CONTROLLER FOR A NONLINEAR SYSTEM 
WITH AN INPUT CONSTRAINT BY USING A 
CONTROL LYAPUNOV FUNCTION II

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Abstract: Malisoff and Sontag proposed a universal control formula for a nonlinear 
system such that the $k$-norm of inputs is less than one, where $1 < k \leq 2$. We have 
generalized the Malisoff’s formula so that it can be applied in any case of $k \geq 1$. 
However, the generalized controller may become discontinuous if $k = 1$ or $k = \infty$. 
In this paper, we propose a new control formula that is continuous except the 
origin in any case of $k \geq 1$. We also confirm the effectiveness of the proposed 
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Keywords: nonlinear systems, constraints, controllers, Lyapunov methods

1. INTRODUCTION

We consider a nonlinear system such that inputs are restricted to the Minkowski ball $U_k$. $U_k$ is a 
subspace of $\mathbb{R}^n$ such that the $k$-norm of inputs is less than one. Lin and Sontag proposed a universal 
control formula with respect to $U_2$ by using a control Lyapunov function (Lin and Sontag, 1991). Malisoff 
and Sontag provided a universal control formula with respect to $U_k$, where $1 < k \leq 2$ (Malisoff and Sontag, 2000). We have generalized the Malisoff’s controller so that it can be applied in 
every case of $k \geq 1$ (N. Kidane and Nishitani, 2005). However, the generalized controller may become discontinuous if $k = 1$ or $k = \infty$. 
Due to discontinuity of the controller, inputs may have chattering.

In this research, we propose a new control formula that is continuous except the origin in any case of 
$k \geq 1$. We show the design scheme briefly.

First, we consider a continuous function $\hat{k}(x)$ and a subspace $U_{\hat{k}} \subset U_k$ such that $U_{\hat{k}}$ is the closure of 
$U_k$. $U_{\hat{k}}$ is similar to $U_k$ (same shape), $U_{\hat{k}}$ becomes a small ball if $P(x) \leq 0$, and 
$U_{\hat{k}} \to U_k$ as $P(x) \to 1$. Second, we stabilize the system by the input 
that minimizes $\dot{V}(x,u)$ in $U_{\hat{k}}$. We also confirm the effectiveness of the proposed controller by computer simulation.

2. PRELIMINARY

In this section, we introduce mathematical notation and some definitions. For a vector $x \in \mathbb{R}^n$, 
k-norm is defined as

$$\|x\|_k = \left( \sum_{i=1}^{n} |x_i|^k \right)^{\frac{1}{k}}. \quad (1)$$

We obtain the following lemma:
Lemma 1. Assume that \( x \in \mathbb{R}^n \) and \( 1 \leq k_1 < k_2 \). Then,
\[
\|x\|_{k_1} \geq \|x\|_{k_2}.
\]
\( \square \)

Proof 1. Let \( e \in \mathbb{R}^n \) be a vector such that \( \|e\|_{k_1} = 1 \), and \( f : \mathbb{R}^n \to \mathbb{R}^n \) be a bijection defined by
\[
f(x) := \left( |x_1|^{k_2} \text{sgn}(x_1), \ldots, |x_n|^{k_2} \text{sgn}(x_n) \right)^T.
\]
Note that \( \|f(e)\|_{k_2} = 1 \). The norm \( \|e\|_{k_2} \) can be written as
\[
\|e\|_{k_2} = \left( \sum_{i=1}^n |e_i|^{k_2} \right)^{\frac{1}{k_2}} = \left( \sum_{i=1}^n |e_i|^{k_2-k_1} \right)^{\frac{1}{k_2}}.
\]
From \( \|e\|_{k_1} = 1 \), we get \( |e_i| \leq 1 \) and \( |e_i|^{k_2-k_1} \leq 1 \). From (3), \( \|e\|_{k_2} \leq 1 \). On the other hand, any vector \( x \in \mathbb{R}^n \) can be written as
\[
x = \|x\|_{k_1} \hat{e} = \|x\|_{k_2} \hat{e},
\]
where \( \hat{e} \in \mathbb{R}^n \) and \( \hat{e} \in \mathbb{R}^n \) are vectors such that \( \|\hat{e}\|_{k_1} = 1 \) and \( \|\hat{e}\|_{k_2} = 1 \). From (4), we get
\[
\|x\|_{k_2} = \|x\|_{k_1} \|\hat{e}\|_{k_2}.
\]
From \( \|\hat{e}\|_{k_2} \leq 1 \) and (5), we obtain (2). \( \|x\|_{k_1} \) and \( \|x\|_{k_2} \) are equal if and only if \( \|x_i\| = \|x\| \) for some \( i \in \{1, \ldots, n\} \). \( \square \)

In this paper, we consider the following affine system:
\[
\dot{x} = f(x) + g(x)u, \quad (6)
\]
where \( x \in \mathbb{R}^n \) is a state vector and \( u \in U \subseteq \mathbb{R}^m \) is an input vector. We assume that \( f : \mathbb{R}^n \to \mathbb{R}^n \) and \( g : \mathbb{R}^n \to \mathbb{R}^{n \times m} \) are continuous mappings and \( f(0) = 0 \). We use the notation \( \mathbb{R}_{>0} := (0, \infty) \) and \( \mathbb{R}_{\geq 0} := [0, \infty) \).

Definition 1. (control Lyapunov function). A smooth proper definite function defined on a neighborhood of the origin \( X \in \mathbb{R}^n \), \( V : X \to \mathbb{R}_{\geq 0} \) is said to be a local control Lyapunov function for system (6) if the condition
\[
\inf_{u \in U} \{ L_f V + L_g V \cdot u \} < 0 \quad (7)
\]
is satisfied for all \( x \in X, x \neq 0 \). Moreover, \( V(x) \) is said to be a control Lyapunov function (clf) for system (6) if \( V(x) \) is a function defined on \( \mathbb{R}^n \) and condition (7) is satisfied for all \( x \in \mathbb{R}^n, x \neq 0 \). \( \square \)

Definition 2. (small control property). A (local) control Lyapunov function is said to satisfy the small control property (scp) if for any \( \delta > 0 \), there is a \( \delta > 0 \) such that, if \( x \neq 0 \) satisfies \( \|x\| < \delta \), then there is some \( u \in U \) with \( \|u\| < \epsilon \) such that \( L_f V + L_g V \cdot u < 0 \). \( \square \)

If there exists no input constraint \( (U \equiv \mathbb{R}^m) \), a smooth radially unbounded positive definite function \( V : \mathbb{R}^n \to \mathbb{R}_{\geq 0} \) is a clf if and only if
\[
L_f V = 0 \implies L_f V < 0, \quad \forall x \neq 0. \quad (8)
\]
We define \( h(x) \) as the right hand side of system (6) with a state feedback law \( u = \beta(x) \);
\[
\dot{x} = f(x) + g(x)\beta(x) := h(x). \quad (9)
\]
If \( \beta(x) \) is continuous except the origin, the closed system has always a Carathéodory solution for each initial state. On the other hand, if \( \beta(x) \) is not continuous, Carathéodory solution do not exist. Hence, we associate (9) with a differential inclusion of the form
\[
\dot{x} \in F(x). \quad (10)
\]
In this paper, we apply the Filippov’s approach
\[
F(x) = \bigcap_{\epsilon>0} \overline{\text{conv}}(h(B_x(x),N}), \quad (11)
\]
where \( B_x(x) \) denotes the open ball of center \( x \) and radius \( \epsilon \), \( \overline{\text{conv}} \) denotes the convex closure of a set, and \( \mu_n \) is the Lebesgue measure of \( \mathbb{R}^n \).

Definition 3. (Lyapunov function). A smooth and positive definite function defined on a neighborhood of the origin \( X \in \mathbb{R}^n \), \( V : X \to \mathbb{R}_{\geq 0} \) said to be a local Lyapunov function for system (10) if the following condition is satisfied for all \( 0 \neq x \in X \):
\[
\frac{\partial V}{\partial x} \cdot v < 0, \quad \forall v \in F(x). \quad (12)
\]
Moreover, \( V(x) \) is said to be a Lyapunov function for system (10) if \( V(x) \) is a radially unbounded function defined on \( \mathbb{R}^n \) and condition (12) is satisfied for all \( 0 \neq x \in \mathbb{R}^n \). \( \square \)

Theorem 1. (Bacciotti and Rosier, 2001) Let \( F \) be a set-valued map such that the local existence of solutions of (10) is insured. If a (local) Lyapunov function exists, then the origin is (locally) asymptotically stabilizable. \( \square \)

3. PREVIOUS WORK

When there is not any input constraint, Sontag proposed a universal control formula for a nonlinear system (Sontag, 1989). In this paper, we consider a nonlinear system such that inputs are restricted to the Minkowski ball of radius 1:
\[
U_k = \left\{ u \in \mathbb{R}^m \mid \|u\|_k = \left( \sum_{i=1}^m |u_i|^k \right)^{\frac{1}{k}} < 1 \right\}, \quad (13)
\]
where \( k \geq 1 \). Lin and Sontag provided a universal control formula with respect to Minkowski ball \( U_2 \) (Lin and Sontag, 1991). Malisoff and Sontag
improved the Lin’s controller in order to apply for the case of $1 < k \leq 2$ (Malisoff and Sontag, 2000). To construct the controller for the case of $k \geq 1$, we have generalized Malisoff’s controller. We introduce important results as the followings (N. Kidane and Nishitani, 2005):

**Theorem 2.** Let $V(x)$ be a local clf for system (6) with input constraint (13), and $a_1 > 0$ be the maximum number such that the condition

$$\inf_{u \in \mathcal{U}_k} \{ L_fV + L_g V \cdot u \} < 0, \quad \forall x \neq 0 \quad (14)$$

is satisfied for all $x \in W = \{ x | V(x) < a_1 \}$. Then, $W$ is a domain in which the origin is asymptotically stabilizable. If $V(x)$ is a clf, then $a_1 = \infty$ and $W = \mathbb{R}^n$. □

**Proposition 1.** We consider system (6) with an input constraint $u \in \hat{U}_k$, where $\hat{U}_k$ is the closure of $U_k$. Let $V(x)$ be a local clf for the system. Then, the input

$$u_i = \begin{cases} 
\frac{|L_{g_i} V|}{\|L_g V\|} \text{sgn}(L_{g_i} V) & (L_g V \neq 0) \\
0 & (L_g V = 0)
\end{cases} \quad (i = 1, \ldots, m) \quad (15)$$

minimizes the derivative $\dot{V}(x, u)$. □

**Lemma 2.** Let $V(x)$ be a local clf for system (6) with input constraint (13), $W$ be a domain in Theorem 2. We define

$$P(x) = \frac{L_fV}{\|L_g V\|} \quad (16)$$

Then,

$$\sup_{x \in \{x \in W | L_g V(x) \neq 0\}} P(x) = 1 \quad (17)$$

□

**Theorem 3.** Let $V(x)$ be a local clf for system (6) with input constraint (13), $W$ be a domain in Theorem 2, $P(x)$ be a function defined by (16), $c > 0$ and $q \geq 1$ are constants. Then, the input

$$u_i = \frac{-P + |P| + c \|L_g V\|_q}{2 + c \|L_g V\|_q} \frac{|L_{g_i} V|}{\|L_g V\|} \text{sgn}(L_{g_i} V) \quad (i = 1, \ldots, m) \quad (18)$$

asymptotically stabilizes the origin in domain $W$. If $m = 1$ or $k = 2$, the input is continuous on $W \setminus \{0\}$. Moreover, if $V(x)$ has the scp, the input is also continuous at the origin. □

If $m \neq 1$ and $k \neq 2$, however, input (18) may have chattering.

For example, in the case of $m \neq 1$ and $k = \infty$, input (18) becomes $u_i = -b_2(x) \text{sgn}(L_{g_i} V)$. It is discontinuous on $\{x | L_g V = 0\}$. In the case of $m \neq 1$ and $k = 1$, $u_i = 0$ when $[L_g V] = \max_{j=1, \ldots, m} |L_{g_j} V|$, and $u_i = -b_2(x) \text{sgn}(L_{g_i} V)$ when the other case $[L_g V] = \max_{j=1, \ldots, m} |L_{g_j} V|$. These controllers may cause chattering in inputs.

Therefore, the closed system may not have Carathéodory solutions in the case of $m \neq 1$ and $k \neq 2$. In this paper, we construct a controller that is continuous except the origin; namely, the controlled system has always a Carathéodory solution for each initial state.

4. CONTROLLER DESIGN

The objective of this paper is to design a stabilizing controller that is continuous except the origin in any case of $k \geq 1$. In our previous work (N. Kidane and Nishitani, 2005), we have proposed controller (18). We show the construction scheme briefly.

First, we consider a subspace $\bar{U}_k \subset \hat{U}_k$ such that $\bar{U}_k$ is similar to $\hat{U}_k$. We consider $P(x) \rightarrow 0$. Second, we design a stabilizing controller by choosing the input that minimizes $\dot{V}(x, u)$ in $\bar{U}_k$. Consider the (hyper) surface $Q : L_g V \cdot u = a_2$ such that $\bar{U}_k \cap Q \neq \emptyset$ and $a_2$ becomes minimum.

When input $u$ coincides contact point between $Q$ and $\bar{U}_k$, $\dot{V}(x, u)$ takes minimum value. Then, the input that minimizes $\dot{V}(x, u)$ in $\bar{U}_k$ is denoted by the contact point between $Q$ and $\bar{U}_k$. In the case of $k = 2$, a subspace $\bar{U}_2 \subset \bar{U}_2$ becomes a ball. Hence, the input that minimizes $\dot{V}(x, u)$ in $\bar{U}_2$ (namely, the contact point between $Q$ and $\bar{U}_2$) moves continuously on the boundary of $\bar{U}_2$. On the other hand, in the case of $k = \infty$, a subspace $\bar{U}_\infty \subset \bar{U}_\infty$ always becomes a rectangle. Hence, the contact point between $Q$ and $\bar{U}_\infty$ jumps from a vertex to another vertex at the moment that the sign of $L_g V$ changes. This causes chattering phenomenon in inputs.

In this section, we propose a stabilizing controller that is continuous except the origin in any case of $k \geq 1$ as the followings: First, we consider a continuous function $k$ and a subspace $\bar{U}_k \subset \hat{U}_k$ that satisfies the following conditions: $\bar{U}_k$ is similar to $\hat{U}_k$, $\bar{U}_k$ becomes a small ball if $P(x) \leq 0$, and $\bar{U}_k \rightarrow \hat{U}_k$ as $P(x) \rightarrow 1$. Second, we stabilizes the system by the input that minimizes $\dot{V}(x, u)$ in $\bar{U}_k$ (See Fig. 1). Note that the subset $\bar{U}_k$ has to be large enough to hold $\dot{V}(x, u) < 0$ $(\forall 0 \neq x \in W)$ under the input constraint $u \in \bar{U}_k$. 
In fact, the monotonicity is not necessary. But, we choose a function $b_i : \mathbb{R}^n \rightarrow \mathbb{R}_{>0}$, which is necessary for avoiding chattering phenomena.

The right hand side of the equation becomes maximum when $|L_g V| = \cdots = |L_{gm} V|$ (See Fig. 2). Assigning the values $|L_g V| = \cdots = |L_{gm} V|$ into (21), we can achieve the following necessary condition to satisfy $\dot{V}(x, u) < 0$:

$$\dot{k} \geq \frac{k}{1 - k \log_m P}.$$  \hfill (22)

Therefore, we have to choose $\dot{k}$ such that inequality (22) is satisfied and $\dot{k}$ becomes to 2 quickly enough to occur no chattering in inputs.

Remark 1. (Choice of $\dot{k}$). The directional vector of input (19) corresponds to the input that minimizes $\dot{V}(x, u)$ in $\dot{U}_k$. And input (19) have to satisfy input constraint (13). If $L_g V \neq 0$, the input such that the directional vector corresponds to the input that minimizes $\dot{V}(x, u)$ in $\dot{U}_k$ and the input exists on the boundary of $\dot{U}_k$ can be written as the following:

$$u_i = - \frac{L_g V_i}{\|L_g V\|} \frac{\text{sgn}(L_g V)}{k_i} \quad (i = 1, \ldots, m). \hfill (20)$$

i) We consider the case of $k \geq 2$. From Lemma 1 and $k < \dot{k}$, input (20) achieves

$$\dot{V}(x) \leq P \|L_g V\| \frac{1}{k_i} - \|L_g V\| \frac{1}{k_i}. \hfill (21)$$

The term in bracket $(\cdot)$ becomes maximum when $|L_g V| = \cdots = |L_{gm} V|$. Assigning the values $|L_g V| = \cdots = |L_{gm} V|$ into (23), we get a necessary condition to satisfy $\dot{V}(x, u) < 0$ as the following:

$$\dot{k} \leq \frac{k}{1 + k \log_m P}. \hfill (24)$$

Therefore, we have to choose $\dot{k}$ such that inequality (24) is satisfied and $\dot{k}$ becomes to 2 quickly enough to avoid chattering in inputs.

Although $b_4(x)$ and $\dot{k}$ are not obtained uniquely, we propose the following selection:

Theorem 4. Let $V(x)$ be a local clf for system (6) with input constraint (13), $W$ be a domain in Theorem 2, $P(x)$ be a function defined by (16), $c > 0$ and $q \geq 1$ are constants, and $m \geq 2$. We define $\dot{k}$ and $k$ as the following:

$$(\dot{k} = \begin{cases} \frac{k}{1 - k \log_m \left\{ P + (1 - P)m \frac{1}{m} \frac{1}{q} \right\} \text{sgn}(k - 2) \quad (P > 0) \\ 2 \quad (P \leq 0) \end{cases}) \hfill (25)$$

$$k = \begin{cases} \frac{\dot{k}}{k} \quad (k \geq 2) \\ k \quad (1 \leq k < 2). \end{cases} \hfill (26)$$

Remark 2. (Choice of $\dot{k}$). The directional vector of input (19) corresponds to the input that minimizes $\dot{V}(x, u)$ in $\dot{U}_k$. And input (19) have to satisfy input constraint (13). If $L_g V \neq 0$, the input such that the directional vector corresponds to the input that minimizes $\dot{V}(x, u)$ in $\dot{U}_k$ and the input exists on the boundary of $\dot{U}_k$ can be written as the following:

$$u_i = - \frac{L_g V_i}{\|L_g V\|} \frac{\text{sgn}(L_g V)}{k_i} \quad (i = 1, \ldots, m). \hfill (20)$$

The right hand side of the equation becomes maximum when $|L_g V| = \cdots = |L_{gm} V|$ (See Fig. 2). Assigning the values $|L_g V| = \cdots = |L_{gm} V|$ into (21), we can achieve the following necessary condition to satisfy $\dot{V}(x, u) < 0$:

$$\dot{k} \geq \frac{k}{1 - k \log_m P}. \hfill (22)$$

Therefore, we have to choose $\dot{k}$ such that inequality (22) is satisfied and $\dot{k}$ becomes to 2 quickly enough to occur no chattering in inputs.
Then, the input
\[
    u_i = \frac{-\langle P + |P| + cL_qV\|q\rangle}{(P + |P|) \left( 1 - m \frac{k_i}{k_i+1} \right) + 2m \frac{k_i}{k_i+1} + cL_qV\|q\rangle} \cdot \frac{\|\check{L}_gV\|^{\frac{k-1}{k}}}{\|L_qV\|^{\frac{k-1}{k}}} \cdot \text{sgn}(L_qV), \quad (L_qV \neq 0)
\]

\[
u_i = 0 \quad \text{for} \quad (L_qV = 0) \quad (i = 1, \ldots, m)
\]

(27)
asymptotically stabilizes the origin in domain \(W\), and it is continuous on \(W \setminus \{0\}\). Moreover, if \(V(x)\) has the scp, the input is also continuous at the origin.

**Proof 2.** In the case of \(L_qV = 0\), input constraint (13) is satisfied clearly. From Theorem 2, we get
\[
\dot{V}(x) = L_qV < 0 \quad \text{for all} \quad x \neq 0 \quad \text{in} \quad W.
\]

We consider the case of \(L_qV \neq 0\). From Lemma 1, note that \(\|\cdot\|_k \leq \|\cdot\|_k^\lambda\) in the case of \(k \geq \lambda\). From the fact and \(P(x) < 1\), we get
\[
\|u\|_k \leq \frac{P + |P| + c\|L_qV\|_q}{P + |P| + (2 - |P|)m \frac{k_i}{k_i+1} + c\|L_qV\|_q} < 1.
\]

Therefore, input constraint (13) is satisfied. If \(\delta < 1\), \(\|L_qV\|_q < \delta\), and \(L_qV < \delta\|L_qV\|_{k/(k-1)}\), then \(\|u\|_k < (2 + c)m \frac{k_i}{k_i+1} \delta\). Furthermore, \(\|u\|_k\) can be made as small as desired when \(\delta\) is taken to be small enough. In the case of \(P(x) \leq 0\), the condition \(V(x) < 0\) is satisfied obviously. We consider the case of \(0 < P(x) \leq 1\).

i) In the case of \(k \geq 2\), input (27) achieves
\[
\dot{V}(x) < (2P + c\|L_qV\|_q) y_1(x)
\]
\[
= \left[2 \left\{ P + (1 - P)m \frac{k_i}{k_i+1} \right\} + \|L_qV\|^{\frac{k-1}{k}} \right] \cdot \|L_qV\|^{\frac{k-1}{k}} \cdot \text{sgn}(L_qV),
\]

where
\[
y_1(x) = \left\{ P + (1 - P)m \frac{k_i}{k_i+1} \right\} \|L_qV\|^{\frac{k-1}{k}} - \|L_qV\|^{\frac{k-1}{k}}.
\]
y1(x) becomes maximum when \(|L_gV| = \cdots = |L_{g_m}V|\). From the values \(|L_gV| = \cdots = |L_{g_m}V|\) and (25), we obtain \(y_1(x) < 0\) and \(V(x) < 0\).

ii) In the case of \(1 \leq k < 2\), input (27) achieves
\[
\dot{V}(x) < (2P + c\|L_qV\|_q) \|L_qV\|^{\frac{k-1}{k}} \cdot \|L_qV\|^{\frac{k-1}{k}}
\]
\[
\cdot y_2(x) / \left[2 \left\{ P + (1 - P)m \frac{k_i}{k_i+1} \right\} + \|L_qV\|^{\frac{k-1}{k}} \right],
\]

where
\[
y_2(x) = \left\{ P + (1 - P)m \frac{k_i}{k_i+1} \right\} \|L_qV\|^{\frac{k-1}{k}} - \|L_qV\|^{\frac{k-1}{k}}.
\]

\[
5. \text{SIMULATION}
\]

In this section, we consider the same example as (N. Kidane and Nishitani, 2005):
\[
\begin{align*}
\dot{x}_1 &= x_1 - 4x_2 + u_1 \\
\dot{x}_2 &= x_2 + u_2
\end{align*}
\]

with an input constraint \(\|u\|_\infty < 1\). We choose a local clf as \(V(x) = \frac{x_1^2 + x_2^2}{2}\). From (16), we get
\[
P = x_1^2 - 4x_1x_2 + x_2^2.
\]

We set \(c = 1\) and \(q = 1\) in (27). Then, the controller
\[
u_i = \begin{cases} 
-\frac{P + |P| + \|x\|_1}{(1 - \|x\|_1) (P + |P|) + \sqrt{2} + \|x\|_1} & (x \neq 0) \\
0 & (x = 0) \\
\end{cases} \\
(i = 1, 2)
\]

(30)
asymptotically stabilizes the origin in domain \(W = \{x | x_1^2 + x_2^2 < 2/9 \}\), where
\[
\hat{k} = -\frac{1}{\log_2 \left( P + (1 - P)^{1/2}\right)}.
\]

Let \(x(0) = (-0.3, 0.3)^T\) be an initial state. Figure 3 and Fig. 4 show the trajectory of the state and the change in the input, respectively. The trajectory converges to zero, and the input constraint \(\|u\|_\infty < 1\) is satisfied. In the example of our previous paper (N. Kidane and Nishitani, 2005), we have admitted chattering phenomenon in input \(u_2\). On the hand, Fig. 4 demonstrates continuous resonance of the input.

6. CONCLUSION

In this paper, we have proposed a stabilizing controller that is continuous except the origin in any case of \(k \geq 1\) as the following: First, we considered a continuous function \(k\) and a subspace \(U_k \subset U_k^\prime\) such that \(U_k^\prime\) is similar to
Fig. 3. Trajectory with input (30)

Fig. 4. Change in input (30)

\( \bar{U}_k \) (same shape), \( \bar{U}_k' \) becomes a small ball if \( P(x) \leq 0 \), and \( \bar{U}_k' \rightarrow \bar{U}_k \) as \( P(x) \rightarrow 1 \). Second, we stabilized the system by the input that minimizes \( \dot{V}(x, u) \) in \( \bar{U}_k' \). We have obtained necessary conditions to hold \( \dot{V}(x, u) < 0 \) (\( \forall 0 \neq x \in W \)). Moreover, we have demonstrated the controller’s effectiveness by computer simulation.

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