DIRECT ADAPTIVE CONTROL FOR NONLINEAR
UNCERTAIN SYSTEMS WITH TIME DELAY

Tomohisa Hayakawa *

* CREST, Japan Science and Technology Agency,
Saitama, 332-0012, JAPAN
tomohisa_hayakawa@ipc.i.u-tokyo.ac.jp

Abstract: A direct adaptive control framework for nonlinear uncertain delay dynamical systems is developed. The proposed framework is Lyapunov-Krasovskii-based and guarantees asymptotic stability with respect to the plant states. Specifically, if the nonlinear system is represented in normal form, then it is shown that nonlinear adaptive controllers can be constructed without requiring knowledge of the system dynamics except the system delay amount. Furthermore, in the case where the system is particularly given in a multivariable second-order form, the adaptive control law is shown to be simplified and constructed without even requiring the information of the delay amount. Finally, a numerical example is provided to demonstrate the efficacy of the proposed approach.

1. INTRODUCTION

The presence of time delay effects in complex, modern controlled systems can severely degrade closed-loop system performance, and in some cases drive the system to instability. Furthermore, it is unavoidable that there exist discrepancies between real-world systems and their system models that are constructed for control purposes. It is easily surmised that the applying controls to a physical system involving coupled sources of these effects may produce highly undesirable system response such as oscillatory behavior, actuator failure, and even chaos.

In the face of such system uncertainties as well as time delays, research on adaptive control methodologies is still far from complete. Specifically, even though recent notable results concerning adaptive controllers is given in Foda & Mahmoud (1998), Wu (2000), Wu (2002), and Niculescu & Annaswamy (2003), these approaches can handle either linear or a very special class of nonlinear systems with known system delays to show ultimate boundedness (practical stability) rather than Lyapunov stability.

In this paper we develop an adaptive control framework for nonlinear uncertain systems in the presence of system time delays. In particular, in the first part of the paper, a Lyapunov-Krasovskii-based direct adaptive control framework is developed that requires the knowledge of the system delay amount and guarantees partial asymptotic stability of the closed-loop system; that is, Lyapunov stability of the overall closed-loop systems states and attraction with respect to the plant states. As a consequence, the adaptive gain states are shown to be bounded. In the case where the nonlinear system is represented in normal form (Isidori 1995) with input-to-state stable internal dynamics (Sontag 1989, Isidori 1995), we construct nonlinear adaptive controllers without requiring knowledge of the system dynamics except the delay amount. In addition, the proposed nonlinear adaptive controllers also guarantee asymptotic stability of the system state if the system dynamics are unknown and the input matrix function is parameterized by an unknown constant sign-definite matrix. Finally, in the second part of the paper, we specialize the aforementioned results to multivariable second-order uncertain nonlinear systems. In this case, we remove the assumption that the system delay amount is known. This implies that the adaptive control framework becomes delay-independent.
The notation used in this paper is fairly standard. Specifically, \( \mathbb{R} \) denotes the set of real numbers, \( \mathbb{R}^{n \times n} \) denotes the set of \( n \times n \) real matrices, \((\cdot)^T\) denotes transpose, and \( I_n \) denotes the \( n \times n \) identity matrix. Furthermore, we write \( \text{tr}(\cdot) \) for the trace operator, \( \| \cdot \| \) for the Euclidean vector norm, and \( \| \cdot \|_F \) for the Frobenius matrix norm. Finally, \( M \otimes N \) denotes the Kronecker product of matrices \( M \) and \( N \).

2. DIRECT ADAPTIVE CONTROL FOR DELAY DYNAMICAL SYSTEMS

In this section we consider the problem of characterizing direct adaptive feedback control laws for nonlinear uncertain systems with time delay. Specifically, consider the nonlinear uncertain delay dynamical system \( \mathcal{G} \) of the form

\[
\dot{x}(t) = f(x(t)) + f_d(x(t), x(t - \tau)) + G(x(t))u(t), \\
x(\theta) = \eta(\theta), \quad -\tau \leq \theta \leq 0, \quad t \geq 0, \quad (1)
\]

where \( x(t) \in \mathbb{R}^n \) is the state vector, \( u(t) \in \mathbb{R}^m \) is the control input, \( f : \mathbb{R}^n \to \mathbb{R}^n \) and satisfies \( f(0) = 0, f_d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) and satisfies \( f_d(0, 0) = 0, \tau \geq 0 \) is a system delay amount, \( G : \mathbb{R}^n \to \mathbb{R}^{n \times m}, \eta(\cdot) \in \mathcal{C} = C([-\tau, 0], \mathbb{R}^n) \) is a continuous vector-valued function specifying the initial state of the system, and \( \mathcal{C}([-\tau, 0], \mathbb{R}^n) \) denotes a Banach space of continuous functions mapping the interval \([-\tau, 0]\) into \( \mathbb{R}^n \) equipped with the topology of uniform convergence.

Note that the state of \( (1) \) at time \( t \) is the piece of trajectories \( x \) between \( t - \tau \) and \( t \), or, equivalently, the element \( x_t \) in the space of continuous functions defined on the interval \([-\tau, 0]\) and taking values in \( \mathbb{R}^n \); that is, \( x_t \in \mathcal{C}([-\tau, 0], \mathbb{R}^n) \), where \( x_t(\theta) \triangleq x(t + \theta), \theta \in [-\tau, 0] \). Furthermore, since for a given time \( t \) the piece of the trajectories \( x_t \) is defined on \([-\tau, 0]\), the uniform norm \( \| x_t \| = \sup_{\theta \in [-\tau, 0]} \| x(t + \theta) \| \) is used for the definitions of Lyapunov and asymptotic stability of \( (1) \) with \( u(t) \equiv 0 \). For further details see Krassovski (1963) and Hale & Verduyn Lunel (1993). The control \( u(\cdot) \) in \( (1) \) is restricted to the class of admissible controls consisting of measurable functions such that \( u(t) \in \mathbb{R}^m, t \geq 0 \). Furthermore, for the nonlinear uncertain system \( \mathcal{G} \) we assume that the required properties for the existence and uniqueness of solutions are satisfied; that is, \( f(\cdot), f_d(\cdot), G(\cdot), \) and \( u(\cdot) \) satisfy sufficient regularity conditions such that \( (1) \) has a unique solution forward in time.

**Theorem 2.1.** Consider the nonlinear uncertain delay dynamical system \( \mathcal{G} \) given by \((1)\). Assume there exist matrices \( K_g \in \mathbb{R}^{m \times s}, K_{dg} \in \mathbb{R}^{m \times s} \), a continuously differentiable function \( V_d : \mathbb{R}^n \to \mathbb{R} \), and continuous functions \( V : \mathbb{R}^n \to \mathbb{R}, \quad G : \mathbb{R}^n \to \mathbb{R}^{n \times m}, \quad \dot{G} : \mathbb{R}^n \to \mathbb{R}^{n \times m}, \quad \dot{F} : \mathbb{R}^n \to \mathbb{R}^n, \quad F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, \) and \( \ell : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) such that \( V_d(\cdot) \) and \( V(\cdot) \) are positive definite, radially unbounded, \( V_d(0) = 0, t(0, 0) = 0, F(0) = 0, F_0(0) = 0 \), and, for all \( x \in \mathbb{R}^n \) and \( x_{d} \in \mathbb{R}^n \),

\[
0 = V'_d(x)f_d(x) + V'_d(x)f_d(x, x_d) + V_d(x) \\
- V_d(x_{d}) + \ell^T(x, x_d)\ell(x, x_d), \quad (2)
\]

where

\[
f_s(x) \equiv f(x) + G(x)\dot{G}(x)K_gF(x),
\]

\[
f_d(x, x_d) \equiv f_d(x, x_d) + G(x)\dot{G}(x)K_{dg}F_d(x, x_d).
\]

Furthermore, let \( Q \in \mathbb{R}^{m \times m}, Q_0 \in \mathbb{R}^{m \times m}, Y \in \mathbb{R}^{\ell \times \ell}, \) and \( Y_d \in \mathbb{R}^{n \times s \times d} \) be positive definite. Then the adaptive feedback control law

\[
u(t) = \dot{G}(x(t))K(t)F(x(t)) + \dot{G}_d(x(t))K_{dg}(t)F_d(x(t), x(t - \tau)), \quad (5)
\]

where \( K(t) \in \mathbb{R}^{m \times s} \) and \( K_{dg}(t) \in \mathbb{R}^{m \times s} \), with update laws

\[
\dot{K}(t) = -\frac{1}{2}Q\dot{G}^T(t)G^T(t)V_u'(x(t)) \\
\cdot F^T(t)y_0, \quad K(0) = K_0,
\]

\[
\dot{K}_{dg}(t) = -\frac{1}{2}Q_d\dot{G}^T_d(t)G^T(t)V_u'(x(t)) \\
\cdot F^T(t)(x(t), x(t - \tau))y_d, \quad K(0) = K_{dg0}, \quad (6)
\]

guarantees that the solution \( x(t), K(t), K_{dg}(t) \equiv (0, K_g, K_{dg}) \) of the closed-loop system given by \((1), (5)-(7)\) is Lyapunov stable and \( t \to \infty \).

**Proof.** Note that with \( u(t), t \geq 0 \), given by \((5)\) it follows from \((1)\) that

\[
\dot{x}(t) = f(x(t)) + f_d(x(t), x(t - \tau)) + G(x(t))\dot{G}(x(t))K(t)F(x(t)) + G(x(t))\dot{G}_d(x(t))K_{dg}(t)F_d(x(t), x(t - \tau)),
\]

\[
x(\theta) = \eta(\theta), \quad -\tau \leq \theta \leq 0, \quad t \geq 0, \quad (8)
\]

or, equivalently,

\[
\dot{x}(t) = f(x(t)) + f_d(x(t), x(t - \tau)) + G(x(t))\dot{G}(x(t))(K(t) - K_{dg})F(x(t)) + G(x(t))\dot{G}_d(x(t))K_{dg}(t) - K_{dg}(t)
\]

\[
\cdot F_d(x(t), x(t - \tau)),
\]

\[
x(\theta) = \eta(\theta), \quad -\tau \leq \theta \leq 0, \quad t \geq 0. \quad (9)
\]

To show Lyapunov stability of the closed-loop system \((6), (7)\), and \((9)\) consider the Lyapunov-Krasovskii functional candidate \( V : \mathcal{C} \times \mathbb{R}^{m \times s} \times \mathbb{R}^{m \times s} \to \mathbb{R} \) given by

\[
V(\psi, K, K_{dg}) = V_0(\psi(0)) + \int_0^\tau V_{sd}(\psi(\theta))d\theta
\]

\[
+ \text{tr} Q^{-1}(K - K_{dg})G^{-1}(K - K_{dg})^T + \text{tr} Q_d^{-1}(K_{dg} - K_{dg})y_d^{-1}(K_d - K_{dg})^T,
\]

\[
\psi(\cdot) \in \mathcal{C}, \quad (10)
\]

where \( \psi(\theta) \equiv x(\theta)(\theta) \). Note that \( V(\psi_0, K_g, K_{dg}) = 0 \), \( \psi_0(\theta) = 0, \theta \in [-\tau, 0] \). Furthermore, note that there exist class \( K_\infty \) functions \( \alpha_1(\cdot), \alpha_2(\cdot), \alpha_3(\cdot) \) such that

\[
V(\psi, K, K_{dg}) \geq \alpha_1(\| \psi(0) \|) + \alpha_2(\| K - K_{dg} \|_F)
\]

\[
+ \alpha_3(\| K_{dg} - K_{dg} \|_F), \quad (11)
\]
Now, letting $x(t)$ denote the solution to (9) and using (2), (6), and (7), it follows that the Lyapunov-Krasovskii directional derivative along the closed-loop system trajectories is given by

$$
\dot{V}(x(t), K(t), \dot{K}(t)) = V'(x(t)) f_a(x(t)) + f_{da}(x(t), x(t) - \tau)
$$

where $f_a(x(t))$ is defined in Hayakawa (2005). The case where (13) possesses input-to-state stable internal dynamics can be handled as shown in Hayakawa et al. (2005).

Next, define $x_i \triangleq \left[ q_i, \ldots, q_i^{(r_i-2)} \right]^T$, $i = 1, \ldots, m$, $x_{m+1} \triangleq \left[ q_i^{(r_i-1)}, \ldots, q_i^{(n_i-1)} \right]^T$, and $x \triangleq \left[ x_1^T, \ldots, x_{m+1}^T \right]^T$, so that (13) can be described as (1) with

$$
G(x) = \begin{bmatrix}
0_{(n-m) \times m} \\
0_{m \times n} \\
0_{m \times m}
\end{bmatrix},
$$

where $A_0 \in \mathbb{R}^{n \times n}$ is a known matrix of zeros and ones capturing the multivariable controllable canonical form representation (Chen 1984), $f_{a_i} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $f_{da_i} : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ are unknown functions such that $f_{a_i}(0) = 0$ and $f_{da_i}(0, 0) = 0$, and $x_a$ denotes the delayed value of $x$. Here, we assume that $f_{a_i}(x)$ and $f_{da_i}(x)$ are unknown and are parameterized as $f_{a_i}(x) = \Theta_0 f_{a_i}(x)$ and $f_{da_i}(x, x_a) = \Theta_2 f_{da_i}(x, x_a)$, where $f_{a_i} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and satisfies $f_{a_i}(0) = 0$, $f_{da_i} : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ and satisfies $f_{da_i}(0, 0) = 0$, and $\Theta_0 \in \mathbb{R}^{m \times q}$ and $\Theta_2 \in \mathbb{R}^{m \times q_a}$ are matrices of uncertain constant parameters.

Next, to apply Theorem 2.1 to the uncertain system (1) with $f(x)$, $f_a(x, x_a)$, and $G(x)$ given by (14), let $K_g \in \mathbb{R}^{m \times s}$ and $K_{da} \in \mathbb{R}^{m \times s_a}$, where $s = q + r$ and $s_a = q_a + r_a$, be given by

$$
K_g = [\Theta_n - \Theta, \Phi_n], \\
K_{da} = [\Theta_{da} - \Theta_2, \Phi_{da}],
$$

where $\Theta_n \in \mathbb{R}^{m \times q}$, $\Theta_{da} \in \mathbb{R}^{m \times q_a}$, $\Phi_n \in \mathbb{R}^{m \times r}$, and $\Phi_{da} \in \mathbb{R}^{m \times r_a}$ are known matrices, and let

$$
F(x) = \begin{bmatrix}
\hat{f}_a(x) \\
\hat{f}_{da}(x, x_a)
\end{bmatrix}, \\
\hat{f}_a(x, x_a) = \begin{bmatrix}
f_{a_i}(x, x_a) \\
\hat{f}_{da_i}(x, x_a)
\end{bmatrix},
$$

where $\hat{f}_a \in \mathbb{R}^m \rightarrow \mathbb{R}^m$ and $\hat{f}_{da_a} : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^{m \times r_a}$, with $\hat{f}_a(0) = 0$ and $\hat{f}_{da_a}(0, 0) = 0$, are arbitrary functions. In this case, it follows that, with $\hat{G}(x) = \hat{G}_a(x) = G_a^{-1}(x)$.

It is important to note that the adaptive control law (5)–(7) does not require explicit knowledge of the gain matrices $K_g$ and $K_{da}$. Theorem 2.1 simply requires the existence of $K_g$ and $K_{da}$ along with the construction of $F(x)$, $\hat{f}_a(x, x_a)$, $\hat{G}(x)$, $\hat{G}_a(x)$, $V_a(x)$, and $V_a(x)$ such that (2) holds. However, no specific structure on the nonlinear dynamics $f(x)$ is required to apply Theorem 2.1. However, if (1) is in normal form with asymptotically stable internal dynamics (Isidori 1995), then we can always construct functions $F : \mathbb{R}^n \rightarrow \mathbb{R}^s$ and $F_a : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^{s_a}$, with $F(0) = 0$ and $F_a(0, 0) = 0$, such that the condition (2) is satisfied. To see this assume that the nonlinear uncertain system $G$ is generated by

$$
q_i^{(r_i)}(t) = f_{u_i}(q(t)) + f_{du_i}(q(t), q(t - \tau)) + \sum_{j=1}^{m} G_{s_i}(q(t)) u_j(t), \\
t \geq 0,
$$

where $q = [q_1, \ldots, q_i^{(r_i-1)}, \ldots, q_m, \ldots, q_m^{(r_m-1)}]^T$, $q(\theta) = \eta(\theta), -\tau \leq \theta \leq 0$, $q_i^{(r_i)}$ denotes the $r_i$th derivative of $q_i$, and $r_i$ denotes the relative degree with respect to the output $q_i$. Here we assume that the square matrix function $G_s(q)$ composed of the entries $G_{s_i}(q)$, $i, j = 1, \ldots, m$, is such that det $G_s(q) \neq 0$, $q \in \mathbb{R}^d$, where $\tau = r_1 + \cdots + r_m$ is the (vector) relative degree of (13). Furthermore, since (13) is in a form where it does not possess internal dynamics, it follows that $\hat{r} = n$. The case where (13) possesses input-to-state stable internal dynamics can be handled as shown in Hayakawa et al. (2005).
Given by (14) and \( G_i(x) = B_iG(x), \) where \( B_i \) is an unknown symmetric matrix and the sign definiteness of \( B_{ii} \) is known. Assume there exist matrices \( K_R \in \mathbb{R}^{m \times s}, K_{Sd} \in \mathbb{R}^{m \times q}, \) a continuously differentiable function \( V_{e} : \mathbb{R}^n \rightarrow \mathbb{R}, \) and continuous functions \( V_{sd} : \mathbb{R}^n \rightarrow \mathbb{R}, \ F : \mathbb{R}^n \rightarrow \mathbb{R}^p, \ F_{d} : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^s, \) and \( \ell : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^p \) such that \( V_{e}(\cdot) \) and \( V_{sd}(\cdot) \) are positive definite, radially unbounded, \( V_{e}(0) = 0, \) \( V_{sd}(0) = 0, \) \( \ell(0,0) = 0, \) \( F_{e}(0) = 0, \) \( F_{d}(0,0) = 0, \) and, for all \( x \in \mathbb{R}^n \) and \( x_d \in \mathbb{R}^n, \) (2) holds. Finally, let \( Y \in \mathbb{R}^{r \times n}, \) and \( Y_d \in \mathbb{R}^{r \times m} \) be positive definite. Then the adaptive feedback control law

\[
\begin{align*}
\dot{u}(t) &= G_i^{-1}(x(t))K_i(t)F(x(t)) \\
&\quad + G_i^{-1}(x(t))K_d(t)F_d(x(t), x(t - \tau)), \quad (22)
\end{align*}
\]

where \( K(t) \in \mathbb{R}^{m \times s} \) and \( K_d(t) \in \mathbb{R}^{m \times q}, \) with update laws

\[
\begin{align*}
\dot{K}(t) &= -\frac{1}{2}B_i^r V_{e}^T(x(t))F^T(x(t))Y, \\
\dot{K}_d(t) &= -\frac{1}{2}B_i^r V_{e}^T(x(t))F_d^T(x(t), x(t - \tau))Y_d, \quad (23)
\end{align*}
\]

guarantees that the solution \((x(t), K(t), K_d(t)) \equiv (0, K_R, K_{Sd})\) of the closed-loop system given by (1), (22)–(24) is Lyapunov stable and \( \ell(x(t), x(t - \tau)) \rightarrow 0 \) as \( t \rightarrow \infty. \) If, in addition, \( F^T(x(t), x_d(t)) > 0, \) \((x, x_d) \in \mathbb{R}^n \times \mathbb{R}^m, (x, x_d) \neq (0, 0), \) then \( \|x\| \rightarrow 0 \) as \( t \rightarrow \infty \) for all \( \eta \in C. \)

**Proof.** The result is a direct consequence of Theorem 2.1. First, let \( G(x) = \hat{G}_d(x) = G_i^{-1}(x) \) so that \( G(x)\hat{G}_d(x) = G(x)G_d(x) = [0_{m \times (n-m)}, B_{ii}]^T. \) Second, since \( Q \) and \( Q_d \) are arbitrary positive-definite matrices, \( Q \in (6) \) and \( Q_d \) in (7) can be replaced by \( q|B_{ii}|^{-1} \) and \( q_d|B_{ii}|^{-1}, \) respectively, where \( q \) and \( q_d \) are positive constants and \( |B_{ii}| = (B_{ii})^2, \) where \( (\cdot)^+ \) denotes the (unique) positive-definite square root. Now, since \( B_{ii} \) is symmetric and sign definite it follows from the Schur decomposition that \( B_{ii} = U_DU_D^T, \) where \( U \) is orthogonal and \( D_{ii} \) is real diagonal. Hence, \( |B_{ii}|^{-1}G(x)^T(x)G(x)T(x) = |B_{ii}|^{-1}G_i^T(x)G_i(x)T(x) = |B_{ii}|^{-1}I_{m_i} \) where \( I_{m_i} = I_{m_i} \) for \( B_{ii} > 0 \) and \( I_{m_i} = -I_{m_i} \) for \( B_{ii} < 0. \) Now, (6) and (7), with \( qY \) and \( q_dY_d \) replaced by \( Y \) and \( Y_d, \) respectively, imply (23) and (24), respectively.

**3. DIRECT ADAPTIVE CONTROL FOR SECOND-ORDER SYSTEMS WITH UNKNOWN TIME DELAY**

In this section we present a result that does not require knowledge of the delay amount \( \tau. \) Specifically, in this section, we consider the nonlinear uncertain matrix second-order delay dynamical system given by (13) with the relative degree given by \( r_1 = \cdots = r_m = 2. \) With \( x_1 \triangleq [q_1, \ldots, q_m]^T, \) \( x_2 \triangleq [q_1, \ldots, q_m]^T, \) and \( x \triangleq [x_1, x_2]^T, \) it follows that the state space representation is equivalently given by (1) with \( n = 2m, \) \( f(x), f_a(x, x_d), \) and \( \hat{G}_d(x) \) given by (14). Note that \( \hat{A} \) in (14) is given by \( \hat{A} = \begin{bmatrix} 0_m & I_m \\ 0_m & 0_m \end{bmatrix}. \) Here, we assume that \( f(x), f_a(x, x_d), \) and \( \hat{G}_d(x) \) are uncertain, \( f_a(x) \) is parameterized as \( f_a(x) = \Theta f_a(x), \) where \( f_a : \mathbb{R}^{2m} \rightarrow \mathbb{R}^s \) and satisfies \( f_a(0,0,0) = 0, \) \( \Theta \in \mathbb{R}^{2m \times q} \) is a matrix of uncertain constant parameters, and \( f_a(\cdot, \cdot) \) belongs to \( F_d, \) where

\[
\begin{align*}
&f_a(x) = f(x) + G(x)\hat{G}(x)K_g F(x) \\
&= \hat{A}x + \hat{f}_a(x) + \begin{bmatrix} 0_{(n-m) \times m} \end{bmatrix} G_s^{-1}(x) \\
&\cdot \left[ \Theta \hat{f}_a(x) + \Phi \hat{f}_a(x) \right] \\
&= \hat{A}x + \begin{bmatrix} 0_{(n-m) \times 1} \end{bmatrix} + \left[ \Theta \hat{f}_a(x) + \Phi \hat{f}_a(x) \right] \\
\end{align*}
\]
\[ \mathcal{F}_d \triangleq \{ f_d : \mathbb{R}^{2m} \times \mathbb{R}^{2m} \to \mathbb{R}^m : f_{da}(0,0) = 0, \quad f_{da}^2(x,x_d) f_{da}(x,x_d) \leq \gamma^2 x_d^2 \} \]

and \( \gamma > 0 \). Furthermore, as in Section 2 we similarly assume that \( G(x) \) is such that \( G(x) \) is unknown and is parameterized as \( G_a(x) = B_a G_n(x) \), where \( G_n : \mathbb{R}^n \to \mathbb{R}^{m \times m} \) is known and satisfies \( \det G_n(x) \neq 0, \ x \in \mathbb{R}^n \), and \( B_a \in \mathbb{R}^{m \times m} \), with \( \det B_a \neq 0 \), is an unknown symmetric sign-definite matrix but the sign definiteness of \( B_a \) is known; that is, \( B_a > 0 \) or \( B_a < 0 \). For the statement of the next result define \( \text{sgn}(B_a) = 1 \) for \( B_a > 0 \), and \( \text{sgn}(B_a) = -1 \) for \( B_a < 0 \).

**Corollary 3.1.** Consider the nonlinear uncertain matrix-second order delay dynamical system \( \dot{y}(t) = F(y(t), y(t - \tau)) \), where \( F(y(t), y(t - \tau)) \) is a positive-definite matrix that is positive definite for \( a = \Theta_d \in \mathbb{R} \). Furthermore, as in Section 2 we similarly define \( \Theta_d \in \mathbb{R} \). Then it follows that

\[ R \triangleq (\Theta_d^T P + \hat{P} \Theta_d + I_{2m} + \gamma -2PB_a B_a^T P) \]

\[ \geq -2a_1 P_{12m} I_{12m} + \lambda_{\max}(I_{2m} + \gamma -2PB_a B_a^T P) \]

\[ \geq 0. \] (28)

Next, note that with \( u(t) \) given by (26) it follows from (1) that

\[ \dot{x}(t) = f(x(t)) + f_a(x(t), x(t - \tau)) + G(x(t)) G_n^{-1}(x(t)) K(t) F(x(t)) \]

\[ x(\theta) = \eta(\theta), \ -\tau \leq \theta \leq 0, \ t \geq 0. \] (29)

or, equivalently,

\[ \dot{z}(t) = \tilde{A}_x x(t) + B_0 f_{da}(x(t), x(t - \tau)) + B_0 B_a (K(t) - K) F(x(t)), \]

\[ x(\theta) = \eta(\theta), \ -\tau \leq \theta \leq 0, \ t \geq 0. \] (30)

To show Lyapunov stability of the closed-loop system (27) and (30) consider the Lyapunov-Krasovskii functional candidate \( V : C \times \mathbb{R}^{m \times m} \to \mathbb{R} \) given by

\[ V(p, K) = \psi^T(0) P \psi(0) + \int_0^T \psi^T(\theta) \psi(\theta) d\theta - T \]

\[ + \text{tr} \{ B_a (K - K) Y^{-1} (K - K)^T \}, \]

\[ \psi(\cdot) \in C, \] (31)

where \( |B_a| = (B_a^2)^{1/2} \) and \( (\cdot)^{1/2} \) denotes the (unique) positive-definite square roots. Note that \( V(0, K) \) is 0, where \( \psi(0) = 0, \ \theta \in [-\tau, 0] \). Furthermore, since \( p_1 > 0 \) and \( \det P = -a_1 p_2^2 - a_2 p_2 p_2 - p_2^2 > -a_1 p_2^2 + \frac{1}{p_2} (a_1 p_2 + p_2) (a_1 p_2 + p_2) = 0 \), \( P \) is positive definite and thus it follows that there exist class \( \hat{K}_\infty \) functions \( \alpha_1(\cdot) \) and \( \alpha_2(\cdot) \) such that

\[ V(p, K) \geq \alpha_1(|\psi(0)|) + \alpha_2(||K - K||_F). \] (32)

Now, letting \( x(t) \) denote the solution to (30) and using (25), (27), and (28), it follows that the Lyapunov-Krasovskii directional derivative along the closed-loop system trajectories is given by

\[ \dot{V}(x(t), K(t)) = 2x^T(t) P \left[ \tilde{A}_x x(t) + B_0 f_{da}(x(t), x(t - \tau)) + B_0 B_a (K(t) - K) F(x(t)) \right] + x^T(t) x(t) \]

\[ -T \]

\[ + 2 \text{tr} \{ B_a (K(t) - K) Y^{-1} K(t) \}, \]

\[ = x^T(t) (\hat{A}_x^T P + P \hat{A}_x) x(t) + 2x^T(t) P B_0 f_{da}(x(t), x(t - \tau)) + x^T(t) P B_0 B_a^T P B_0 B_a \]

\[ + x^T(t) x(t) - x^T(t) x(t) \]

\[ -2 \text{tr} \left[ (K(t) - K) F(x(t)) x^T(t) P B_0 B_a \right] \]

\[ \leq x^T(t) (\hat{A}_x^T P + P \hat{A}_x) x(t) \]

\[ + \gamma^2 x^T(t) P B_0 B_a^T P x(t) \]

\[ + x^T(t) x(t) \]

\[ \leq -x^T(t) x(t), \]

\[ \leq 0, \] (33)

which proves that the solution \( x(t), K(t) \equiv (0, K_0) \) to (27) and (30) is Lyapunov stable. Furthermore, since \( R > 0 \), it follows from Theorem 3.1 of Hale & Verduyn Lunel (1993, p. 143) that \( \|x_0\| \to 0 \) as \( t \to \infty \) for all \( \eta(\cdot) \in C, f_{da}(\cdot, \cdot) \in \mathcal{F}_d \), and \( \tau \in [0, \infty) \).
A direct adaptive nonlinear control framework for adaptive stabilization of multivariable nonlinear uncertain delay dynamical systems was developed. Using Lyapunov-Krasovskii functionals the proposed framework was shown to guarantee partial asymptotic stability of the closed-loop system. Furthermore, in the case where the nonlinear system is represented in normal form with input-to-state stable zero dynamics, the nonlinear adaptive controllers were constructed without knowledge of the system dynamics. Specifically, in the case of matrix second-order nonlinear systems, the adaptive controller does not even require the knowledge of the delay amount. Finally, an illustrative numerical example was presented to show the utility of the proposed direct adaptive stabilization scheme.

REFERENCES