A SUM OF SQUARES APPROXIMATION OF NONNEGATIVE POLYNOMIALS

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Abstract: We show that every real nonnegative polynomial $f$ can be approximated as closely as desired by a sequence of polynomials $\{f_{\epsilon}\}$ that are sums of squares. Each $f_{\epsilon}$ has a simple and explicit form in terms of $f$ and $\epsilon$. Copyright © 2005 IFAC

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1. INTRODUCTION

The study of relationships between nonnegative and sums of squares (s.o.s.) polynomials, initiated by Hilbert, is of real practical importance in view of numerous potential applications, notably in polynomial programming. Indeed, checking whether a given polynomial is nonnegative is a NP-hard problem whereas checking it is s.o.s. reduces to solving a (convex) Semidefinite Programming (SDP) problem for which efficient algorithms are now available. For instance, recent results in real algebraic geometry, most notably in [Schm¨udgen, 1991], [Putinar, 1993], [Jacobi and Prestel, 2001] have provided s.o.s. representations of polynomials, positive on a compact semialgebraic set; the interested reader is referred to [Prestel and Delzell, 2001], and [Scheiderer, 2003] for a nice account of such results. This in turn has permitted to develop efficient SDP-relaxations in polynomial optimization; see e.g. [Lasserre, 2001, 2002], [Parrilo, 2003], [Schweighofer, 2004], and the many references therein. See also [Henrion and Lasserre, 2004] for control applications.

So, back to a comparison between nonnegative and s.o.s. polynomials, on the negative side, [Blekherman, 2004] has shown that if the degree is fixed, then the cone of nonnegative polynomials is much larger than that of s.o.s. However, on the positive side, a denseness result states that the

\text{cone of s.o.s. polynomials is } \text{dense} \text{ in the space of polynomials that are nonnegative on } [-1, 1]^n \text{ (for the norm } \|f\|_1 = \sum_{\alpha} |f_{\alpha}| \text{ whenever } f \text{ is written } \sum_{\alpha} f_{\alpha} x^\alpha \text{ in the usual canonical basis); see e.g. Theorem 5, p. 122 in [Berg, 1980].}

**Contribution.** We show that every nonnegative polynomial $f$ is almost a s.o.s., namely we show that $f$ can be approximated by a sequence of s.o.s. polynomials $\{f_{\epsilon}\}$, in the specific form

\[ f_{\epsilon} = f + \epsilon \sum_{k=0}^{r_{\epsilon}} \sum_{j=1}^{n} x_{i,j}^{2k+1}, \]

for some $r_{\epsilon} \in \mathbb{N}$, so that $\|f - f_{\epsilon}\|_1 \to 0$ as $\epsilon \downarrow 0$.

This result is in the spirit of the previous denseness result. However we here provide in (1) an explicit converging approximation with a very specific (and simple) form; namely it suffices to slightly perturbate $f$ by adding a small coefficient $\epsilon > 0$ to each square monomial $x_{i,j}^{2k}$ for all $i = 1, \ldots, n$ and all $k = 1, \ldots, r$, with $r$ sufficiently large.

To prove this result we combine

- (generalized) Carleman’s sufficient condition for a moment sequence $y = \{y_{\alpha}\}$ to have a representing measure $\mu$ (i.e., such that $y_{\alpha} = \int x^\alpha d\mu$ for all $\alpha \in \mathbb{N}^n$), and

- a duality result from convex optimization.
As a consequence, we may thus define a procedure to approximate the global minimum of a polynomial \( f \). It consists in solving a sequence of SDP-relaxations which are simpler and easier to solve than those defined in [Laszerre, 2001].

2. NOTATION AND DEFINITIONS

For a real symmetric matrix \( A \), the notation \( A \succeq 0 \) (resp. \( A \succ 0 \)) stands for \( A \) positive semidefinite (resp. positive definite). The sup-norm \( \| x \|_\infty \) of a vector \( x \in \mathbb{R}^n \), is denoted by \( \| x \|_\infty \). Let \( \mathbb{R}[x_1, \ldots, x_n] \) be the real polynomials, and let

\[
v_r(x) := (1, x_1, \ldots, x_n, x_1^2, x_1x_2, \ldots, x_n^2) \tag{2}
\]

be the canonical basis for the \( \mathbb{R} \)-vector space \( \mathcal{A} \), of real polynomials of degree at most \( r \), and let \( s(r) \) be its dimension. Similarly, \( v_\infty(x) \) denotes the canonical basis of \( \mathbb{R}[x_1, \ldots, x_n] \) as a \( \mathbb{R} \)-vector space, denoted \( \mathcal{A} \). So a vector in \( \mathcal{A} \) has always finitely many zeros.

Therefore, a polynomial \( p \in \mathcal{A} \) is written

\[
x \mapsto p(x) = \sum_\alpha p_\alpha x^\alpha = \langle p, v_r(x) \rangle, \quad x \in \mathbb{R}^n,
\]

where \( x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2} \cdots x_n^{\alpha_n} \) for some vector \( p = \{p_\alpha\} \in \mathbb{R}^{s(r)} \), the vector of coefficients of \( p \) in the basis (2).

Extending \( p \) with zeros, we can also consider \( p \) as a vector indexed in the basis \( v_\infty(x) \) (i.e. \( p \in \mathcal{A} \)). If we equip \( \mathcal{A} \) with the usual scalar product \( \langle \cdot, \cdot \rangle \) of vectors, then for every \( p \in \mathcal{A} \),

\[
p(x) = \sum_{\alpha \in \mathbb{N}^n} p_\alpha x^\alpha = \langle p, v_\infty(x) \rangle, \quad x \in \mathbb{R}^n.
\]

Given a sequence \( y = \{y_\alpha\} \) indexed in the basis \( v_\infty(x) \), let \( L_y : \mathcal{A} \to \mathbb{R} \) be the linear functional

\[
x \mapsto L_y(p) := \sum_{\alpha \in \mathbb{N}^n} p_\alpha y_\alpha = \langle p, y \rangle.
\]

Given a sequence \( y = \{y_\alpha\} \) indexed in the basis \( v_\infty(x) \), the moment matrix \( M_r(y) \in \mathbb{R}^{s(r) \times s(r)} \) with rows and columns indexed in the basis \( v_r(x) \) in (2), satisfies

\[
[M_r(y)(1,1) = y_\alpha \text{ and } M_r(y)(i,1) = y_\beta] \implies M_r(y)_{i,j} = y_{\alpha+i\beta}.
\]

For instance, with \( n = 2 \),

\[
M_2(y) = \begin{bmatrix}
00 & 01 & 02 & 03 & 04 \\
01 & 02 & 03 & 04 & 05 \\
02 & 03 & 04 & 05 & 06 \\
03 & 04 & 05 & 06 & 07 \\
04 & 05 & 06 & 07 & 08
\end{bmatrix}.
\]

An sequence \( y = \{y_\alpha\} \) has a representing measure \( \mu_y \) if

\[
y_\alpha = \int_{\mathbb{R}^n} x^\alpha \, d\mu_y, \quad \forall \alpha \in \mathbb{N}^n. \tag{3}
\]

In this case one also says that \( y \) is a moment sequence. In addition, if \( \mu_y \) is unique then \( y \) is said to be a determinate moment sequence.

The matrix \( M_r(y) \) defines a bilinear form \( \langle \cdot, \cdot \rangle_y \) on \( \mathcal{A} \), by

\[
(q,p)_y := \langle q, M_r(y)p \rangle = L_y(qp), \quad q,p \in \mathcal{A},
\]

and if \( y \) has a representing measure \( \mu_y \) then

\[
(q,M_r(y)q) = \int_{\mathbb{R}^n} q(x)^2 \, d\mu_y(\text{dx}) \geq 0, \tag{4}
\]

so that \( M_r(y) \succeq 0 \).

Next, given a sequence \( y = \{y_\alpha\} \) indexed in the basis \( v_\infty(x) \), let \( y_{2k}^{(i)} := L_y(x_{2k}^i) \) for every \( i = 1, \ldots, n \) and every \( k \in \mathbb{N} \). That is, \( y_{2k}^{(i)} \) denotes the element in the sequence \( y \), corresponding to the monomial \( x_{2k}^i \).

Of course not every sequence \( y = \{y_\alpha\} \) has a representing measure \( \mu_y \) as in (3). However, there exists a sufficient condition to ensure that it is the case. The following result stated in [Berg, 1980] is from [Nussbaum, 1966], and is re-stated here, with our notation.

**Theorem 1.** Let \( y = \{y_\alpha\} \) be an infinite sequence such that \( M_r(y) \succeq 0 \) for all \( r = 0, 1, \ldots \). If

\[
\sum_{k=0}^{\infty} (y_{2k}^{(i)})^{-1/2k} = \infty, \quad i = 1, \ldots, n, \tag{5}
\]

then \( y \) is a determinate moment sequence.

The condition (5) in Theorem 1 is called Carleman's condition as it extends to the multivariate case the original Carleman's sufficient condition given for the univariate case.

3. PRELIMINARIES

Let \( B_M \) be the closed ball

\[
B_M = \{ x \in \mathbb{R}^n \mid \| x \|_\infty \leq M \}. \tag{6}
\]

Proposition 2. Let \( f \in \mathbb{R}[x_1, \ldots, x_n] \) be such that

\[-\infty < f^* := \inf_{x} f(x). \]

Then, for every \( \epsilon > 0 \) there is some \( M \in \mathbb{N} \) such that

\[
f_M^* := \inf_{x \in B_M} f(x) < f^* + \epsilon, \quad \forall M \geq M.
\]
Equivalently, \( f^*_M \downarrow f^* \) as \( M \to \infty \).

**Proof.** Suppose it is false. That is, there is some \( e_0 > 0 \) and an infinite sequence \( \{M_k\} \subset \mathbb{N} \), with \( M_k \to \infty \), such that \( f_{M_k} \geq f^* + \epsilon_0 \) for all \( k \). But let \( x_0 \in \mathbb{R}^n \) be such that \( f(x_0) < f^* + \epsilon_0 \). With any \( M_k \geq ||x_0||_\infty \), one obtains the contradiction \( f^* + \epsilon_0 \leq f_{M_k} \leq f(x_0) < f^* + \epsilon_0 \).

To prove our main result (Theorem 5 below), we first introduce the following related optimization problems.

\[
P: \quad f^* := \inf_{x \in \mathbb{R}^n} f(x), \quad (7)
\]

and for \( 0 < M \in \mathbb{N} \), the problem \( P_M \)

\[
\inf_{\mu \in \mathcal{P}(\mathbb{R}^n)} \left\{ \int f \, d\mu \mid \int \sum_{i=1}^{n} e_i^2 \, d\mu \leq M \right\}, \quad (8)
\]

where \( \mathcal{P}(\mathbb{R}^n) \) is the space of probability measures on \( \mathbb{R}^n \). The respective optimal values of \( P \) and \( P_M \) are denoted \( \inf P = f^* \) and \( \inf P_M, \) or \( \min P \) and \( \min P_M \) if the minimum is attained.

**Proposition 3.** Let \( f \in \mathbb{R}[x_1, \ldots, x_n] \) be such that \( -\infty < f^* := \inf_x f(x) \), and consider the two optimization problems \( P \) and \( P_M \) defined in (7) and (8) respectively. Then, \( \inf P_M \downarrow f^* \) as \( M \to \infty \). If \( f \) has a global minimizer \( x^* \in \mathbb{R}^n \), then \( \min P_M = f^* \) whenever \( M \geq ||x^*||_\infty \).

**Proof.** Let \( \mu \in \mathcal{P}(\mathbb{R}^n) \) be admissible for \( P_M \). As \( f \geq f^* \) on \( \mathbb{R}^n \) then it follows immediately that \( \int f \, d\mu \geq f^* \), and so, \( \inf P_M \geq f^* \) for all \( M \).

As \( B_M \) is closed and bounded, it is compact and so, with \( f_{M_k} \) as in Proposition 2, there is some \( \hat{x} \in B_M \) such that \( f(\hat{x}) = f_{M_k} \). In addition let \( \mu \in \mathcal{P}(\mathbb{R}^n) \) be the Dirac probability measure at the point \( \hat{x} \). As \( ||\hat{x}||_\infty \leq M \),

\[
\int \sum_{i=1}^{n} e_i^2 \, d\mu = \sum_{i=1}^{n} 0(\hat{x}_i)^2 \leq neM^2,
\]

so that \( \mu \) is an admissible solution of \( P_M \) with value \( \int f \, d\mu = f(\hat{x}) = f_{M_k} \), which proves that \( \inf P_M \leq f_{M_k} \). This latter fact, combined with Proposition 2 and with \( f^* \leq \inf P_M \), implies \( \inf P_M \downarrow f^* \) as \( M \to \infty \), the desired result. The final statement is immediate by taking as feasible solution for \( P_M \), the Dirac probability measure at the point \( x^* \in B_M \) (with \( M \geq ||x^*||_\infty \)). As its value is now \( f^* \), it is also optimal, and so, \( P_M \) is solvable with optimal value \( \min P_M = f^* \).

Proposition 3 provides a rationale for introducing the following Semidefinite Programming (SDP) problems. Let \( 2r_f \) be the degree of \( f \) and for every \( r_f \leq r \in \mathbb{N} \), consider the SDP problem

\[
Q_r \left\{ \begin{array}{l}
\min_{y} \sum_{i=1}^{n} y_i \\
\text{s.t.} \quad M_r(y) \geq 0 \\
\quad \sum_{k=0}^{n} \frac{y_{2k}}{k!} \leq neM^2, \\
y_0 = 1,
\end{array} \right. \quad (9)
\]

and its associated dual SDP problem

\[
Q^*_r \left\{ \begin{array}{l}
\max_{\lambda \geq 0} \gamma - \lambda \sum_{r} \sum_{k=0}^{n} \frac{x_{2k}}{k!} \\
\text{s.t.} \quad f - \gamma \geq \lambda \sum_{r} \sum_{k=0}^{n} \frac{x_{2k}}{k!}, \\
q \quad \text{s.o.s. of degree} \leq 2r,
\end{array} \right. \quad (10)
\]

with respective optimal values \( \inf Q_r \), and sup \( Q^*_r \) (or min \( Q_r \), and max \( Q^*_r \) if the optimum is attained, in which case the problems are said to be solvable). For more details on SDP theory, the interested reader is referred to the survey paper [VandenBerghe and Boyd, 1996].

The SDP problem \( Q_r \) is a relaxation of \( P_M \), and we next show that in fact

- \( Q_r \) is solvable for all \( r \geq r_0 \),
- its optimal value \( \min Q_r \to \inf P_M \) as \( r \to \infty \), and
- \( Q^*_r \) is also solvable with same optimal value as \( Q_r \), for every \( r \geq r_f \).

This latter fact will be crucial to prove our main result in the next section. Let \( l_\infty \) (resp. \( l_1 \)) be the Banach space of bounded (resp. summable) infinite sequences with the sup-norm (resp. the \( l_1 \)-norm).

**Theorem 4.** Let \( f \in \mathbb{R}[x_1, \ldots, x_n] \) be of degree \( 2r_f \), with global minimum \( f^* > -\infty \), and let \( M > 0 \) be fixed. Then :

(i) For every \( r \geq r_f \), \( Q_r \) is solvable, and \( \min Q_r \uparrow \inf P_M \) as \( r \to \infty \).

(ii) Let \( y^{(r)} = \{y^{(r)}_j\} \) be an optimal solution of \( Q_r \) and complete \( y^{(r)} \) with zeros to make it an element of \( l_\infty \). Every (pointwise) accumulation point \( y^* \) of the sequence \( \{y^{(r)}\}_{r \in \mathbb{N}} \) is a determinate moment sequence, that is,

\[
y^*_\alpha = \int x^{\alpha} \, d\mu^*, \quad \alpha \in \mathbb{N}^n, \quad (11)
\]

for a unique probability measure \( \mu^* \), and \( \mu^* \) is an optimal solution of \( P_M \).

(iii) For every \( r \geq r_f \), max \( Q^*_r = \min Q_r \).

For a proof see [Lasserre, 2004].
So, one can approximate the optimal value $f^*$ of $\mathbb{P}$ as closely as desired, by solving SDP-relaxations $\{Q_\epsilon\}$ for sufficiently large values of $r$ and $M$. Indeed, $f^* \leq \inf Q_\epsilon \leq f_M^*$, with $f_M^*$ as in Proposition 2. Therefore, let $\epsilon > 0$ be fixed, arbitrary. By Proposition 3, we have $f^* \leq \inf P_M \leq f^* + \epsilon$ provided that $M$ is sufficiently large. Next, by Theorem 4(i), one has $\inf Q_\epsilon \geq \inf P_M - \epsilon$ provided that $r$ is sufficiently large, in which case, we finally have $f^* - \epsilon \leq \inf Q_\epsilon \leq f^* + \epsilon$.

Notice that the SDP-relaxation $Q_\epsilon$ in (9) is simpler than the one defined in [Lasserre, 2001]. Both have the same variables $y \in \mathbb{R}^{r+1}$, but the former has one SDP constraint $M_r(y) \geq 0$ and one scalar inequality (as one substitutes $y_0$ with 1) whereas the latter has the same SDP constraint $M_r(y) \geq 0$ and one additional SDP constraint $M_r(\theta y) \geq 0$ for the localizing matrix associated with the polynomial $x \mapsto \theta(x) = M^2 - \|x\|^2$. This results in a significant simplification.

4. SUM OF SQUARES APPROXIMATION

Let $\mathcal{A}$ be equipped with the norm

$$f \mapsto \|f\| := \sum_{\alpha \in \mathbb{N}^n} |f_\alpha|, \quad f \in \mathcal{A}.$$  

**Theorem 5.** Let $f \in \mathbb{R}[x_1, \ldots, x_n]$ be nonnegative with global minimum $f^*$, that is,

$$0 \leq f^* \leq f(x), \quad x \in \mathbb{R}^n.$$  

(i) There is some $r_0 \in \mathbb{N}, \lambda_0 \geq 0$ such that, for all $r \geq r_0$ and $\lambda \geq \lambda_0$,

$$f + \lambda \sum_{k=0}^r \sum_{j=1}^n \frac{x_j^{2k}}{k!} \quad \text{is a sum of squares.} \quad (12)$$

(ii) For every $\epsilon > 0$, there is $r_\epsilon \in \mathbb{N}$ such that,

$$f_{\epsilon} := f + \epsilon \sum_{k=0}^{r_\epsilon} \sum_{j=1}^n \frac{x_j^{2k}}{k!} \quad \text{is a sum of squares.} \quad (13)$$

Hence, $\|f - f_{\epsilon}\| \rightarrow 0$ as $\epsilon \downarrow 0$.

For a proof see [Lasserre, 2004].

**Remark 6.** Theorem 5(ii) is a denseness result in the spirit of Theorem 5, p. 122 in [Berg, 1980] which states that the cone of s.o.s. polynomials is dense (also for the norm $\|f\|_1$) in the cone of polynomials that are nonnegative on $[-1,1]^n$. However, notice that Theorem 5(ii) provides an explicit converging sequence $\{f_\epsilon\}$ with a simple and very specific form.

**Ex:** Let $f \in \mathbb{R}[x,y]$ be the Motzkin polynomial

$$(x,y) \mapsto f(x,y) := 1 + x^4 y^2 + x^2 y^4 - 3x^2 y^2.$$  

Then, as proved by B. Reznick,

$$f + (n-1)^{-1}(xy)^{2n}/n^n, \quad \text{is s.o.s.} \forall n \geq 3.$$  

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