ANALYSIS OF RANDOM REFERENCE TRACKING IN SYSTEMS WITH SATURATING ACTUATORS

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Abstract: This paper develops a method for analysis of random reference tracking in feedback systems with saturating actuators. The development is motivated by the frequency domain approach to linear systems, where the bandwidth and resonance peak of the sensitivity function are used to predict the quality of step reference tracking. Similarly, based on the so-called saturating random sensitivity function, we introduce tracking quality indicators and show that they can be used to determine both the quality of random reference tracking and the nature of track loss under actuator saturation. Copyrights © 2005 IFAC

Keywords: Reference tracking, Saturating actuators, Tracking quality indicators

1. INTRODUCTION

1.1 Motivation

As it is well known, the quality of reference tracking in linear systems is determined by the loop transfer function. In systems with saturating actuators, this is not the case. Indeed, for example, consider the SISO feedback system and the reference signal shown in Figure 1, where the latter is a realization of a colored noise process with power spectral density \( S_r(\omega) = \frac{6}{1+5(\omega/0.5)^2} \). The quality of tracking for several \( C(s) \) and \( P(s) \), satisfying \( C(s)P(s) = \frac{75}{s(s+10)} \), is illustrated in Figure 2. (In Figure 2(a), the reference and the output signals practically coincide.) Clearly, the nature of tracking errors in each of the three cases is qualitatively different, which supports the above assertion.

The track loss in systems with saturating actuators may occur due to a number of different reasons. These include those that occur in linear systems plus those due to actuator saturation. To illustrate these reasons, consider again the system and the reference signal of Figure 1 and select \( C(s) \) and \( P(s) \), which result in different patterns of track loss but with the same standard deviation of the tracking error, \( \sigma_e \). The results are shown in Figure 3 (for \( \sigma_e = 0.67 \)). As one can see, track loss in Figures 3 (a)–(c) is due to static unresponsiveness, dynamic lagging, and oscillatory behavior, respectively. These reasons take place in the purely linear case as well (see Eun et al. 2003). Track loss in Figures 3 (d)–(g) is due to saturation, namely, amplitude truncation without controller wind-up, amplitude truncation with the controller wind-up, amplitude truncation with the controller wind-up...

Fig. 1. Feedback control system with saturating actuator and reference signal
The goals of this paper are to analyze what determines the quality of tracking in systems with saturating actuators and quantify under which conditions one or another type of track loss takes place.

In the case of linear systems, the quality of step input tracking is often characterized in the frequency domain by the Sensitivity (S) function, specifically, by its d.c. gain, bandwidth, and resonance peak. Recently, this approach has been extended to tracking random inputs by introducing the notion of random sensitivity (RS) function (Eun et al. 2003). In particular, it has been shown that the d.c. gain, bandwidth, and resonance peak of the RS function characterize the quality of random reference tracking in linear systems in the same manner as the S function characterizes the quality of tracking steps. In the current paper, we extend this approach to systems with saturating actuators. This is accomplished by introducing and analyzing the so-called saturating random sensitivity (SRS) function. Due to actuator nonlinearity, the SRS function depends not only on the frequency but also on the “amplitude” of the signals involved and, therefore, is a function of two independent variables. We provide a method for calculating the SRS using a quasi-linearization technique known as stochastic linearization (Roberts and Spanos 1990). In (Gökçek et al. 2001), stochastic linearization has been used for analysis and design of systems with saturating actuators from the point of view of disturbance rejection. In this paper, we use it in the framework of reference tracking.

1.3 Related Literature and Paper Outline

Systems with saturating actuators have been studied for a long time (see recent monographs (Saberi et al. 2000, Hu and Lin 2001, Kapila 2002)). However, just a few publications have been devoted to reference tracking. These include (Yakubovich et al. 1999) where tracking domains have been investigated, (Saberi et al. 2000) where asymptotic output tracking has been studied, (Goldfarb and Sirithanapipat 1999) where tracking domains have been analyzed, and (Eun et al. 2004a) where the notion of system type has been extended to feedback control with saturating actuators. However, no general methods for analysis of quality of random reference tracking in systems with saturating actuators exist. This paper is intended to contribute to this end.

To accomplish this, Section 2 introduces the SRS and its characteristics: d.c. gain, bandwidth, resonance frequency and resonance peak. In Section 3, we use these characteristics to define dimensionless tracking quality indicators and diagnostic flow charts. Finally, in Section 4 the conclusions are given. Due to space limitations the proofs are not included here and can be found in (Eun et al. 2004b).
2. SATURATING RANDOM SENSITIVITY FUNCTION

2.1 Random Reference Signals

Similar to (Eun et al. 2003), the class of random reference signals, considered in the work, is defined as the scaled steady state output of the third order Butterworth filter driven by a standard white Gaussian process. The transfer function of this filter is given by

\[ F(s; \Omega) = \sqrt{\frac{3}{\Omega}} \left( \frac{\Omega^3}{s^3 + 2\Omega s^2 + 2\Omega^2 s + \Omega^3} \right), \tag{1} \]

where the d.c. gain is selected so that, for all 3-dB bandwidths \( \Omega \), the standard deviation of the output is 1. Thus, the reference signals considered in this work are given by

\[ r(t) = \sigma_r r(t; \Omega), \tag{2} \]

where \( r(t; \Omega) \) is the output of (1) and \( \sigma_r \) is the “amplitude” or, more precisely, the standard deviation of \( r(t) \).

Clearly, higher order Butterworth filters can be considered instead of (1). However, as it turns out, the results remain quite similar to those obtained using (1) (see also (Eun et al. 2003)) and, thus, for the sake of simplicity, we consider band-limited reference signals \( r(t) \) defined by (1) and (2).

2.2 System Model

Consider the system shown in Figure 4 with reference signal (1), (2) and \( \text{sat}_\alpha(u) \) defined by

\[ \text{sat}_\alpha(u) = \begin{cases} 
\alpha & \text{if } \alpha < u, \\
u & \text{if } -\alpha \leq u \leq \alpha, \\
-\alpha & \text{if } u < -\alpha.
\end{cases} \tag{3} \]

Due to the nonlinearity, exact analysis of this system requires solving the Fokker-Plank equation, which is possible only in a few special cases. Therefore, a simplification is necessary. We use for this purpose the method of stochastic linearization (Roberts and Spanos 1990). According to this method, the saturation function is replaced by a linear function, the slope of which depends on the standard deviation of the signal at the input of the saturation. This method is akin to the method of describing functions and ensures similar accuracy.

Using stochastic linearization, the nonlinear system of Figure 4 can be replaced by the quasi-linear system shown in Figure 5, where the equivalent gain \( N(\sigma_u) \) is given by (Gökeck et al. 2001)

\[ N(\sigma_u) = \frac{\alpha}{\sqrt{2\pi} \sigma_u}, \tag{4} \]

\[ \text{erf}(\xi) = \frac{2}{\sqrt{\pi}} \int_0^\xi \exp(-t^2) \, dt. \tag{5} \]

The system of Figure 5 is quasi-linear since \( N \) depends on the standard deviation of \( \hat{u} \).

The results reported in this paper are obtained using the simplified model of Figure 5. However, the system of Figure 4 is also used – to verify that the results derived are applicable to the original nonlinear system as well.

2.3 Definition and Properties of the Saturating Random Sensitivity Function

If \( N \) in Figure 5 were a constant gain equal to 1, the sensitivity and the random sensitivity functions of the closed loop system would be given by (Eun et al. 2003)

\[ S(s) = \frac{1}{1 + P(s)C(s)}. \tag{6} \]

\[ RS(\Omega) = \left\| \frac{F(s; \Omega)}{1 + P(s)C(s)} \right\|_2. \tag{7} \]

These functions are extended to the case of the quasi-linear system of Figure 5 by defining the saturating random sensitivity (SRS) as follows:

\[ SRS(\Omega, \sigma_r) = \frac{\left\| \frac{F(s; \Omega)}{1 + N P(s)C(s)} \right\|_2}{\sigma_r}, \tag{8} \]

\[ N = \text{erf} \left( \frac{\alpha}{\sqrt{2}\sigma_r} \right). \tag{9} \]

As one can see, physically \( SRS(\Omega, \sigma_r) \) represents the ratio of the standard deviations of the error signal \( \hat{e}(t) \) and reference signal \( r(t) \), i.e.,

\[ SRS(\Omega, \sigma_r) = \frac{\sigma_{\hat{e}}}{\sigma_r}. \]

Asymptotic properties of \( SRS(\Omega, \sigma_r) \) are as follows:

**Theorem 1.** Assume that the closed loop system of Figure 5 is asymptotically stable for all \( N \in (0, 1] \), \( P(s) \) is strictly proper and \( C(s) \) is proper. Then,

(i) for any \( \Omega > 0 \),

\[ \lim_{\sigma_r \to 0} SRS(\Omega, \sigma_r) = \left\| \frac{F(s; \Omega)}{1 + P(s)C(s)} \right\|_2; \tag{10} \]

(ii) for any \( \sigma_r > 0 \),

\[ \lim_{\Omega \to \infty} SRS(\Omega, \sigma_r) = 1; \tag{11} \]
\[
\lim_{\Omega \to 0} SRS(\Omega, \sigma_r) = \left| \frac{1}{1 + NP(0)C(0)} \right|.
\]

where \(N\) satisfies
\[
N = \text{erf} \left( \sqrt{2} \sigma_r \frac{\alpha}{C(0)} \right).
\]

Clearly, statement (10) implies that for small reference signals, \(SRS(\Omega, \sigma_r)\) practically coincides with \(RS(\Omega)\). Statement (11) indicates that for large \(\Omega\) the functions \(SRS(\Omega, \sigma_r)\), \(RS(\Omega)\) and \(S(s)\) are practically identical, and no tracking takes place. Finally, since \(N \leq 1\), statement (12) shows that for low frequencies \(SRS(\Omega, \sigma_r)\) is typically larger than \(RS(\Omega)\) and, thus, the presence of saturation impedes tracking.

Figures 6 and 7 illustrate the \(SRS\) functions for all systems of Figures 2 and 3, respectively. As it will be shown in Section 3, these functions define the nature of tracking and track loss in the corresponding systems.

2.4 Shape Characteristics

Although a complete description of \(SRS(\Omega, \sigma_r)\) requires a two-dimensional surface, a compact (but incomplete) description can be given in terms of

| Fig. 6. \(SRS(\Omega, \sigma_r)\) for systems of Figure 2. |
| (a) | (b) | (c) |

| Fig. 7. \(SRS(\Omega, \sigma_r)\) for systems of Figure 3. |
| (d) | (e) | (f) |

Table 1. Trackable Domains for systems of Figure 2

| \(|TD|\) | (a) | (b) | (c) |
|-------|-----|-----|-----|
| \(\infty\) | \(\infty\) | 1.5 |

Table 2. Trackable Domains for systems of Figure 3

| \(|TD|\) | (a) | (b) | (c) | (d) | (e) | (f) | (g) |
|-------|-----|-----|-----|-----|-----|-----|-----|
| 3.75  | \(\infty\) | \(\infty\) | 0.53 | 0.40 | \(\infty\) | \(\infty\) |

characteristics, similar to those used to describe the \(S(s)\) and \(RS(\Omega)\) functions. Namely, introduce

(i) saturating random d.c. gain:
\[
SR_{dc} = \lim_{\Omega \to 0, \sigma_r \to 0} SRS(\Omega, \sigma_r),
\]

(ii) saturating random bandwidth:
\[
SR_{\Omega BW}(\sigma_r) = \min \{\Omega | SRS(\Omega, \sigma_r) = 1/\sqrt{2}\},
\]

(iii) saturating random resonance frequency:
\[
SR_{\Omega r}(\sigma_r) = \arg \max_{\Omega > 0} SRS(\Omega, \sigma_r),
\]

(iv) saturating random resonance peak:
\[
SR_{M r}(\sigma_r) = \sup_{\Omega > 0} SRS(\Omega, \sigma_r).
\]

For the \(SRS\) functions of Figures 7 (a) and (d), \(SR_{dc}\) are 0.67 and 0.019, respectively, while for all others it is 0. Clearly, one might expect that tracking of even small and slowly changing signals in the system of Figure 7 (a) is poor, and the track loss is due to static unresponsiveness.

The \(SR_{\Omega BW}\) for all systems of Figures 2 and 3 are shown in Figures 8 and 9, respectively. In all cases \(SR_{\Omega BW}\) is monotonically decreasing in \(\sigma_r\), but systems of Figure 2 (c) and Figures 3 (a), (d), (e) result in \(SR_{\Omega BW}\) with almost infinite roll-off rate. This phenomenon can be explained using the notion of Trackable Domain (TD) introduced in (Eun et al. 2004a). Indeed, it has been shown in (Eun et al. 2004a) that the set of step inputs that can be tracked by a system with saturating actuators and its size can be quantified, respectively, as
\[
TD = \left\{ r_0 \in \mathbb{R} : |r_0| < \left| \frac{1}{C_0} + P_0 \right| \alpha \right\},
\]

\[
|TD| = \left| \frac{1}{C_0} + P_0 \right| \alpha,
\]

where \(r_0\) is the size of the step and \(C_0\) and \(P_0\) are d.c. gains of the controller and plant, respectively. Trackable domains for all systems of Figures 2 and 3 are given in Tables 1 and 2, respectively. Clearly, systems of Figure 2 (c) and Figures 3 (a), (d), (e) have finite trackable domains and, therefore, their bandwidth must drop to 0 for \(\sigma_r\) sufficiently large, no matter how small \(\Omega\) is.
The tracking quality indicators for this system and reference signal are:

\[ I_0 = \frac{\sigma_r}{|\overline{TD}|}, \quad I_1 = SR_{dc}, \quad I_2 = \frac{\Omega}{SR\Omega_{BW}(\sigma_r)}, \quad I_3 = \min \left( \frac{\Omega}{SR\Omega_r(\sigma_r)}, SRM_r(\sigma_r) - 1 \right) \]

Clearly, \( I_0 \) quantifies the “size” of the reference signal vis-a-vis the trackable domain; large \( I_0 \) implies that amplitude truncation must take place. Indicator \( I_1 \) quantifies the level of static responsiveness; large \( I_1 \) implies that responsiveness, even to small and slow signals, is poor. Indicator \( I_2 \) quantifies the bandwidth of the reference signal in units of the closed loop bandwidth; large \( I_2 \) implies that dynamic lagging must take place. Finally, \( I_3 \) characterizes oscillatory properties of the response; large \( I_3 \) implies that oscillations must be present.

Although indicators \( I_1-I_3 \) are proper extensions of the corresponding tracking quality indicators for linear systems (Eun et al. 2003), they may be large due to either linear or nonlinear part of the system. The two cases can be discriminated by the value of the equivalent gain, \( N \), defined by (9). Specifically if \( N \) is close to 1, the phenomenon is caused by the linear part of the system, otherwise, it is due to saturation.

Based on the above discussion, the nature of tracking quality and reasons for track loss can be diagnosed using the flow charts shown in Figure 10. Each of them includes a qualitative term “large”.

Based on our experience, an indicator can be viewed as large if

\[ I_0 > 0.4, \quad I_1 > 0.1, \quad I_2 > 0.4, \quad I_3 > 0.2. \]

Consider, for example, the system of Figure 4 with \( C(s) = 5/s, \ P(s) = 15/(s + 10) \) and \( r(t) = 1.5 \ r(t; 10) \). The tracking quality indicators for this system and reference signal are:

\[ I_0 = 1, \ I_1 = 0, \ I_2 = 4.705, \ I_3 = 0.078, \]

while \( N = 0.47 \). Thus, using Figure 10 (a) we determine that tracking is poor due to the amplitude truncation with wind-up. Using Figure 10 (b), we conclude that there is no loss of tracking due to unresponsiveness. Using Figure 10 (c), we expect lagging due to saturation (i.e., nonlinear lagging). These conclusions are supported by the traces of \( y(t) \) (obtained by simulating the system of Figure 4) shown in Figure 11.

Table 3 presents the tracking quality indicators and the conclusions as to the nature of tracking and track loss for all systems considered in Section 1.

Remark: The diagnostics approach, described above, leads to qualitatively correct results in the majority of cases analyzed. However, it is not always the case. Typically, this approach fails when \( C(s) \) and \( P(s) \) are such that the usual sensitivity function, \( S(s) \), does not predict the step response well. An
Table 3. Diagnosed quality of tracking in systems of Figures 2 and 3.

<table>
<thead>
<tr>
<th>Fig.</th>
<th>$I_0$</th>
<th>$I_1$</th>
<th>$I_2$</th>
<th>$I_3$</th>
<th>$N$</th>
<th>$C(s)$ pole at $s = 0$?</th>
<th>Track qual. &amp; track loss</th>
</tr>
</thead>
<tbody>
<tr>
<td>2(a)</td>
<td>0</td>
<td>0</td>
<td>0.187</td>
<td>0.017</td>
<td>2</td>
<td>No</td>
<td>good</td>
</tr>
<tr>
<td>2(b)</td>
<td>0</td>
<td>0</td>
<td>0.791</td>
<td>0.003</td>
<td>0.049</td>
<td>No</td>
<td>nonlinear lag.</td>
</tr>
<tr>
<td>2(c)</td>
<td>0.667</td>
<td>0</td>
<td>0.187</td>
<td>0.006</td>
<td>1</td>
<td>No</td>
<td>static unresponsiveness</td>
</tr>
<tr>
<td>3(b)</td>
<td>0</td>
<td>0</td>
<td>0.797</td>
<td>0.756</td>
<td>1</td>
<td>Yes</td>
<td>linear oscill.</td>
</tr>
<tr>
<td>3(c)</td>
<td>1.873</td>
<td>0.019</td>
<td>0.200</td>
<td>0.001</td>
<td>0.014</td>
<td>No</td>
<td>ampl. trunc. without windup</td>
</tr>
<tr>
<td>3(d)</td>
<td>1.25</td>
<td>0</td>
<td>0.02</td>
<td>0.106</td>
<td>Yes</td>
<td>ampl. trunc. without windup and nonlinear lag.</td>
<td></td>
</tr>
<tr>
<td>3(e)</td>
<td>0</td>
<td>0</td>
<td>1.080</td>
<td>0.002</td>
<td>0.043</td>
<td>No</td>
<td>nonlinear lag.</td>
</tr>
<tr>
<td>3(f)</td>
<td>0</td>
<td>0</td>
<td>1.025</td>
<td>0.406</td>
<td>0.182</td>
<td>No</td>
<td>nonlinear osc.</td>
</tr>
</tbody>
</table>

Fig. 10. Diagnostic flow charts for analysis of tracking quality in systems with saturating actuators.

Example of this type, where neither linear nor saturating cases are well characterized by their sensitivity functions, can be found in (Eun et al. 2004b).

4. CONCLUSIONS

This paper provides a simple method for analysis of random reference tracking in systems with saturating actuators. The method mimics the classical frequency domain approach to step reference tracking in linear systems. Indeed, it is based on the indicators, which are similar to bandwidth and resonance peak, used in the linear case, and which allow one to predict the quality of random reference tracking and nature of track loss in systems with saturating actuators.

The method developed in this paper offers control system designers a quick and easy way to predict system performance without resorting to lengthy and expensive numerical simulations. In addition, it illuminates reasons for track loss, which might be useful for developing improvement measures.

REFERENCES


Fig. 11. Tracking $r(t) = 1.5 r(t; 10)$ in system of Figure 4 with $C(s) = 5/s$, $P(s) = 15/(s + 10)$ and $\alpha = 1$. 