Abstract: Although structural constraints such as model order and time delay have been incorporated in the continuous time system identification since its origin, the constraints on the estimated parameters were rarely enforced. This paper proposes a continuous time system identification approach with constraints. It shows that by incorporating physical parameter information known \textit{a priori} as hard constraints, the traditional parameter estimation schemes are modified to minimize a quadratic cost function with linear inequality constraints. Using the structure of Frequency Sampling Filters as the vehicle, the paper shows that the constraints can be readily imposed on continuous time frequency response estimation and step response estimation. In particular, \textit{a priori} knowledge in both time-domain and frequency domain is utilized simultaneously as the constraints for the optimal parameter solution. A Monte-Carlo simulation study with 100 noise realization is used to demonstrate the improvement of the estimation results in terms of continuous time frequency response and continuous time step response. 

Keywords: Continuous time system identification, nonparametric model, frequency response, step response, constraints.

1. INTRODUCTION

Although structural constraints such as model order and time delay have been incorporated in continuous time system identification since its origin, the constraints on the estimated parameters were rarely enforced. In order to pre-specify a set of constraints in the estimation, \textit{a priori} knowledge about the continuous time system parameters is required. This \textit{a priori} knowledge may come from a system that is partially known or from experiments that have been performed independently.
The contributions of this paper include

- Derive an algorithm for estimation of continuous time frequency response and step response with constraints. A priori knowledge of frequency response and step response information is simultaneously utilized as either linear equality or inequality constraints. This algorithm also includes discrete noise models in the estimation scheme.

- Monte-Carlo simulation studies on how the constraints affect the variance and bias of the continuous time parameter estimation schemes.

2. ESTIMATION OF NON-PARAMETRIC MODELS WITH CONSTRAINTS

The popular non-parametric representation in system identification consists of frequency response, step response and impulse response. It is called non-parametric (in contrast to parametric) in the sense that the process dynamics is captured by a set of response coefficients, instead of description of process poles and zeros. It is known that step response representation is invariant between system descriptions in continuous time and discrete time at the sampling instant, and continuous time frequency response representation can be closely approximated by the discrete frequency representation up to its Nyquist frequency. Therefore, unlike the parametric representation, the continuous time non-parametric presentations can be readily approximately by its discrete counter-parts, leading to the estimation from discrete data.

2.1 The Estimation Problem

In the sequel, we will show the unified framework of estimating continuous time non-parametric representations using discrete time systems approach. Assume that the continuous time system is stable with transfer function \( G_c(s) \). The system is sampled uniformly with an interval \( \Delta t \), and the system has a settling time \( T_s \) such that when \( t \geq T_s \), the impulse response \( h(t) \approx 0 \). The corresponding discrete parameter to \( T_s \) is \( N = \frac{T_s}{\Delta t} \). The discrete transfer function of the system can be represented in terms of the frequency response coefficients via the frequency sampling filters expression (Wang and Cluett, 2000):

\[
G(z) = \sum_{l=-n/2}^{n/2} G(e^{j\Omega})H^l(z) \tag{1}
\]

where \( n \) is an odd number to represent the number of frequencies included in the frequency sampling filters model; \( \Omega \) is the fundamental sampling frequency defined by \( \Omega = \frac{2\pi}{N} \). The \( l \)th frequency sampling filter is given as

\[
H^l(z) = \frac{1}{N} \frac{1 - z^{-N}}{1 - e^{j\Omega} z^{-1}} = \frac{1}{N}(1 + e^{j\Omega}z^{-1} + ... + e^{j(N-1)\Omega}z^{-(N-1)})
\]
At \( z = e^{j\Omega} \), \( H'(z) = 1 \), Equation (1) can also be written in terms of real and imaginary parts of the discrete frequency response \( G(e^{j\Omega}) \) (Bitmead and Anderson, 1981) as

\[
G(z) = \frac{1}{N} \frac{1 - z^{-N}}{1 - z^{-1}} G(e^{j\Omega}) \\
+ \sum_{l=1}^{\frac{N}{2}} [\text{Re}(G(e^{j\Omega})) F^l_R(z) \\
+ \text{Im}(G(e^{j\Omega})) F^l_I(z)] \tag{2}
\]

where \( F^l_R(z) \) and \( F^l_I(z) \) are the \( l \)th second order filters given by

\[
F^l_R(z) = \frac{1}{N} \frac{2(1 - \cos(\Omega) z^{-1})(1 - z^{-N})}{1 - 2\cos(\Omega) z^{-1} + z^{-2}} \\
F^l_I(z) = \frac{1}{N} \frac{2\sin(\Omega) z^{-1}(1 - z^{-N})}{1 - 2\cos(\Omega) z^{-1} + z^{-2}}
\]

The frequency sampling filters model can be regarded as a hybrid structure between a continuous time system and a discrete time system when the sampling interval \( \Delta t \) is sufficiently small. For the continuous time frequency \( \omega \leq \frac{\pi}{N} \), the continuous time frequency response \( G_c(j\omega) \approx G(e^{j\Omega}) \). Therefore, the coefficients of the discrete model are corresponding to continuous time frequency response at \( \omega = 0, \frac{\pi}{2N}, \frac{\pi}{N}, ..., \frac{\pi}{N} \).

Suppose that \( u(k) \) is the process input, \( y(k) \) is the process output and \( v(k) \) is the disturbance signal. The output \( y(k) \) can be expressed in a linear regression form by defining the parameter vector and the regressor vector as

\[
\theta = \begin{bmatrix} G(e^{j\Omega}) \\
\text{Re}(G(e^{j\Omega})) \\
\text{Im}(G(e^{j\Omega})) \\
\text{Re}(G(e^{j\Omega} \frac{\pi}{2N})) \\
\text{Im}(G(e^{j\Omega} \frac{\pi}{N})) \end{bmatrix} \\
\phi(k) = \begin{bmatrix} f(k)^0 \\
f(k)^1_{R} \\
f(k)^1_{I} \\
f(k)^2_{R} \\
f(k)^2_{I} \end{bmatrix}
\]

where

\[
f(k)^0 = \frac{1}{N} \frac{1 - z^{-N}}{1 - z^{-1}} u(k) \\
f(k)^l_{R} = F^l_R(z) u(k); f(k)^l_{I} = F^l_I(z) u(k)
\]

for \( l = 1, 2, ..., \frac{N}{2} - 1 \). This allows us to write the linear regression with correlated residuals as

\[
y(k) = \phi(k)^T \theta + v(k) \\
v(k) = \frac{\epsilon(k)}{D(z)} \tag{3}
\]

where \( \epsilon(k) \) is a white noise sequence with zero mean and standard deviation \( \sigma \). Given a set of sampled finite amount of data

\[
\{y(1), y(2), y(3), ..., y(M)\} \\
\{u(1), u(2), u(3), ..., u(M)\}
\]

we can obtain an estimate of the frequency sampling filter model and an estimate of the noise model \( \frac{1}{D(z)} \) using the generalized Least Squares method (Clarke, 1967, Soderstrom, 1974). More specifically, in the core estimation algorithm, we let

\[
y_D(k) = D(z)y(k); \phi_D(k) = D(z)\phi(k)
\]

The estimation of \( \hat{\theta} \) is obtained by minimizing the quadratic performance index

\[
J = \sum_{k=1}^{M} [y_D(k) - \phi_D(k) \theta]^2 \\
= \theta^T \sum_{k=1}^{M} [\phi_D(k) \phi_D(k)^T] \theta \\
- 2\theta^T \sum_{k=1}^{M} [\phi_D(k) y_D(k)] + \text{const} \tag{4}
\]

\( \hat{\theta} \) is estimated from the error sequence \( e(k) = y(k) - \phi(k)^T \hat{\theta} \), \( k = 1, 2, 3, ..., M \). The generalized Least Squares method is based on an iterative procedure and the iteration stops after the estimated parameters converge.

In order to obtain the estimated step response from the estimated frequency parameter vector \( \hat{\theta} \), it can be easily verified (Wang and Cluett, 2000) that the step response of the system at the sample \( m \) is in a linear relation to the frequency parameter vector \( \theta \) via

\[
g_m = Q(m)^T \hat{\theta} \tag{5}
\]

where

\[
Q(m) = \begin{bmatrix} m + 1 \\
2\text{Re}(S(1,m)) \\
2\text{Im}(S(1,m)) \\
\vdots \\
2\text{Re}(S(n-1,2m)) \\
2\text{Im}(S(n-1,2m)) \end{bmatrix}
\]

\[
S(l,m) = \frac{1}{N} \frac{1 - e^{j(l)(m+1)}}{1 - e^{jlm}}, l = 1, 2, ..., \frac{N}{2} - 1.
\]

3. ESTIMATION WITH CONSTRAINTS

3.1 Imposing Constraints

Constraints on Frequency Parameters
Suppose that the continuous time system is known
at frequency $\gamma(< \frac{\pi}{2\Delta \tau})$. By converting it to the discrete frequency $\gamma\Delta t$, from Equation (2) the frequency information can be expressed as

$$G(e^{\gamma\Delta t}) = L(e^{\gamma\Delta t})^T \theta$$

where

$$L(e^{\gamma\Delta t}) = \begin{bmatrix} F(e^{\gamma\Delta t})^0 \\ F(e^{\gamma\Delta t})^1 \\ \vdots \\ F(e^{\gamma\Delta t})^\frac{T}{R} \\ F(e^{\gamma\Delta t})^\frac{T}{I} \end{bmatrix}$$

This equation is then split into real and imaginary parts

$$\text{Re}(G(e^{\gamma\Delta t})) = \text{Re}(L(e^{\gamma\Delta t}))^T \theta$$

$$\text{Im}(G(e^{\gamma\Delta t})) = \text{Im}(L(e^{\gamma\Delta t}))^T \theta$$

If the frequency information is known quite accurately, then equality constraints based on equations (7) and (8) can be imposed in the solutions. This is particularly useful when the system has strong resonance, and the critical frequency information is used in the constraints to ensure good fitting. If the frequency information is known within certain bounds, then the inequality constraints can be imposed as

$$\text{Re}(G(e^{\gamma\Delta t}))_{\text{min}} \leq \text{real}(L(e^{\gamma\Delta t})^T \theta$$

$$\leq \text{Re}(G(e^{\gamma\Delta t}))_{\text{max}}$$

$$\text{Im}(G(e^{\gamma\Delta t}))_{\text{min}} \leq \text{Im}(L(e^{\gamma\Delta t}))^T \theta$$

$$\leq \text{Im}(G(e^{\gamma\Delta t}))_{\text{max}}$$

**Constraints on Step Response Parameters**

Constraints on step response parameters will be based on equation (5). Given the a priori information about some step response coefficients $g_m$, $0 < m \leq N - 1$, the equality constraint is formulated as

$$g_m = Q(m)^T \theta$$

where $Q(m)$ is defined by Equation (5). For inequality constraints, with specification of minimum and maximum of step responses, say $g_{m} \leq g_m \leq \overline{g}_m$, then the inequality constraint on a step response coefficient $g_m$ is formulated as

$$g_{m} \leq Q(m)^T \theta \leq \overline{g}_m$$

**3.2 Solution of the Estimation Problem with Constraints**

The estimation problem with constraints is essentially to minimize the quadratic cost function

$$J = \theta^T \sum_{k=1}^{M} [\phi_D(k)\phi_D(k)^T] \theta - 2\theta^T \sum_{k=1}^{M} [\phi_D(k)y_D(k)] + \text{cons}$$

subject to equality constraints

$$M_i \theta = \gamma_i$$

and inequality constraints

$$M_2 \theta \leq \gamma_2$$

By defining $E = \sum_{k=1}^{M} [\phi_D(k)\phi_D(k)^T]$ and $F = -2\sum_{k=1}^{M} [\phi_D(k)y_D(k)]$, $M = [M_1^T \; M_2^T]^T$, $\gamma = [\gamma_1^T \; \gamma_2^T]$, the necessary conditions for this optimization problem (Kuhn-Tucker condition) are (Luenberger, 1984)

$$E \Delta U + F + M^T \lambda = 0$$

$$M_2 \Delta U - \gamma \leq 0$$

$$\lambda^T (M_2 \Delta U - \gamma) = 0$$

$$\lambda \geq 0$$

where the vector $\lambda$ contains the Lagrange multipliers. These conditions can be expressed in a simpler form in terms of the set of active constraints. Let $S_{\text{act}}$ denote the index set of active constraints. Then the necessary conditions become

$$E \Delta U + F + \sum_{i \in S_{\text{act}}} \lambda_i M_i^T = 0$$

$$M_i \Delta U - \gamma_i = 0 \quad i \in S_{\text{act}}$$

$$M_i \Delta U - \gamma_i < 0 \quad i \notin S_{\text{act}}$$

$$\lambda_i \geq 0 \quad i \in S_{\text{act}}$$

$$\lambda_i = 0 \quad i \notin S_{\text{act}}$$

where $M_i$ is the $i$th row of the $M$ matrix. It is clear that if the active set were known, the original problem could be replaced by the corresponding problem having equality constraints only. Alternatively, suppose an active set is guessed and the corresponding equality constrained problem is solved. Then if the other constraints are satisfied and the Lagrange multipliers turn out to be nonnegative, that solution would be correct. In the case that only equality constraints are involved, the optimal solution has a closed-form as

$$\begin{bmatrix} E \\ M_1 \\ 0 \end{bmatrix} \begin{bmatrix} \theta \\ \lambda_1 \end{bmatrix} = -\begin{bmatrix} F \\ \gamma_1 \end{bmatrix}$$

Explicitly:

$$\lambda_1 = -(M_1 E^{-1} M_1^T)^{-1} (\gamma_1 + M_1 E^{-1} F)$$

$$\hat{\theta} = -E^{-1} (F + M_1^T \lambda_1)$$
When inequality constraints are required, an iterative algorithm is needed to solve the quadratic programming problem (Luenberger1984).

4. MONTE-CARLO SIMULATION STUDY

This section is to illustrate the strength of the proposed approach in the estimation of non-parametric models through a Monte-Carlo simulation study. The relay experiment proposed by Astrom and Hagglund (Astrom and Hagglund, 1984) is particularly suitable for continuous time identification as the sampling interval in the experiment can be chosen as small as desired. Wang et al. (1999) extended this experiment to include identification of more general class of models other than simple frequency response points. The same set of design parameters as in Wang and Gawthrop (2000) is used here to generate the input excitation signal. In the Monte-Carlo simulation study, a white noise sequence with standard deviation of 0.8 is used to generate $e(k)$ and the disturbance $\xi(k) = \frac{0.1}{1-0.9z^{-1}}$. 100 realizations of the white noise sequence are generated by changing the seed of the generator from 1 to 100.

The system used for simulation is given by the transfer function

$$G(s) = \frac{e^{-3s}}{(s^2 + 0.4s + 1)(s + 1)^3}$$

(17)

The sampling interval for this system is $\Delta t = 0.1$ second. The settling time $T_s$ is estimated as 40 seconds, hence the number of samples to steady state $N = \frac{T_s}{\Delta t} = 400$. By using frequency sampling filters to parametrize this system, the number of frequency required is 63, yielding the number of parameters in the FSF model as $n = 125$. Figures 1-3 show the magnitude and phase of the frequency responses when estimated without constraints. From the distribution of the responses, it is seen that the estimation of the non-parametric models is unbiased. However, the variances are large both for the frequency response and step response. To introduce equality constraints on the estimation, a priori knowledge about the system is required. The a priori knowledge for this system is assumed as time delay being approximately 1 seconds, the gain being 1, and the first pair of frequency response $G_e(\frac{2\pi}{T}) = 0.5305 - j0.8317$. The a priori knowledge about the steady state gain of the system is translated into the constraint on the first parameter of the FSF model while the a priori knowledge about the frequency response information is translated into two equality constraints on the second and third parameters of the FSF model. Note that a frequency information at an arbitrary frequency can be translated into constraints in a linear combination of the parameters of the FSF model. Similarly the a priori information about time delay is translated into a set of linear equality constraints in terms of the parameters of the FSF model. Four constraints have been put on the time delay at the sampling instant $k = 0, 3, 6, 9$. The reason for not using every sampling instant is because the solution is ill-conditioned when constraint is imposed on every sampling instant. As it is seen from the Monte-Carlo simulation study, this approach is adequate for this purpose. With the equality constraints imposed on the estimated parameters, the Generalized Least Squares method is modified to have the constraints on the system parameters, but not on the noise model parameters. Figures 4-6 show the magnitude and phase of the frequency responses when estimated with constraints. In comparison with the estimation results obtained
5. CONCLUSIONS

This paper discussed estimation of continuous time nonparametric models with constraints, where a priori knowledge is incorporated to improve the estimation results. A Monte-Carlo simulation study is used to demonstrate the improvement of the estimation in a noise environment. The authors are currently working on analysis of the bias and variances when constraints are introduced in the estimation algorithm and the extension of this estimation to continuous time transfer function models. With recent work in unstable systems using FSF model (Gawthrop and Wang, 2004), the authors envisage the extension of this work to unstable systems.

REFERENCES


