AN OPTIMAL EXTRAPOLATOR FOR REDUCING PHASE DELAY OF SAMPLE DATA-HOLD

Reza Shahnazi *, Hamid Khaloozadeh **

*Department of Electrical Engineering Ferdowsi University of Mashhad, Mashhad, Iran
**Department of Electrical Engineering, K.N.T University of Technology, Tehran, Iran

Abstract: The objective of this paper is to minimize the phase delay of the zero-order hold (ZOH) an usual sample-data hold device. In this paper an optimized data holding scheme whose phase delay is significantly less than of the ZOH is presented. The method is based on using optimal extrapolators, which has a better result than extrapolators proposed by Yekutiel (1980) and Belicynski and Kozinski (1984). This method is more general than the other methods. Copyright © 2005 IFAC

Keywords: Sample and hold, Optimization, Bandwidth, Discrete systems, Digital filter.

1. INTRODUCTION

Discrete time control systems may operate partly in discrete time and partly in continuous time. Thus in such control systems some signals appear as discrete-time functions and other signals as continuous-time functions. In most cases continuous time systems controlled by a digital computer, so a data hold circuit will be used to convert the discrete-time signal into a continuous-time signal (Ogata, 1987). The most frequently employed holding device is the “zero-order-hold” (ZOH), which implies that the circuit holds the amplitude of the sample from one sampling instant to the next.

A technique which often was employed for designing digital controllers is designing a continuous controller in S-plane and transfers it to Z-plane by bilinear transformation. In such cases the intrinsic phase delay by the ZOH has direct reduction in phase margin. To reduce phase delay Yekutiel (1980), Belicynski and Kozinski (1984) and Leonard (1999) proposed a digital filter, which can be achieved by modification the digital controller software. The method which Yekutiel (1980) proposed was named Piece-wise Constant Higher-Order-Hold (PC-HOH), which includes a standard ZOH hardware and modifying the output level of it by the digital filter. The filter is obtained by using Taylor’s series expansion (order m) as an extrapolator polynomial for the new output value, \( y(nT + \Delta) \), and is as follows

\[
G_m(z) = \frac{y_m(z)}{y(z)} = \sum_{i=0}^{m} \frac{\Delta^i}{i!} \left(1 - \frac{z^{-1}}{T}\right)^i \quad (1)
\]

which \( y^{(i)}(nT) \) denotes the i-th time derivative of \( y(t) \). In PC-HOH, \( \Delta = T/2 \) is assumed, so \( G_m(z) \) will be as follows

\[
G_m(z) = \sum_{i=0}^{m} \frac{1}{i!} \left(1 - \frac{z^{-1}}{2}\right)^i \quad (2)
\]

Table 1 shows the transfer functions for \( m=0, 1, 2 \). It is obvious \( m=0 \) represents the standard ZOH.
Belicynski and Kozinski (1984) proposed another method based on Newton Extrapolation Polynomial. The Newton Extrapolation Polynomial Method (NEPM) sets its output at the mT-th moment to a level equal to the predicted value $y[(m + q)T]$ where $0 \leq q < 1$ and holds it for one sampling period. The prediction value of $y[(m + q)T]$, denoted $\hat{y}[(m + q)T]$, can be obtained by evaluating the mth-order Newton Extrapolation Polynomial by defining $y(iT) = y_i$ and $y[(m + q)T] = \hat{y}_{m+q}$, and finally, the proposed filter is obtained as follows

$$G_f(z) = \frac{\hat{y}(z)}{y(z)} = \sum_{k=0}^{m} B^m_k(q)z^{-k} \quad (3)$$

where

$$B^m_k(q) = (-1)^k \begin{pmatrix} q + m \choose q - k \end{pmatrix} \begin{pmatrix} m - k \choose k \end{pmatrix} \quad (4)$$

Repeatedly Belicynski and Kozinski (1984) assumed that $q=0.5$ or $\Delta = qT = T/2$. Table 2 represents filter transfer functions for $m=0, 1, 2$ using NEPM. It is shown that the NEPM has better result than PC-HOH.

<table>
<thead>
<tr>
<th>Order $(m)$</th>
<th>$G_f(z)$ for $\Delta = T/2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>$1.5 - 0.5z^{-1}$</td>
</tr>
<tr>
<td>2</td>
<td>$1.625 - 0.75z^{-1} + 0.125z^{-2}$</td>
</tr>
</tbody>
</table>

Table 2 Digital filter transfer functions related to NEPM

The method which Leonard (1999) proposed, is not based on extrapolation polynomials but he used the digital phase lead filter as follows

$$F(z) = \frac{1 + bz^{-1}}{1 + b} \quad -1 < b < 1 \quad (5)$$

Condition $-1 < b < 1$ guarantees that $F(z)$ is a minimum phase filter. Since $F(1) = 1$, the low frequencies are not changed with $F(z)$.

As it is known the ZOH transfer function is as follows

$$\text{ZOH}(j\omega) = \frac{1 - \exp(-j\omega T)}{j\omega} \quad (6)$$

see for example (Ogata, 1987; Phillips and Harbor, 1988) that has a phase lag of $0.5\omega T$. The usual Digital to Analog Converters (DAC) contains a sampler that has a gain $1/T$, so, compounding the frequency responses of (5) and (6) reach to:

$$H(j\omega) = \frac{1}{T} \text{ZOH}(j\omega) \frac{1 + b\exp(-j\omega T)}{1 + b} \quad (7)$$

In order to eliminate the phase delay of ZOH efficiently, Leonard (1999) proposed minimizing a cost function $J(b)$ which is defined as follows

$$J(b) = \int_{-\pi}^{\pi} \left| \Phi(\omega, b) \right|^2 d\omega \quad (8)$$

where $\Phi(\omega, b)$ is the phase of $H(j\omega)$ and $\omega_{BW}$ is the closed-loop bandwidth of the plant under control. This method is called Optimal Filter Method (OFM), which is compounding standard ZOH and optimal filter. From Sampling Theorem and Aliasing Phenomenon, the relation between sampling frequency and $\omega_{BW}$ can be found as

$$\omega_s = 6 \text{ to } 25 \omega_{BW} = k\omega_{BW} \quad (9)$$

Table 3 represents the optimal value of $b$ for different $k$. It is shown that OFM has better result than PC-HOH and NEPM, (for $m=1$). One advantage of OFM besides of a better phase delay compensation is, the OFM take into account the $\omega_{BW}$ and the ratio of sampling-time to $\omega_{BW}$ ($k = \omega_s / \omega_{BW}$) of the system to its cost function.

In the next section the proposed method which is called Optimal Extrapolator is presented. Section 3 presents the stability analysis of the proposed method. Frequency responses comparison among the different methods is done in section 4. An example is simulated in section 5.

2. MAIN RESULT

The main contribution of this paper is in defining a cost function which has been involved by $\Delta(0 \leq \Delta < T)$ or $q(0 \leq q < 1)$ and the best $\Delta$ or $q$ is computed by minimization this cost function. As it was seen in (1) the digital filter transfer function related to PC-HOH in $Z$-plane is as follow

$$G_m(z) = \frac{y_m(z)}{y(z)} = \sum_{i=0}^{m} \frac{\Delta^i}{i!} \left(\frac{1 - z^{-1}}{T}\right)^i$$

The frequency behavior of $G_m(z)$ $m = 1, 2, \ldots$ is obtained by substitution of $z = \exp(j\omega T)$ in (1), so

$$G_m(\exp(j\omega T)) = \sum_{i=0}^{m} \frac{\Delta^i}{i!} \left(1 - \exp(-j\omega T)\right)^i$$

(10)
Considering (6), since a DAC contains a sampler which has a gain $1/T$, so the compound transfer function of ZOH and proposed method for $m=1,2$ is as follows:

$$H_{11}(j\omega) = \frac{1}{T} ZOH(j\omega) \sum_{i=0}^{\infty} \frac{\Delta_i}{i!} \frac{1-\exp(-j\omega T)}{T}$$ (11)

$$H_{21}(j\omega) = \frac{1}{T} ZOH(j\omega) \sum_{i=0}^{\infty} \frac{\Delta_i}{i!} \left(1-\exp(-j\omega T)\right)^i$$ (12)

The phase of $H_{11}(j\omega)$ has been shown by $\Phi_{11}(\omega, \Delta_i)$ for $i=1,2$, respectively as below:

$$\Phi_{11}(\omega, \Delta_1) = -0.5\omega T + \tan^{-1}\left(\frac{\Delta_1 \sin(\omega T)}{T + \Delta_1(1-\cos(\omega T))}\right)$$ (13)

$$\Phi_{21}(\omega, \Delta_2) = -0.5\omega T + \tan^{-1}\left(\frac{2T\Delta_1 \sin(\omega T) + \Delta_2 (2 \sin(\omega T) - \sin(2\omega T))}{2T^2 + 2T\Delta_2 (1-\cos(\omega T)) + \Delta_2^2 (1+\cos(2\omega T)) - 2 \cos(\omega T)}\right)$$ (14)

Like (8) a cost function as follows can be defined:

$$J_d(\Delta_i) = \int_{0}^{\text{norm}} \left|\Phi_{11}(\omega, \Delta_i)\right|^2 d\omega \quad i=1,2$$ (15)

The objective is to minimize $J_d(\Delta_i)$ for $i=1,2$.

Now, for minimizing $J_d(\Delta_i)$ for $i=1,2$ with respect to variation of $\Delta_i$ in the interval [0, T] the function quad for integration and function min for finding the minimum from MATLAB software has been used. For better comparison $\Delta_i$ was normalized in interval [0,1], which the normalized value of $\Delta_i$ is denoted by $\Delta_{n_i}$. Table 4 shows $\Delta_{n_i}$ for $i=1,2$ and for different $k$ in the interval [0,1]. It can be seen from Table 4 that for small $k$ the optimal $\Delta$ is far from $T/2$ but when $k$ goes larger it will be close to $T/2$. This proposed holding device is denoted by $P_1$.

The above approach can be repeated for NEPM. From (3) we have:

$$G_f(z) = \frac{\bar{y}(z)}{y(z)} = \sum_{k=0}^{m} B^m_k(q) z^{-k}$$

It was mentioned that in NEPM for finding transfer functions $q=0.5$ was assumed, but it is not an optimal $q$. Like (11) and (12) the compound transfer functions for $m=1,2$ are as follows:

$$H_{12}(j\omega) = \frac{1}{T} ZOH(j\omega) \sum_{k=0}^{\infty} B^m_k(q_1) \exp(-kj\omega T)$$ (16)

$$H_{22}(j\omega) = \frac{1}{T} ZOH(j\omega) \sum_{k=0}^{\infty} B^m_k(q_2) \exp(-kj\omega T)$$ (17)

The phase of $H_{12}(j\omega)$ has been shown by $\Phi_{12}(\omega, q_i)$ for $i=1,2$, respectively, therefore:

$$\Phi_{12}(\omega, q_1) = -0.5\omega T + \tan^{-1}\left(\frac{q_1 \sin(\omega T)}{q_1(1-\cos(\omega T)) + 1}\right)$$ (18)

$$\Phi_{22}(\omega, q_2) = -0.5\omega T + \tan^{-1}\left(\frac{q_2 (q_2+1) \sin(\omega T) - 1/2q_2 (q_2+1) \sin(2\omega T)}{q_2 (q_2+1) \sin(\omega T) - 1/2q_2 (q_2+1) \sin(2\omega T) + 1/2q_2 (q_2+1) \sin(2\omega T)}\right)$$ (19)

Like (15) a cost function as follows can be defined:

$$J_d(\alpha_i) = \int_{0}^{\text{norm}} \left|\Phi_{12}(\omega, q_i)\right|^2 d\omega \quad i=1,2$$ (20)

Again the objective is to minimize $J_d(q_i)$ for $i=1,2$ with respect to $q_i$. Table 5 shows optimal $q_i$ for different values of $k$. This proposed holding device is denoted by $P_2$.

Since $\Delta = qT$ so (13) is the same as (18). Therefore $P_1$ and $P_2$ are the same for $m=1$.

Investigating the Table 3,4 it is seen that $P_1$, $P_2$ and the OFM have the same zero (zero at $z=q/(q+1)=-b$) and the same pole (pole at $z=0$) for $m=1$.

One advantage of these two methods is to permit of using lower sampling rate. It can be seen that the closed-loop bandwidth of system is involved in $P_1$ and $P_2$. Table 6 and 7 represents the digital filter transfer functions related to $P_1$ and $P_2$ using the optimal $\Delta_i$ and $q_i$ for $k=6$, and $m=0, 1, 2$.

The higher order filters can be obtained as above, easily.

<table>
<thead>
<tr>
<th>$k$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b$</td>
<td>-0.619</td>
<td>-0.547</td>
<td>-0.477</td>
<td>-0.432</td>
<td>-0.405</td>
<td>-0.374</td>
<td>-0.359</td>
<td>-0.345</td>
<td>-0.340</td>
<td>-0.338</td>
</tr>
</tbody>
</table>

Table 3 The parameters of the OFM method for different $k$
### Table 4 Proposed method parameters for PC-HOH and different k (m=1,2)

<table>
<thead>
<tr>
<th>k</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta_{in} )</td>
<td>0.9997</td>
<td>0.9995</td>
<td>0.9110</td>
<td>0.7582</td>
<td>0.6769</td>
<td>0.6151</td>
<td>0.5601</td>
<td>0.5350</td>
<td>0.5020</td>
<td>0.5086</td>
<td>0.5152</td>
</tr>
<tr>
<td>( \Delta_{2in} )</td>
<td>0.7503</td>
<td>0.6182</td>
<td>0.5326</td>
<td>0.4967</td>
<td>0.4835</td>
<td>0.4798</td>
<td>0.4888</td>
<td>0.5231</td>
<td>0.5020</td>
<td>0.4954</td>
<td>0.5073</td>
</tr>
</tbody>
</table>

### Table 5 The parameters of the proposed method for NEPM and different k (m=1,2)

<table>
<thead>
<tr>
<th>k</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>( q_1 )</td>
<td>0.9997</td>
<td>0.9995</td>
<td>0.9110</td>
<td>0.7582</td>
<td>0.6769</td>
<td>0.6151</td>
<td>0.5601</td>
<td>0.5350</td>
<td>0.5020</td>
<td>0.5086</td>
<td>0.5152</td>
</tr>
<tr>
<td>( q_2 )</td>
<td>0.67</td>
<td>0.53</td>
<td>0.45</td>
<td>0.42</td>
<td>0.42</td>
<td>0.43</td>
<td>0.45</td>
<td>0.45</td>
<td>0.55</td>
<td>0.49</td>
<td>0.50</td>
</tr>
</tbody>
</table>

### 3. STABILITY ANALYSIS

As mentioned in section 2 and from equation (1) the transfer functions for \( m=1 \) and 2 are:

\[
1 + \frac{1 - z^{-1}}{T} \Delta \quad 0 \leq \Delta < T
\]

\[
1 + \frac{1 - z^{-1}}{T} \Delta + \left(1 - \frac{1 - z^{-1}}{T}\right)^2 \Delta^2 \quad 0 \leq \Delta < T
\]

respectively. As it is known \( \Delta = qT \) which \( 0 \leq q < 1 \), so equations (21) and (22) will become as follows

\[
\frac{(q + 1)z - q}{z} \quad 0 \leq q < 1
\]

\[
\frac{(q^2 + q + 2)z^2 - 2q(q + 1)z + q^2}{2z^2} \quad 0 \leq q < 1
\]

Obviously (23) is the same as (3) for \( m=1 \). The transfer function of (3) for \( m=2 \) is

\[
\frac{(q + 1)(q + 2)z^2 - 2q(q + 2)z + q(q + 1)}{2z^2}
\]

It is seen, the poles of the transfer functions are in the unit circle since \( 0 \leq q < 1 \). The zero of (23) is in the unit circle because \( z = q/(q + 1) \). Considering Jury's lemma it is obvious that the numerator polynomial of (24) and (25) satisfies the stability conditions. It can be seen that the transfer functions of the proposed methods for \( m=1,2 \) are both minimum phase for any \( q \) in the interval \([0,1]\), so they are easy to implement. Also from Table 6 and 7 it can be seen that they are close to each other, for \( m=2 \), so it can be said that the optimization methods reach to the same results.

### Table 6 Digital filter transfer functions related to \( P_1 \)

<table>
<thead>
<tr>
<th>Order(( m ))</th>
<th>( G(z) ) for ( k=6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>( 1.679 - 0.669z^{-1} )</td>
</tr>
<tr>
<td>2</td>
<td>( 1.7182 - 1.0164z^{-1} + 0.2982z^{-2} )</td>
</tr>
</tbody>
</table>

### Table 7 Digital filter transfer functions related to \( P_2 \)

<table>
<thead>
<tr>
<th>Order(( m ))</th>
<th>( G(z) ) for ( k=6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>( 1.679 - 0.669z^{-1} )</td>
</tr>
<tr>
<td>2</td>
<td>( 1.7182 - 1.0164z^{-1} + 0.2982z^{-2} )</td>
</tr>
</tbody>
</table>

### 4. FREQUENCY RESPONSE

In this part the comparison of the frequency responses of different methods is shown. The closed-loop bandwidth \( \omega_{BW} = 1.66 \) and \( k=6 \) has been assumed.

Figs. 1 and 2 show the Bode plots of ZOH, \( P_1 \), \( P_2 \) for \( m=1 \) and OFM. It can be seen that \( P_1 \), \( P_2 \) for \( m=1 \) and OFM as mentioned in section 2 have same frequency response. It is also can be seen that the phase delay has been compensated. Figs. 3 and 4 represent the Bode plots of ZOH, PC-HOH, NEPM, \( P_1 \) and \( P_2 \) for \( m=1 \). It can be seen that \( P_1 \) and \( P_2 \) has better phase delay compensation.

Figs. 5 and 6 represents the Bode plots of ZOH, PC-HOH and \( P_1 \) for \( m=2 \), and Figs. 7 and 8 represent the Bode plots of ZOH, NEPM and \( P_2 \) for \( m=2 \). It can be seen that, although for \( m=2 \), PC-HOH and NEPM has higher phase than \( P1 \) and \( P2 \) but they are not optimal with respect to cost functions defined in (15) and (20), so optimal extrapolators have better result than PC-HOH, NEPM and it concludes OFM, too.

Magnitude distortion as will be seen in Figs. 1, 3, 5 and 7 is acceptable, because in most applications an anti-aliasing filter is employed to attenuate high frequencies and also frequencies near Nyquist frequencies in order to avoid aliasing consequences, see for example (Astorn and Wittenmark, 1990; Landau, 1993; Leonard, 1999), therefore reducing the phase lag on bandwidth, will be more important. Clearly, as can be seen in Figs. 2, 4, 6 and 8 by increasing the extrapolation order (\( m \)) this aim can be achieved.
Fig. 1. Frequency Response-Magnitude $P_1$, $P_2$ and OFM

Fig. 2. Frequency Response-phase $P_1$, $P_2$ and OFM

Fig. 3. Frequency Response-Magnitude $P_1$, $P_2$, PC-HOH and NEPM for $m=1$

Fig. 4. Frequency Response-Phase $P_1$, $P_2$, PC-HOH NEPM for $m=1$

Fig. 5. Frequency Response-Magnitude $P_1$, PC-HOH for $m=2$

Fig. 6. Frequency Response-Phase $P_1$, PC-HOH for $m=2$

Fig. 7. Frequency Response-Magnitude $P_2$, NEPM for $m=2$

Fig. 8. Frequency Response-Phase $P_2$, NEPM for $m=2$
5. EXAMPLE

Consider the lead/lag compensated double-integrator plant which has been shown in Fig. 9. This system is used as a benchmark in such cases. If the controller digitized by Tustin transformation then it leads to $G(z) = \frac{0.37z}{0.7950(z - 0.037)}$.

The open loop bandwidth is $\omega_{oc} = 1$ rad/s. The sampling frequency has been selected as $\omega_s = 10$ rad/s. So the closed-loop bandwidth $\omega_{BW} = \omega_s / 6 = 1.66 > 1$ is obtained. The controller is followed by one of three holding devices 1) The standard ZOH 2) $P_1$ for $m=1$ 3) $P_1$ for $m=2$. Table 8 demonstrates the system's characteristic. Phase delay compensation can be deduced when optimal extrapolator is added to the classical ZOH.

6. CONCLUSIONS

In this paper a new method for reducing the phase delay of ZOH was presented. The proposed method that is called Optimal Extrapolator provides better result than PC-HOH and NEPM. The proposed method for $m=1$ has as the same results as OFM. Furthermore the proposed method has a general framework for higher order extrapolator polynomials.

### Table 8 Results of system using different controllers

<table>
<thead>
<tr>
<th>System</th>
<th>Phase Margin [deg]</th>
<th>Gain Margin [dB]</th>
<th>Open Loop BW (0 dB) [rad/s]</th>
<th>Closed Loop BW (0 dB) [rad/s]</th>
<th>Closed Loop Peak/Freq. [dB]/[rad/s]</th>
<th>Step Over shoot [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Continuous</td>
<td>50.03</td>
<td>Inf</td>
<td>1</td>
<td>1.24</td>
<td>2.58/0.63</td>
<td>27.72</td>
</tr>
<tr>
<td>Digital + ZOH</td>
<td>31.78</td>
<td>8.65</td>
<td>1.01</td>
<td>1.71</td>
<td>5.2/1.05</td>
<td>51.94</td>
</tr>
<tr>
<td>Digital+P1 (m=1)</td>
<td>48.49</td>
<td>6.79</td>
<td>1.24</td>
<td>2.6</td>
<td>2.2/0.660</td>
<td>29.57</td>
</tr>
<tr>
<td>Digital+P1 (m=2)</td>
<td>50.14</td>
<td>7.49</td>
<td>1.08</td>
<td>1.58</td>
<td>2.54/0.666</td>
<td>24</td>
</tr>
</tbody>
</table>

### REFERENCES


