Abstract: A general optimal control problem for ABS is formulated and analyzed, with the elimination of excessive slip and reduction of braking distance taken into account. Analytical formulas for singular optimal solutions are derived. These results are applied to a simple laboratory model of ABS. The results of simulations are compared with those obtained by a gain-scheduling approach. Copyright © 2005 IFAC

Keywords: ABS, brake control, friction, optimal control, singularities.

1. INTRODUCTION

For the last twenty years intensive development of control systems for car brakes has been observed. The antilock brake system (ABS) fulfils two basic tasks in a car. First, it prevents the wheels from locking, by keeping the slip below the maximum admissible level. Second, it should reduce the braking distance to its minimum possible value. In a typical situation (on dry asphalt) the maximum friction force between the tire and the road occurs at a certain moderate slip, when the wheel is not locked. In the first ABS systems, various on-off controllers were applied (Hattwig 1993, Maier 1995). Later PID and gain-scheduled PID controllers, as well as their robustified versions were introduced. (Johansen et al., 2001) proposed a robust gain-scheduled LQ controller where Sontag’s procedure is used to stabilize the system. Some strategies based on optimization and off-line trajectory planning are presented in (Johansen, 2001). In a real car it is necessary to estimate the car velocity and parameters of the friction curve in the presence of disturbances and various effects which are difficult to model (Petersen, 2003). To this end, the extended Kalman filter is frequently used.

The laboratory ABS model (LABS) used in this work allows precise identification of friction mechanisms and does not require state estimation. This gives a possibility to determine optimal control basing on a mathematical model and apply it in practice. The paper is organized as follows. Section 2 contains a short description of the laboratory setup and the state equations, also in the scaled version. In the next section the optimal control problem is formulated, optimality conditions are given and singular optimal controls are analyzed. Next, the optimal solution is calculated by the MSE method (Szymkat and Korytowski, 2003). Section 4 presents a comparison with the gain-scheduled LQ controller based on the results of (Johansen et al., 2001, 2003). Results of the real life laboratory experiments are discussed in section 5. The paper ends with conclusions.
accelerate it. During the braking process, the latter motor is switched off. Both motors are steered by PWM signals with frequency 7.0 [kHz].

The relative pulse width of the PWM signal of the upper wheel braking motor is the control variable. The peripheral velocity of the lower wheel can be identified with the speed of the vehicle and the angular velocity of the upper wheel can be identified with the angular velocity of the rotating wheel of the vehicle.

\[
\begin{align*}
\dot{x}_1 &= S(\lambda)(c_1 x_1 + c_2) + c_3 x_1 + c_4 + (c_5 S(\lambda) + c_{16}) x_3 \\
\dot{x}_2 &= S(\lambda)(c_{21} x_1 + c_{22}) + c_{23} x_2 + c_{24} + c_{25} S(\lambda) x_3 \\
\dot{x}_3 &= c_{31} (u - x_3), \quad 0 \leq u \leq u_{\text{max}}
\end{align*}
\]

where

\[
S(\lambda) = \frac{w_4 \lambda^p}{a + \lambda^p} + w_5 \lambda^3 + w_7 \lambda^2 + w_1 \lambda .
\]

The dependence of the friction coefficient on the slip is presented in Fig. 2.

![Diagram of LABS](image)

**Fig. 1. Diagram of LABS**

We use the following notations (see Fig. 1). The angular velocities of the upper and lower wheel [rad/s] are denoted by \( x_1 \) and \( x_2 \), respectively, \( x_1 = M_1 \) is the braking moment [Nm] of the upper wheel, \( r_1 \) and \( r_2 \) are the radii of the upper and lower wheel [m], \( J_1 \) and \( J_2 \) are the moments of inertia of the upper and lower wheel [kgm²], \( d_1 \) and \( d_2 \) are the coefficients of viscous friction in the upper and lower wheel bearing [Nm/s], \( F_n \) is the force with which the upper wheel presses the lower one [N], \( \mu(\lambda) \) is the coefficient of friction between the wheels, \( \lambda = \frac{r_2 x_2 - r_1 x_1}{r_2 x_2} \) is the relative difference of peripheral velocities of the wheels, or the wheel slip (\( r_2 x_2 \geq r_1 x_1 \)). \( M_{10} \) and \( M_{20} \) are the moments of static friction [Nm] of the upper and lower wheel, respectively, \( M_g \) is the moment of gravity acting on the rocker arm, \( L \) is the distance between the point of contact of the wheels and the axis of the rocker arm [m], \( \varphi \) is the angle between the normal at the point of contact of the wheels and the rocker arm [rad], \( u \) is the control of the disk brake. The values of model parameters are given in Appendix A.

It is assumed that the friction is proportional to the pressing force \( F_n \) with the proportionality coefficient \( \mu(\lambda) \).

### 2.1 State equations

The dynamics of the system is described by the following state equations

\[
\begin{align*}
\dot{x}_1 &= S(\lambda)(c_1 x_1 + c_2) + c_3 x_1 + c_4 + (c_5 S(\lambda) + c_{16}) x_3 \\
\dot{x}_2 &= S(\lambda)(c_{21} x_1 + c_{22}) + c_{23} x_2 + c_{24} + c_{25} S(\lambda) x_3 \\
\dot{x}_3 &= c_{31} x_3 + u
\end{align*}
\]

\[
\begin{align*}
q_1 &= c_{14} + c_{16} \gamma = 0, \quad \frac{\gamma}{\alpha} c_{16} \beta = -1, \\
c_{13} \delta &= -1, \quad \frac{\gamma}{\alpha} c_{24} = -1, \quad c_{13} \delta \kappa \beta = 1.
\end{align*}
\]

We rewrite the state equations in the new notations, omitting the bars over symbols

\[
\begin{align*}
\dot{x}_1 &= S q_1 - x_1 + (c_{15} S - 1) x_3 \\
\dot{x}_2 &= S q_2 + q_3 + c_{23} S x_3 \\
\dot{x}_3 &= c_{31} x_3 + u
\end{align*}
\]

\[
\begin{align*}
q_1 &= c_{14} + c_{12}, \quad q_2 = c_{21} x_1 + c_{22}, \quad q_3 = c_{23} x_2 - 1 \\
\lambda &= 1 - x_1 x_2^{-1}, \quad S(\lambda) = \frac{\lambda^p}{a + \lambda^p} + w_5 \lambda^3 + w_7 \lambda^2 + w_1 \lambda.
\end{align*}
\]

The constraints on control now take the form

\[
\frac{\gamma}{\kappa} \leq \bar{u} \leq \bar{u}_{\text{max}}, \quad \bar{u}_{\text{min}} = -\frac{\gamma}{\kappa} \bar{u}_{\text{max}} = \frac{\bar{u}_{\text{max}} - \gamma}{\kappa}.
\]
3. OPTIMAL CONTROL

3.1 Control task

The goal of control is to reduce the velocity in time $T$ in such a way that an adequate compromise is ensured between excessive slip, braking distance and accuracy of reaching the target state. These requirements are expressed by the following performance index

$$Q(u, T) = \frac{1}{2} \| x(T) - x_f \|_Q^2 + \frac{T}{2} \rho \left[ \eta r_x x - r_1 x_1 \right]_2^2 + \rho_1 \int_T^0 r_x x_2 d\tau$$

The first term in (5) is a penalty for the error in reaching the target, the second penalizes for excessive slip, and may be interpreted as a measure of probability of losing steering qualities by the vehicle. Excessive slip is likely to cause a complete loss of steerability of the vehicle. The third term is proportional to braking distance. The parameters $r$ and $r_1$ are nonnegative weighting coefficients. The control horizon $T$ can be free or fixed. Notice that the task of slip stabilization can be obtained as a special case after putting $r_1 = 0$. Using the scaled variables introduced in section 2.2, we write the performance index in the form

$$Q(u, T) = \frac{1}{2} \| \bar{x}(T) - \bar{x}_f \|_Q^2 + \frac{T}{2} \rho \left[ \eta \bar{x}_2 - \bar{x}_1 \right]_2^2 + \rho_1 \int_T^0 \bar{x}_2 d\tau$$

3.2 Optimality conditions

We use Pontryagin’s maximum principle. The Hamiltonian is as follows

$$H = H_0(x, \psi) + H_1(x, \psi) u$$

$$H_0 = \psi_1 (S q_1 - x_1 + (c_{31} S - 1) x_3) + \psi_2 (S q_2 + q_1 + c_{25} S x_3) + \psi_3 c_{31} x_3 - \frac{1}{2} \rho (x_2 - x_1)_2^2 - \rho_1 x_2$$

$$H_1 = \psi_3.$$

The control maximizing the Hamiltonian satisfies

$$u(t) = \begin{cases} u_{\max} \cdot \phi(t) > 0 \\ u_{\min} \cdot \phi(t) < 0 \end{cases}$$

where

$$\phi(t) = \psi_3 = H_1 |_x.$$

Write the adjoint equations

$$\psi_1 = -(S q_1 + c_{11} S - 1 + c_{31} S x_3) \psi_1 -$$

$$- (S q_2 + c_{21} S + c_{25} S x_3) \psi_2 - \rho (x_2 - x_1)_+$$

$$\psi_2 = -(S q_2 + c_{11} S x_3) \psi_1 -$$

$$- (S q_2 + c_{31} S x_3) \psi_2 + \rho (x_2 - x_1)_+ + \rho_1$$

$$\psi_3 = -(c_{11} S - 1) \psi_1 - c_{25} S \psi_2 - c_{31} \psi_3$$

with terminal conditions

$$\psi_1(T) = x_1(T)^f - x_1(T), \quad \psi_2(T) = x_2(T) - x_2(T)$$

$$\psi_3(T) = 0.$$

It can be proved that a singularity of the second order usually occurs in the optimal solution of the considered problem. In the interval of singularity we have $\phi(t) \equiv 0$, i.e., $\psi_1(t) \equiv 0$. By differentiating this identity four times we obtain an expression for the optimal singular control as a function of state, $u(t) = u_s(x(t))$. The adjoint variables are eliminated using the relationships $\phi = \psi = \psi_3 \equiv 0$. To save space we omit the detailed computations which are lengthy and laborious.

3.3 Experiments

Let $\eta = 0.7$, $\rho = 1000$, $\rho_1 = 0$, $T = 0.0064$. The search for optimal control is started in the class of bang-bang functions (Fig. 3). Large density of control switchings indicates the singularity interval and allows the structure of optimal control to be determined.
In the next stage, the MSE method was used to calculate the optimal solution (Szymkat and Korytowski, 2003). The results are shown in Fig. 4.

Let now $\eta = 0.7$, $\rho = 1000$, $\rho_1 = 0$, $T = 0.0064$. As before, a bang-bang approximation of optimal control was used to establish the structure of optimal solution. The results obtained with the use of the explicit representation of singular control are shown in Fig. 5. The next example shows (Fig. 6) the consequences of diminishing the weighting coefficients $\rho$ 14 times, which means that exceeding the value of slip 0.7 is much less penalized.

In the fourth experiment the slip was stabilized at the value 0.3. It was thus assumed $\eta = 0.7$, $\rho = 1$, $\rho_1 = 0$, $T = 0.01$, and the term in the performance index penalizing for missing the target was skipped. The results are shown in Fig. 7.
4. COMPARISON WITH GAIN-SCHEDULED LQ CONTROLLER

This section is devoted to the gain-scheduled LQ (GSLQ) controller of (Johansen et al., 2001, 2003). We consider the model (2.1), neglecting the friction in the bearings and static friction. The state equations take the form

\[ \dot{x}_1 = c_{12} S(\lambda) + c_{15} S(\lambda) + c_{16} x_3 \]
\[ \dot{x}_2 = c_{22} S(\lambda) + c_{25} S(\lambda) x_3 \]
\[ \dot{x}_3 = c_{31} (u - x_1) \]

with

\[ c_{12} = \frac{M_s r_1}{J_1}, \quad c_{15} = \frac{r_2}{J_1}, \quad c_{16} = -\frac{1}{J_1} \]
\[ c_{22} = -\frac{M_s r_2}{J_2}, \quad c_{25} = -\frac{r_2}{J_2} \]

Substituting \( \lambda = \frac{r_2 x_2 - r_1 x_3}{r_2 x_2} \) and treating the velocity \( x_2 \) as a disturbance we obtain the equations

\[ \dot{x}_1 = f(\lambda, x_2, x_3) \]
\[ \dot{x}_3 = c_{31} (u - x_1) \]
\[ f = x_2^{-1} ((1-\lambda)(c_{22} S(\lambda) + c_{25} S(\lambda) x_3) - \overline{c}_{12} - \overline{c}_{15} S(\lambda) - \overline{c}_{16} \rho_b x_3) \]

The model linearized at the equilibrium point \( \bar{x}_0 = (\lambda_0, x_2, 0, 0) \) has the form

\[ \Delta \dot{x}_1 = f'_3(\bar{x}_0) \Delta \dot{x}_3 + f'_3(\bar{x}_0) \Delta x_3 \]
\[ \Delta \dot{x}_3 = c_{31} (\Delta u - \Delta x_1) \]
\[ \Delta \dot{x}_4 = \Delta \dot{\lambda} \]
\[ \Delta \dot{\lambda} = \lambda - \lambda_0, \quad \Delta u = u - u_0, \quad \Delta x_1 = x_3 - x_{30} \]

This model is non-stationary and its coefficients depend on the velocity \( x_3 \), which is taken as a disturbance. In the controller synthesis it will be assumed that \( x_2 \) is constant. The third equation is introduced into the model so that the controller has the integrating (astatic) property. The controller synthesis consists of determining a gain matrix \( K = [K_1, K_2, K_3] \) which minimizes a quadratic performance index

\[ J = \int_0^\infty (\Delta x^T W(x_2) \Delta x + \Delta u^T \Delta u) dt \]
\[ \Delta x = \text{col}(\Delta \lambda, \Delta x_3, \Delta x_4), \quad \Delta u = -K \Delta x \]
\[ W(x_2) = x_2^{3/2} \text{diag}(w_1, w_2, w_3) \]
\[ w_1 = 0.1, \quad w_2 = 10^{-4}, \quad w_3 = 15, \quad R = 1. \]

It is assumed that the set value of the slip is equal to \( \lambda_0 = 0.2 \), which corresponds to the braking moment \( x_{30} = 5.7156 \) [Nm]. The gain matrices are determined by solving an appropriate Riccati equation, for the velocity \( x_2 \) in the range from 1 to 180 [rad/s]. The dependence of the controller gains on \( x_2 \) is shown in Fig. 8.

This gain-scheduled LQ controller is confronted with the solution, optimal according to the performance index (5) with the parameters \( \eta = 0.8, \rho = 1, \quad \rho_1 = 0, \quad T = 0.01 \). Notice that the state trajectory generated by the optimal controller (black line in Fig. 9) exhibits better transient behavior than the GSLQ controller (blue line). The corresponding slip trajectories are shown in Fig. 10.
5. EXPERIMENTAL VALIDATION

In the experiment the slip was stabilized at the value 0.3. It was thus assumed $\eta = 0.7$, $\rho = 1$, $\rho_1 = 0$, $T = 1.2$, and the term in the performance index penalizing for missing the target was skipped. The MSE method was used to calculate the optimal solution. Next, the optimal control was applied to the LABS. The results of the open loop experiments are shown in Fig. 11 and 12. A measured time history of the slip and optimal control are shown in Fig. 11. The trajectory of LABS is shown in Fig. 12. It is easy to see that the quality of the slip stabilization is very good.

6. CONCLUSIONS

The Laboratory ABS model (LABS) is a simple and convenient tool for experimental verification of different antilock brake control methods. The optimal control setting of the antilock braking problem, with the control time, terminal error, excessive slip and braking distance accounted for in the performance index, provides efficient control algorithms which can be used in practice. This is possible due to the careful identification of LABS. The MSE method of optimal control calculation has proved well-suited for this application. An interesting feature of the optimal solutions in the considered problem is the presence of singularities that may be analytically treated.

Further research should answer the important question how the obtained results can be incorporated into an adaptive real-time control scheme, resulting in a robust, reliable practical solution.

REFERENCES


