STATE DEPENDENT SWITCHING CONTROL FOR INVERTED PENDULUM SYSTEM

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Abstract: A switching control strategy for inverted pendulum systems is proposed based on the energy function and its transitions. We deal with a simplified second order model of cart-pendulum systems. Assume a virtual energy whose potential energy has an opposite sign. Paying attention to the motions of the pendulum, the energy changes are analyzed theoretically. According to these analyses, the conditions of a simple switching controller are derived and the stability of the switching controlled system is guaranteed. The role of the parameters in controllers is examined in numerical simulations. Copyright© 2005 IFAC

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1. INTRODUCTION

The control design based on the Lyapunov stability theory has often been used for stabilizing nonlinear systems. In these approaches, mechanical energy of systems is a useful candidate of Lyapunov functions because it helps us to intuitively recognize the physical characteristics of controlled behavior (e.g., Bloch, Chang, Leonard and Marsden(2001), Angeli (2001)). However, the design processes of continuous controllers often become comparatively complex because of the nonlinearity of the system, and obtained nonlinear controllers mostly become complex. On the contrary, by using the switching process in control, simple controllers for inverted pendulum systems, which are nonlinear systems, have been proposed (Åström and Furuta(2000), Shiriaev, Egeland and Ludvigsen(1998), Liberzon (2003)). These controllers swing the pendulum back and forward so that the mechanical energy of the pendulum increases. In many methods, two controllers are switched for the swinging-up phase and the stabilization phase. Therefore, another difficulty appears: when should the controllers be switched?

In this paper, a comparatively simple switching control law for inverted pendulum system is proposed. In the proposed method, the controllers are switched according to the angle of the pendulum. The pendulum goes to the lowest position due to gravity when control is not applied. Therefore, we assume a virtual energy function obtained reversing the sign of potential energy. This function becomes smallest when the pendulum remains stationary at the upright equilibrium. If there were controllers to always decrease this function, the states would converge to the upright equilibrium. However, there are not unbounded continuous control inputs which always decrease this virtual energy function. Consequently, the stability of the controlled system will be guaranteed by focusing on the typical modes of motion: rotation, swinging motion and convergence. The transitions of energy function are analyzed theoretically and the conditions of switching control law are clarified. In practically use, this switching control law can be used without consideration of transitions among these modes of motion and does not require the calculation of the energy function or another function for input switching. The simplest
controllers are shown and the influences of the gain parameters on system behavior are investigated in numerical simulations.

2. DEFINITIONS AND THE MODEL OF SYSTEM

Consider the simplified inverted pendulum system described in Åström and Furuta (2000) and others. The equation of motion for this system is given as

$$ J \frac{d^2 \theta}{dt^2} - mg l \sin \theta + mlv \cos \theta = 0, \quad (1) $$

where $\theta$ is an angle between the pendulum and the vertical axis, $g$ is the acceleration of gravity, $m$ is the mass of the pendulum, $l$ is the length of a rod from the pivot point to the center of mass, and $J$ is the moment of inertia with respect to the pivot point. The external control force is horizontally applied at the pivot point. This force implies the acceleration of cart in the case of cart-pendulum systems. As described in Åström and Furuta (2000), the equation of motion (1) is normalized by introducing $t = \sqrt{mg l/J} r$, $u = v/g$.

$$ \dot{\theta} - \sin \theta + \frac{u}{\cos \theta} = 0, \quad (2) $$

where $\dot{\theta} = d\theta/dt$. The total mechanical energy of this system, which consists of kinetic energy and potential energy, is obtained as,

$$ E(\theta, \dot{\theta}) = \frac{1}{2} \dot{\theta}^2 + (\cos \theta - 1), \quad (3) $$

where $\dot{\theta} = d\theta/dt$. The purpose of this study is to stabilize the pendulum at the upright equilibrium. Therefore, the following definitions are used in this paper.

**Definition 1** The upright equilibriums are the states such that

$$ \dot{\theta} = 0, \quad \cos \theta = 1. \quad (4) $$

**Definition 2** The pendent equilibriums are the states such that

$$ \dot{\theta} = 0, \quad \cos \theta = -1. \quad (5) $$

**Definition 3** The inverted pendulum system (2) is asymptotically stable when almost all states except the pendent equilibriums converge to any one of the upright equilibriums, i.e.,

$$ \lim_{t \to \infty} \dot{\theta} = 0, \quad \lim_{t \to \infty} \cos \theta = 1 \quad (6) $$

are satisfied.

3. PROPOSED SWITCHING CONTROL LAW

When applying inputs decreasing the energy of system (2), the pendulum is stabilized at the pendent equilibriums, where the potential energy becomes the minimum. Paying attention to this fact, we assume a virtual energy which is obtained if gravity worked in the opposite direction. Moreover, the pendulum will be stabilized at the upright equilibriums if control inputs are applied to decrease this virtual energy. Thus, in this paper, we suppose the following function $E_d$ which is obtained by reversing the sign of potential energy in equation (3).

$$ E_d(\theta, \dot{\theta}) = \frac{1}{2} \dot{\theta}^2 - \cos \theta \quad (7) $$

The time-derivative of $E_d$ is obtained as

$$ \dot{E}_d = \dot{\theta} (\dot{\theta} + \sin \theta) = \dot{\theta} (2 \sin \theta - u \cos \theta). \quad (8) $$

The derivative $\dot{E}_d$ is controlled by the control input $u$ on the right-hand side of (8). However, $\dot{E}_d$ cannot be controlled when $\theta = 0$ or $\cos \theta = 0$, because the coefficient of $u$ becomes 0.

The upright equilibriums defined as **Definition 1** are not only one. That is, all states satisfying $\theta = \pm 2n\pi$, $n = 0, 1, 2, \ldots$ are candidates for the upright equilibriums. Therefore, in this study, we confine control inputs to the class satisfying

$$ u(\theta) = u(\theta + 2\pi). \quad (9) $$

The feedback controllers satisfying (9) perform the same role in the neighborhoods of any upright equilibrium. From this fact, the controlled system has a possibility of stabilizing the pendulum at an appropriate upright equilibrium among $\theta = \pm 2n\pi$, $n = 0, 1, 2, \ldots$ according to the initial states. As a class of control inputs satisfying (9), we choose the inputs as

$$ u = 2F(\theta) \sin \theta, \quad (10) $$

where $F(\theta)$ is a bounded function and $F(\theta) = F(\theta + 2\pi)$. Applying (10) to (8), we obtain

$$ \dot{E}_d = 2 \dot{\theta} \sin \theta (1 - F(\theta) \cos \theta). \quad (11) $$

In order to satisfy $\dot{E}_d \leq 0$, we have to determine the function $F(\theta)$ so that $(1 - F(\theta) \cos \theta)$ has the opposite sign of $\dot{\theta} \sin \theta$. When $\dot{\theta} \sin \theta < 0$, it is appropriate to select the function $F(\theta)$ such that

$$ F(\theta) \cos \theta \leq 1, \quad \text{for} \ \dot{\theta} \sin \theta < 0. \quad (12) $$

On the other hand, when $\dot{\theta} \sin \theta \geq 0$, the conditions for $\dot{E}_d \leq 0$ is rewritten as

$$ F(\theta) \cos \theta \geq 1, \quad \text{for} \ \dot{\theta} \sin \theta \geq 0. \quad (13) $$

There are no bounded functions $F(\theta)$ satisfying (13) at the points where $\cos \theta = 0$. Therefore, we propose the controllers satisfying conditions (12) and (13) in as wide a range as possible. Moreover, outside of this range, the controller is switched to 0 taking into consideration the conditions for finite inputs. To summarize, a proposed switching law of the function

$$ u(\theta) = \begin{cases} 2f(\theta) \sin \theta, & \text{if} \ \dot{\theta} \sin \theta < 0 \\ 2f(\theta) \sin \theta, & \text{if} \ \dot{\theta} \sin \theta \geq 0 \end{cases} \quad (14) $$

where $f(\theta) = \frac{1}{2} \frac{\theta}{\cos \theta - 1}$ is a bounded function. These functions are plotted in Fig. 1. The input switching in clockwise rotation.

![Fig. 1 Input switching in clockwise rotation](image-url)
\( F(\theta) \) is described as follows.
\[
F(\theta) = \begin{cases} 
  f_1(\theta) & \text{if } \theta \sin \theta < 0 \\
  f_2(\theta) & \text{if } \theta \sin \theta \geq 0, \cos \theta > \delta \\
  0 & \text{if } \theta \sin \theta \geq 0, \cos \theta \leq \delta
\end{cases}
\] (14)
where \( 0 < \delta < 1 \), \( f_1(\theta) \cos \theta \leq 1 \) for \( \theta \) and \( f_2(\theta) \cos \theta \geq 1 \) for \( \cos \theta > \delta \). Furthermore, the points where \( f_1(\theta) \cos \theta = 1 \) or \( f_2(\theta) \cos \theta = 1 \) are at most finite points in the range \( -\pi < \theta < \pi \).

Since \( \dot{E}_d \) has a positive sign when \( F(\theta) = 0 \), the global stability cannot be discussed by using \( \dot{E}_d \) as a candidate of the Lyapunov function. In the next section, stability is discussed by focusing on the characteristic motion modes of pendulum systems and investigating the change of \( \dot{E}_d \).

4. PARAMETER CONDITIONS FOR STABILITY

Motions of system (2) from arbitrary states \( \theta (\theta, \dot{\theta}) \) are divided into two motions: the pendulum makes a full rotation through \( \theta \pm 2\pi n \), and the pendulum stops before making a full rotation. The latter is a swinging motion such that the motion direction is reversed due to gravity whenever the pendulum stops. Since system (2) has no friction, the total mechanical energy (3) is kept when \( \dot{\theta} = 0 \). Therefore, the system depends only on initial states and the pendulum repeats eternally either rotations or swinging motions, or remains stationary at the equilibrium points. In this section, transitioning the motion modes of the pendulum as shown in Fig. 2, the conditions for stabilization are derived using a switching control law proposed in the previous section.

Using switching control law (10)/(14), an invariant set (Bacciotti and Rosier(2001)) appears in the neighborhood of each upright equilibrium.

**Lemma 1** Applying switching control law (10)/(14) to system (2), \( O = \{ \theta | \dot{E}_d < -\delta \} \) is an invariant set. □

**Proof** Let \( \theta \in O \) at \( t = 0 \). Then, \( \cos \theta > \delta \) because the states satisfy \( E_d < -\delta \). In such case, control input function \( F(\theta) \) is \( f_1(\theta) \) or \( f_2(\theta) \). Since the input functions \( f_1(\theta) \) and \( f_2(\theta) \) satisfy conditions (12) and (13) and \( E_d < -\delta \), it is evidently satisfied that \( E_d < -\delta \), i.e., \( \theta \in O \), for \( t \geq 0 \).

From the LaSalle’s invariance principle, the following lemma is obtained.

**Lemma 2** Applying switching control law (10)/(14) to system (2), arbitrary states in the invariant set \( O \) converge to an upright equilibrium if \( f_1(\theta) \cos \theta > 1/2 \) for \( \theta \in O \). □

**Proof** From LaSalle’s invariance principle, the states in \( \theta \in O \) converge into the largest invariant set contained in \( \{ \theta \in O | \dot{E}_d = 0 \} \), because \( \dot{E}_d \leq 0 \) (Imura (2000)). Therefore, we shall show that the largest invariant set consists of only the upright equilibriums. From equation (11), \( N \) is easily rewritten as
\[
N = \{ \theta \in O | \dot{\theta} = 0 \} \cup \{ \theta \in O | \sin \theta > 0 \} \cup \{ \theta \in O | F(\theta) \cos \theta = 1 \}.
\] (15)

Under the conditions that the points where \( F(\theta) \cos \theta = 1 \) are at most finite points, the states \( \theta \) in the set \( \{ \theta | F(\theta) \cos \theta = 1 \} \) are discrete points and are not continuous points. Thus, the states in invariant sets contained in the set \( N \) surely satisfy \( \dot{\theta} = 0 \). It is required for \( \dot{\theta} = 0 \) to satisfy \( \ddot{\theta} = 0 \). Namely, from equation (2), it is necessary that the right-hand side of the following equation becomes 0.
\[
\ddot{\theta} = \sin \theta (1 - 2F(\theta) \cos \theta)
\] (16)

Since this condition is satisfied when \( \sin \theta = 0 \), the set \( \{ \theta | \sin \theta = 0, \dot{\theta} = 0 \} \) is the invariant set. On the other hand, the set \( \{ \theta | F(\theta) \cos \theta = 1, \dot{\theta} = 0 \} \) is not the invariant set, because \( \theta \neq 0 \).

Furthermore, when using the switching control, there is a possibility that the states remain on the switching surfaces by switching indefinitely between different control modes. This phenomenon is called *livelock* (van der Schaft and Schumacher(2000)). If the states satisfy \( \ddot{\theta} \leq 0 \) after the input function is switched from \( f_2(\theta) \) to \( f_1(\theta) \), then the pendulum cannot escape from the switching point which is not the upright equilibrium and *livelock* occurs. From equation (2), \( \ddot{\theta} \) is obtained as follows.
\[
\ddot{\theta} = \dot{\theta} \sin \theta (1 - 2f_1(\theta) \cos \theta), \quad i = 1, 2
\] (17)

Using \( f_1(\theta) \), \( \ddot{\theta} \) is positive when \( f_1(\theta) \cos \theta > 1/2 \). Therefore, the states escape from the switching points if \( f_1(\theta) \cos \theta > 1/2 \). Furthermore, *livelock* never occurs on the other switching surfaces except the upright position.

From the above-mentioned, the following theorem is obtained in the neighborhood of the upright equilibrium.

**Theorem 1** Assume that \( f_1(\theta) \cos \theta > 1/2 \) for \( \theta \in O \). Then, the states of system (2) with (10)/(14) converge to the upright equilibrium if \( E_d < -\delta \) is satisfied once. □

Consider the case where the states are outside of the invariant set \( O \) described in Lemma 1. In this case, the pendulum rotates or swings. We consider the conditions for decreasing energy \( E_d \). From the form of the time-derivative \( \dot{E}_d \) given as (11), the change of

![Fig. 2 Illustration of motion modes and state transitions by proposed method](image-url)
Proof system (2) is asymptotically stable only if

\[ \cos \theta_1 = \cos \theta_2 + \int_{\theta_1}^{\theta_2} f_1(\theta) \sin 2\theta d\theta \]  

(21)

This equation indicates that \( \theta_2 \) is closer to the upright position than \( \theta_1 \) if the energy change \( \Delta E \)

\[ \Delta E = \int_{\theta_1}^{\theta_2} f_1(\theta) \sin 2\theta d\theta \]  

(22)

is negative. The following lemma concerned with \( \Delta E \) is obtained.

Lemma 4 Using switching control law (10)(14), system (2) is asymptotically stable only if

\[ \int_{\theta_1}^{\theta_2} f_1(\theta) \sin 2\theta d\theta < 0 \]  

(23)

is satisfied for all \( \theta \) satisfying (21) for \(-1 < \cos \theta < \delta \). □

Proof Assume \( E_0 \geq 0 \) and consider the initial states such that \( \cos \theta = \delta \), \( \delta \sin \theta \geq 0 \) and \( E_0 > 1 \). Then, if the pendulum makes a full rotation without satisfying \( \dot{\theta} = 0 \), \( |\dot{\theta}| \) does not become smaller than the initial value because \( E_0 \geq 0 \). Furthermore, from condition (9), the pendulum repeats the rotations and \( |\dot{\theta}| \) never decreases. Therefore, to satisfy \( \dot{\theta} = 0 \) before a full rotation is required so that the states converge to the upright equilibria. However, using switching control law (10)(14), the energy increases from the initial state to the lowest position, and then decreases as shown in Fig. 3. Under the assumption of \( E_0 \geq 0 \), the energy during a full rotation satisfies \( E_f > 1 \) and \( \dot{\theta} \) never becomes 0. Namely, the pendulum cannot stabilize at the upright position from the initial state if \( E_0 \geq 0 \). Therefore, condition (20) implying \( E_0 < 0 \) is required.

Next, swinging motions are considered. Assume that the angular velocity \( \dot{\theta} \) becomes 0 when \( \theta = \theta_1 \) and \( \theta = \theta_2 \). When \( \theta \in \Omega \) at \( \theta = \theta_1 \), \( \cos \theta < \delta \). Thus, the input function \( F(\theta) \) is switched to 0 at \( \theta = \theta_1 \). The pendulum goes downward and the control function is switched to \( f_1(\theta) \) at the lowest point. Noticing that the energy is \(-\cos \theta \) when \( \dot{\theta} = 0 \), the following equation concerned with the energy change from \( \theta_1 \) to \( \theta_2 \) is obtained.

\[ -\cos \theta_2 = -\cos \theta_1 + \int_{\theta_1}^{\theta_2} f_1(\theta) \sin 2\theta d\theta \]  

(21)

This equation indicates that \( \theta_2 \) is closer to the upright position than \( \theta_1 \) if the energy change \( \Delta E \)

\[ \Delta E = \int_{\theta_1}^{\theta_2} f_1(\theta) \sin 2\theta d\theta \]  

(22)

is negative. The following lemma concerned with \( \Delta E \) is obtained.

Lemma 4 Using switching control law (10)(14), system (2) is asymptotically stable only if

\[ \int_{\theta_1}^{\theta_2} f_1(\theta) \sin 2\theta d\theta < 0 \]  

(23)

is satisfied for all \( \theta \) satisfying (21) for \(-1 < \cos \theta < \delta \). □

Proof Assume \( E_0 \geq 0 \) and consider the initial states such that \( \cos \theta = \delta \), \( \delta \sin \theta \geq 0 \) and \( E_0 > 1 \). Then, if the pendulum makes a full rotation without satisfying \( \dot{\theta} = 0 \), \( |\dot{\theta}| \) does not become smaller than the initial value because \( E_0 \geq 0 \). Furthermore, from condition (9), the pendulum repeats the rotations and \( |\dot{\theta}| \) never decreases. Therefore, to satisfy \( \dot{\theta} = 0 \) before a full rotation is required so that the states converge to the upright equilibria. However, using switching control law (10)(14), the energy increases from the initial state to the lowest position, and then decreases as shown in Fig. 3. Under the assumption of \( E_0 \geq 0 \), the energy during a full rotation satisfies \( E_f > 1 \) and \( \dot{\theta} \) never becomes 0. Namely, the pendulum cannot stabilize at the upright position from the initial state if \( E_0 \geq 0 \). Therefore, condition (20) implying \( E_0 < 0 \) is required.

Fig. 3 Energy transition in the proof of Lemma 3

Fig. 4 Illustration of pendulum motion in the proof of Lemma 4

Fig. 5 Illustrations of pendulum motion in the proofs of Lemmas 5 and 6
The following lemmas concerned with transitions between rotations and swinging motions are obtained.

**Lemma 5** Assume that Lemma 4 is satisfied. Then, it is not satisfied that \( \theta \notin O, \dot{\theta} = 0 \) after a full rotation of the pendulum.

**Proof** In a full rotation motion, the input is switched to 0 when the pendulum goes downward through the point where \( \cos \theta = \delta \). Then, the pendulum evidently reaches to the lowest position without stopping. Therefore, the pendulum never stops in the range where \( \cos \theta \leq \delta \) after a full rotation as shown in Fig. 5a).

**Lemma 6** Assume that Lemma 3 is satisfied. Then, the pendulum never makes a full rotation after \( \theta \notin O, \theta = 0 \).

**Proof** Assume that \( \dot{\theta} \) becomes 0 with the point where \( \cos \theta = -\delta \). Then, the energy becomes smaller than the allowable minimum value -\( \delta \) at this angle. Namely, \( \dot{\theta} \) becomes 0 before the pendulum reaches this angle.

Under the conditions described in the above lemmas, the states of the system transition as shown in Fig. 2. Therefore, the following theorem is obtained.

**Theorem 2** Assume that \( f_1(\theta), f_2(\theta) \) and \( \delta \) satisfy the conditions in Theorem 1 and Lemmas 3, 4. Then, applying switching control law (10)(14), controlled system (2) is asymptotically stable.

**Proof** When Lemmas 3 and 4 are satisfied, transitions between swinging and rotation motion modes never occur. Therefore, the energy \( Ed \) decreases by repeating one of these motions and then, \( Ed \) surely becomes smaller than -\( \delta \). From Theorem 1, the states converge to the upright equilibrium.

**Remark** The pendulum actually rotates decreasing its kinetic energy when an initial angular velocity is comparatively large. On the other hand, the pendulum swings up by increasing its potential energy when the initial angular velocity is small. In addition, in this case, the pendulum is attracted to the neighborhood of the upright equilibrium without redundantly increasing the energy.

## 5. SIMPLE CONTROLLERS

### 5.1 The simplest class of switching controllers

Simple controllers satisfying the above-mentioned conditions are shown. At first, let the functions \( f_i(\theta) \) and \( f_i(\theta) \) be constant gains \( k_1 \) and \( k_2 \), respectively. Since the infimum of \( k_i \cos \theta \) in the range of \( \theta \in O \) is \( k_i \delta \), the conditions in Lemma 2 are calculated as,

\[
k_i \delta > 1/2.
\]

On the other hand, the conditions in Lemmas 4 is calculated as

\[
k_i (1 - \cos^2 \theta_2) < 0.
\]

Condition (26) indicates \( k_1 < 0 \), because \( (1 - \cos^2 \theta) \) is always nonnegative. This condition conflicts with condition (25). Namely, there is no parameter \( k_i \) simultaneously satisfying all conditions in Theorem 2, in the case of \( f_i(\theta) = k_i, i = 1,2 \).

Next, let functions \( f_i(\theta) \) and \( f_i(\theta) \) be trigonometric functions. Assume \( k_i \sin \theta(\theta, i = 1, 2 \). Then, there is no condition for stabilizing the pendulum.

Therefore, we assume the functions \( k_i \cos \theta, i = 1, 2 \) as \( f(\theta) \). The conditions are represented as follows.

\[
k_i \delta > 1, k_1 > 1/(2 \delta^2)
\]

\[
k_2 > 1 - \delta^2)
\]

\[
-k_1 (1 + \cos \theta) < 0
\]

Reducing these conditions, the conditions for stabilization are simplified as,

\[
1 > k_1 > 2/(\delta^2), k_2 > 1/(2 \delta^2)
\]

We can easily find the parameters \( k_1, k_2 \) satisfying (32). From the first inequality in (32), the conditions of a switching surface are obtained as \( \delta > 1/\sqrt{2} \).

The maximum value of input \( u \) is \( u_{\max} = 2k_2 \delta \sqrt{1 - \delta^2} \), when \( \cos \theta = \delta \). Under condition (32) of parameters \( k_1, k_2 \), the maximum input \( u_{\max} \) has lower bound as

\[
u_{\max} > 2 \sqrt{1 - \delta^2}.
\]

The parameter minimizing this lower bound of \( u_{\max} \) is obtained as \( \delta = 0.7625 \) from a derivation of the right-hand side of (33). The lower bound of \( u_{\max} \) is 3.048 when \( k_1 = 0.860 \) and \( k_2 = 3.090 \).

### 5.2 Simulation results
Simulation results in the case of $\dot{\theta} = \sqrt{3}/2$ are shown in Figs. 6 and 7. Condition (32) indicates that $1 > k_2 > 2/3$ and $k_1 < 0.706 k_1$ when $\dot{\theta} = \sqrt{3}/2$. $k_2$ is chosen as 6.0. The differences of behaviors from the initial values $\theta = 0.99 \pi$ [rad] and $\dot{\theta} = 0.0$ [rad/s] are compared in changing $k_1$ in 0.70, 0.80, and 0.90. In this case, the pendulum swings up early on as shown in Fig. 6a). In Fig. 7a), the points where the energy trajectories contact with the curve $E_\varphi = \cos \theta$ indicate the points where the kinetic energy becomes 0 and the pendulum motion stops. The energy in these points decreases in proportion to $k_1$. As $k_1$ is larger, angular velocity becomes larger early on and the pendulum swings up to higher positions. When the pendulum does not swing up to a sufficiently high position, for example at $t = 3.0$[s], the input is switched to 0. The pendulum goes through the lowest position and swings up in contrary direction. On the other hand, after $t = 6$[s], the pendulum goes over the upright position and the input function is switched to $f_2(\dot{\theta})$. After the pendulum motion is stopped by a large $f_2(\dot{\theta})$, the input function is switched to $f_1(\theta)$ again. Repeating these switches in the set O, the states converge to the upright equilibriums. As for the range of $\theta \in O$, the larger parameters $k_1$, $k_2$ are, the larger the absolute values of $\dot{E}_\varphi$ becomes. Therefore, the states converge fast after $E_\varphi < -\delta$ is satisfied.

The pendulum rotates as shown in Fig. 6b) when the initial states are $\theta = \pi/3$[rad] and $\dot{\theta} = 2.5$[rad/s]. In this simulation, $k_1$ is 0.7 and $k_2$ is changed in 4.1, 4.3 and 4.5. From Lemma 3, as $k_2$ becomes larger, the energy change $|E_\varphi|$ during a full rotation becomes larger and the states reach into the set O during a few rotations as shown in Fig. 7b). Furthermore, it is verified that the input becomes large if $k_2$ is large.

6. CONCLUSIONS

A switching control strategy for inverted pendulum systems was proposed based on the energy function and its transitions. If gravity worked in the reverse direction, the pendulum would remain at the upright equilibrium. Thus, we assumed a virtual energy whose potential has an opposite sign. Then, a switching control law was designed so that this virtual energy decreases in as wide a range as possible. Paying attention to motions of the pendulum, the energy changes were analyzed theoretically. The conditions of simple switching controller were derived and the stability of the switching controlled system was guaranteed. In simple examples, the influences of gain parameters in the controllers on the motion of the system were shown.

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![Fig. 6 Illustrations of swinging or rotation motions](image)

![Fig. 7 State trajectories, input signals and energy transitions in swinging or rotation motions](image)