_1 = \arctan(\theta \omega_c / 2) + \arctan[(4\lambda + \theta)/(2\lambda^2 \omega_c)] + \pi \]  

(12)

and the phase angle of \( -e^{\theta s} \) at \( \omega_c \) is

\[ \phi_1 = \pi + \theta \omega_c \]  

(13)

From Fig.2, it is seen that the stability condition that the locus of \( H(s) \) arrives at the intersection earlier than that of \( -e^{\theta s} \) can be satisfied only when the phase angle of \( H(s) \) at \( \omega_c \) is larger than that of \( -e^{\theta s} \) (i.e. \( \phi_1 - \phi_2 > 0 \)). Taking \( \phi_1 - \phi_2 = 0 \) and substituting (11)-(13) into it yield

\[ \phi_1 - \phi_2 = \arctan(\theta \omega_c / 2) + \arctan[(4\lambda + \theta)/(2\lambda^2 \omega_c)] \]

\[ -\theta \omega_c \]  

(14)

It is clear that Eq.(14) is only related to \( \beta \) (i.e. \( \lambda / \theta \)). The solution \( \lambda / \theta = 0.0735 \) of Eq.(14) can be obtained by using the function \( \text{fzero} \) of the Matlab. Fig.3 shows that the requirement \( \phi_1 - \phi_2 > 0 \) can just be satisfied only if \( \lambda / \theta > 0.0735 \). Moreover, from (8), it is known that the denominator degree of \( H(s) \) is larger than its numerator. Therefore, in terms of the criterion of dual-locus diagram in

![Fig. 2 The loci of $H(s)$ in (8) and $-e^{\theta s}$](image)

Corollary 1, the range of the adjustable parameter \( \lambda \) for which the analytical PID controller can stabilize the first-order stable plant with time delay is that \( \lambda > 0.0735 \theta \).

### 3.2 Integrating process with time delay

The model of the integrating process with time delay is described as

\[ G(s) = \frac{k}{s} e^{-\theta s} \]  

(15)

If the approximation \( e^{-\theta s} = (1 - \theta s / 2)/(1 + \theta s) \) is adopted in Zhang (1999), the analytical PID controller designed for the plant with transfer function (15) has the following parameters
The other five roots of (19) are either negative or complex values. Assume that $\beta = \lambda / \theta$, then the Eq.(20) can be rewritten as
$$\omega_e = f(\beta) / \theta$$
where $f(\beta)$ denotes the function with respect to $\beta$.

The phase angle of $H(s)$ at $\omega_e$ is
$$\varphi_e = \arctan\left[\frac{(12\beta + 4\theta)\omega_e}{4 - (6\lambda + 4\theta)\omega_e}\right]$$
$$\arctan\left[\frac{4\beta^2 f(\beta)}{12\beta^2 + 6\lambda + 1}\right] - \arctan f(\beta) = 0$$
(25)

It is clear that Eq.(25) is only related to $\beta$ (i.e. $\lambda / \theta$). The solution of Eq.(25) is that $\lambda / \theta = 0.3614$, so the requirement $\varphi_e - \pi > 0$ can be satisfied only if $\lambda / \theta > 0.3614$. Thus, in order to stabilize the integrating plant with the transfer function in (15) using the analytical PID controller, the value of the adjustable parameter $\lambda$ must be larger than 0.3614.$\theta$.

3.3 Unstable process with time delay

For the unstable process with the following transfer function
$$G(s) = \frac{k}{\tau s + 1}e^{\sigma\tau}$$
the values of $K_c$, $T_T$, $T_D$ and $T_e$ in the analytical PID controller (4) are presented by (Zhang, 2002b).

$$T_e = \frac{4\lambda^3}{12\lambda^2 + 6\lambda \theta + \theta^2}, \quad T_T = 3\lambda + \theta$$
$$T_D = \frac{6\lambda \theta + \theta^2}{4\lambda T_e}, \quad K_c = \frac{1}{K 12\lambda^2 + 6\lambda \theta + \theta^2}$$

The characteristic equation of the system in Fig.1 is given by
$$1 + \left(\frac{6\lambda \theta + \theta^2}{4\lambda T_e}\right)s^2 + \left(\frac{12\lambda + 4\lambda}{4\lambda T_e}\right)s + 4 = 0$$
(16)

In terms of Eq. (3), we have
$$H(s) = \frac{6\lambda \theta + \theta^2}{4\lambda} s^2 + \left(\frac{12\lambda + 4\lambda}{4\lambda T_e}\right)s + 4$$
(17)

The loci of $H(s)$ and $-e^{\sigma\tau}$ are shown in Fig.4. Since the transfer function $H(s)$ has two poles at the origin, its Nyquist plot shifts clockwise from $\pi$ to $-\pi$ with the infinite radius when $s$ changes from $-j0$ to $j0$. Similar to the case of the stable process, only the positive values of $\omega$ are considered. The frequency $\omega_e$ satisfies the equation
$$\left[\frac{6\lambda \theta + \theta^2}{4\lambda T_e}\right]\left(j \omega_e\right)^2 + j \omega_e\left(\frac{12\lambda + 4\lambda}{4\lambda T_e}\right) + 4 = 0$$
(18)

Simplifying (18), we have
$$a \omega_e^6 + b \omega_e^4 + c \omega_e^2 - 1 = 0$$
(19)

where
$$a = \lambda^6$$
$$b = 3(6\lambda^2 + 6\lambda \theta + \theta^2)^2$$
$$c = -(18\lambda^2 + 6\lambda \theta + \theta^2)^2$$

Eq. (19) has six analytical roots, among which the valid solution for $\omega_e$ is
$$\omega_e = \sqrt[3]{\frac{d + e + \sqrt[3]{d^2 + e^2}}{3a}}$$
(20)

where
$$d = -\frac{q}{2} \quad \text{and} \quad e = \sqrt[3]{\frac{q^2}{2} + \left(\frac{p}{3}\right)^3}$$
(21)

$p$ and $q$ in (21) are given by the following equations, respectively.
$$p = \frac{c - b^2}{3a^2} \quad \text{and} \quad q = \frac{2b^3 - bc - \frac{1}{a}}{27a^3}$$
(22)
Taking $\beta = \lambda / \tau$ and $m = \theta / \tau$, the valid solution for $\omega_c$ can be obtained from Eq. (29).

$$\omega_c = \frac{1}{\tau} \sqrt{-b + \sqrt{b^2 - 4ac}}$$

(30)

where

$$a = (\beta + m)^2$$
$$b = (\beta + m)^2 - (\beta^2 + 2\beta m + m)^2$$
$$c = -(1-m)^2$$

The phase angle of $H(s)$ at $\omega_c$ is

$$\varphi_1 = \arctan \left( \frac{(\beta^2 + 2\beta m + m)\omega_c}{1-m} \right)$$

(31)

+ $\arctan(\tau\omega_c) + \frac{\pi}{2}$

and the phase angle of $-e^{j\omega}$ at $\omega_c$ is given in (13). Assume that $\varphi_1 - \varphi_2 = 0$ for $\beta = \beta_0$. Then, from (30) and (31), we have

![Fig. 5. The loci of $H(s)$ in (28) and $-e^{j\omega}$](image)

![Fig. 6. The relation between $\lambda / \tau$ and $\theta / \tau$](image)

4. CONCLUSIONS

In this paper, the problems of stabilizing processes with time delay using analytical PID controllers have been considered. By employing the dual-locus diagram method, the stabilizing range of the adjustable parameter in the analytical PID controller is respectively presented for three different cases: stable, integrating and unstable time-delayed processes. For the stable and integrating processes, the stabilizing range depends on the delay of the plant, and for the unstable processes, it is related to both the delay and time constant of the plant. The presented stabilizing solutions provide convenience for the tuning of the analytical PID controller.

ACKNOWLEDGMENT

This paper is supported by National Science Foundation of China (60274032), SRDFP (20030248040), SRSP (04QMH1405), and the Alexander von Humboldt Research Fellowship.

REFERENCES

Åström, K. J. and Haagglund, T (1995), *PID controllers: theory, design, and tuning*, Research Triangle Park, NC, USA.


