AN EXACT STABILITY TEST FOR PLANAR AND MULTI-MODAL PIECEWISE LINEAR SYSTEMS

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Abstract: A necessary and sufficient stability condition for planar and multi-modal piecewise linear systems is derived. The condition is given in terms of poles and zeros of subsystems, and it is computationally tractable. In addition, three numerical examples are illustrated in order to clarify differences between a class of linear time-invariant systems and a class of piecewise linear systems from the viewpoint of stability. Copyright ©2005 IFAC

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1. INTRODUCTION

In the last decade, a lot of techniques have been developed for analysis and controller synthesis for hybrid dynamical systems. Each result depends on the mathematical model that represents behavior of hybrid dynamical systems. One of the typical models is the piecewise linear system (PLS). The system consists of some pairs of linear time-invariant dynamics and a switching rule given by a linear function of the continuous state. Study on PLSs is important as a first step to establish hybrid control theory, because the hybrid dynamics is the simplest in all classes of hybrid dynamical systems.

Unlike linear time-invariant systems, checking stability of a given PLS is a very hard problem due to its hybrid nature. In fact, there exist no systematic ways to check stability of the class of PLSs exactly, although many results have been obtained on stability for several classes of hybrid dynamical systems which include the class of PLSs (see (Decarlo et al., 2000; Lygeros et al., 2003) and the references therein). Most of the results on stability are extensions of Lyapunov’s theorem, where we need to show the existence of a Lyapunov function to guarantee the stability. The Lyapunov methods provide not only sufficient conditions but also necessary conditions for stability under hybrid natures. Actually, the converse theorems ensure the existence of a Lyapunov function when the system is asymptotically stable (Michel and Hu, 1999). On the other hand, we must restrict available classes of Lyapunov functions within a class of piecewise quadratic functions (Rantzer and Johansson, 2000; Gonçalves et al., 2003) or a class of sums of squares (Prajna and Papachristodoulou, 2003) to give systematic ways of finding the Lyapunov functions. This makes the stability conditions conservative. Therefore, we need a new approach to get an explicit necessary and sufficient stability condition for the class of PLSs.

In recent years, direct analysis of behavior of hybrid states has led to exact stability tests for several classes of hybrid dynamical systems. Xu

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2. PLANAR MULTI-MODAL PIECEWISE LINEAR SYSTEMS

We consider a class of planar and multi-modal PLSs represented by

\[
\dot{x} = \begin{cases} 
A_1x, & \text{if } x \in S_1, \\
A_2x, & \text{if } x \in S_2, \\
\vdots & \\
A_mx, & \text{if } x \in S_m,
\end{cases}
\tag{1}
\]

where \(A_i \in \mathbb{R}^{2 \times 2}, S_i \) is a convex cone of the form 
\(S_i := \{x \in \mathbb{R}^2 \mid C_ix \geq 0\}, \) and \(C_i \in \mathbb{R}^{2 \times 2} (i = 1, \ldots, m). \) A matrix \(A_i\) may be equal to another matrix \(A_j\) as seen in Figure 1.

Fig. 1. Planar multi-modal piecewise linear model.

First, the state space of the system (1) is said to be proper, if all the following statements hold:

\[
\begin{align*}
S_i & \neq \mathbb{R}^2, \quad \forall i, \\
\text{int}S_i & \neq \emptyset, \quad \forall i, \\
\bigcup_{i=1}^m S_i & = \mathbb{R}^2, \\
\text{int}(S_i \cap S_j) & = \emptyset, \quad \forall (i, j), \ (i \neq j).
\end{align*}
\]

It is clear that they are quite natural and hence they are not restrictive under memoryless nonlinearities.

Second, the system (1) is said to be well-posed, if the system has a unique solution for each initial state. Definition of solutions can be chosen from the concepts defined in (Imura and van der Schaft, 2000; Imura, 2002; Imura, 2003). The choice does not influence our results shown below. On the other hand, we do not treat systems with sliding modes. In fact, Theorem 7 provided in Section 5 does not hold under existence of sliding modes.

The solution from an initial state \(x_0\) is denoted by \(x(t, x_0)\) where the initial time is always set 0.

Finally, we suppose det \(C_i \neq 0\) for all \(i\) throughout this paper. The assumption does not make the resulting stability condition conservative, because each system whose state space is proper satisfies the assumption after additional partition of the state space with the \(x_1\)-axis and the \(x_2\)-axis. An alternative to avoiding the non-singular assumption is found in (Iwatani and Hara, 2004a) where it needs complicated notation.

Remark 1. Consider a class of planar and bimodal PLSs represented by

\[
\dot{x} = \begin{cases} 
A_1x, & \text{if } cx \geq 0, \\
A_2x, & \text{if } cx \leq 0,
\end{cases}
\tag{2}
\]

where \(c \in \mathbb{R}^{1 \times 2} \) (Iwatani and Hara, 2004b). It is clear that no bimodal systems of the form (2) satisfy the non-singular assumption.
3. STABILITY ANALYSIS

This section provides the basic concepts of stability analysis for the system (1).

We first investigate trajectories of the system (1). Suppose that the system is well-posed and the state space is proper. Then the system has one of the following two properties: (T-i) Infinitely many events occur on each trajectory as illustrated in Figure 2-(i). (T-ii) The number of events which take place on each trajectory is finite as depicted in Figure 2-(ii). The two properties are closely connected with transitive modes and weak transitive modes defined as follows.

Definition 2. (i) A mode $i$ is said to be transitive, if $\forall x_0 \in S_i \setminus \{0\}$, $\exists t > 0$, $x(t, x_0) \not\in S_i$.

(ii) A mode $i$ is said to be weakly transitive, if one of the following two statements holds for all $x_0 \in S_i \setminus \{0\}$:

(a) $\exists t > 0$, $x(t, x_0) \not\in S_i$.

(b) $\lim_{t \to \infty} x(t, x_0) = 0$ and $\forall t \geq 0$, $x(t, x_0) \in S_i$.

We first discuss the relationship between transitive modes and the property (T-i). To this end, we define a set of vectors on the boundary $\partial S_i$ as follows:

$$\mathbb{B}_i := \{x_0 \in \partial S_i \setminus \{0\} \mid \exists \varepsilon > 0, \forall t \in (0, \varepsilon), x(t, x_0) \in S_i\}.$$  

where the set $\mathbb{B}_i$ is called the inward boundary (see Figure 3-(ii)). Then, we can show that a mode $i$ is transitive, if and only if $\mathbb{B}_i \not= \emptyset$ and

$$\exists \tau_i > 0, \forall x_0 \in \mathbb{B}_i, \{x(\tau_i, x_0) \in \partial S_i \text{ and } \forall t \in (0, \tau_i), x(t, x_0) \in \text{int} S_i\}. \quad (3)$$

(see Figure 3-(ii)). This implies that (T-i) holds if and only if all modes are transitive. Then

$$\forall x_0 \in \mathbb{R}^2, \exists \rho \in \mathbb{R}, \rho x_0 = x \left( \sum_{i=1}^{m} \tau_i, x_0 \right), \quad (4)$$

where each $\tau_i$ is defined by (3). Clearly, the origin is globally asymptotically stable if and only if $\rho < 1$ for all $x_0$.

Theorem 3. Consider the system (1). Suppose that the system is well-posed and the state space is proper. Then the following statements hold.

(i) Suppose that all modes are transitive. Then the origin is globally asymptotically stable, if and only if $x(\sum_{i=1}^{m} \tau_i, x_0) < x_0$ holds for all $x_0 \in \mathbb{R}^2 \setminus \{0\}$.

(ii) Suppose that there exists a mode which is not transitive. Then the origin is globally asymptotically stable, if and only if all modes are weakly transitive.

Theorem 3 will be written in terms of poles and zeros of subsystems in Section 5. The stability condition in Theorem 7 shown there is computationally tractable as seen in an algorithm described below the theorem.

4. TRANSFER FUNCTION REPRESENTATION

This section introduces a $2 \times 2$ matrix transfer function for each mode. Each function is the Laplace transform of initial value responses for

\begin{align*}
\begin{array}{c}
\text{(i)} \\
\text{(ii)}
\end{array}
\end{align*}
two initial states on \( \partial S_i \). The poles and zeros of the functions characterize a necessary and sufficient stability condition in the next section.

We first partition \( C_i^{-1} \) as \([z_{i1}, z_{i2}] := C_i^{-1}\). Every vector \( z_{ij} \) is on \( \partial S_i \). We then consider a transfer function of the form

\[
T_i(s) = C_i(sI - A_i)^{-1}C_i^{-1} = \frac{1}{s^2 + \alpha_is + \beta_i} \begin{bmatrix} s + \gamma_{i11} & \gamma_{i12} \\ \gamma_{i21} & s + \gamma_{i22} \end{bmatrix}.
\]

Clearly, \( T_i(s) \) represents the Laplace transform of initial value responses whose initial values are on \( \partial S_i \). We can easily check if \( z_{ij} \) is on the inward boundary \( B_i \) as follows.

**Proposition 4.** Consider the system (1). Assume \( \det C_i \neq 0 \). Then \( z_{ij} \in B_i \) holds, if and only if \( \gamma_{ijk} \geq 0 \) where \( j \in \{k \in \{1,2\} \mid k \neq j \} \).

The following definition chooses two special zeros associated with an initial state on \( B_i \) from \( \gamma_{ijk} \) \( (j = 1, 2, k = 1, 2) \) in \( T_i(s) \), if it exists:

\[
\gamma_i := \gamma_{ip,j}, \quad \delta_i := \gamma_{iq,p}.
\]

where

\[
p_i := \begin{cases} 1, & \text{if } \gamma_{i21} \geq 0, \\ 2, & \text{otherwise}, \end{cases} \quad q_i \in \{j \in \{1,2\} \mid j \neq p_i \}.
\]

Note that \( z_{ip} \) is on the inward boundary \( B_i \), if \( B_i \neq \emptyset \). Conversely, \( B_i = 0 \), if \( \delta_i < 0 \).

5. AN EXPLICIT NECESSARY AND SUFFICIENT STABILITY CONDITION

An explicit necessary and sufficient stability condition for the system (1) is given in terms of poles and zeros of \( T_i(s) \), where we often omit the index \( i \) which expresses a mode from symbols to simplify the notation.

The following lemma provides a necessary and sufficient condition for a mode \( i \) to be transitive or weakly transitive.

**Lemma 5.** Consider the system (1) and assume \( \det C_i \neq 0 \). Then the following statements hold.

(i) The mode \( i \) is transitive, if and only if it holds that

\[
\beta > \begin{cases} \frac{\alpha^2}{4}, & \text{if } \alpha \leq 2\gamma, \\ \frac{\alpha - \gamma^2}{\alpha - \beta}, & \text{if } \alpha \geq 2\gamma. \end{cases}
\]

(ii) The mode \( i \) is weakly transitive, if and only if it holds that

\[
\beta > \begin{cases} \frac{\alpha^2}{4}, & \text{if } \delta \geq 0 \text{ and } \alpha \leq 2\gamma, \\ \frac{\alpha - \gamma^2}{\alpha - \beta}, & \text{if } \delta \geq 0 \text{ and } \alpha \geq 2\gamma, \\ \frac{\gamma^2}{\alpha - \beta}, & \text{if } \delta < 0, \end{cases}
\]

where \( \gamma := \max(0, \gamma) \) and \( \gamma := \min(0, \gamma) \).

The following lemma plays an important role to make \( \rho \) defined in (4) or equivalently (i) in Theorem 3 explicit.

**Lemma 6.** Consider the system (1). Suppose that \( \det C_i \neq 0 \) and (9) holds for \( i \). Define \( \tau \) according to (3). Then, \( x(\tau, z_p) = \eta z_q \) holds, where \( \eta = \frac{\beta}{\sqrt{\beta - \alpha^2} + \gamma^2} \exp \left( -\frac{\alpha \theta}{\sqrt{4\beta - \alpha^2}} \right) \), if \( \alpha^2 < 4\beta \),

\[
\eta = \frac{2\delta}{\alpha - 2\gamma} \exp \left( \frac{-\alpha \theta}{\alpha - 2\gamma} \right), \quad \text{if } \alpha^2 = 4\beta,
\]

\[
\eta = \frac{\delta \exp \left( \frac{\lambda_2 \log \left| \lambda_2 + \gamma - \lambda_1 \log |\lambda_1 + \gamma| \right|}{\lambda_1 - \lambda_2} \right)}{\lambda_1 - \lambda_2}, \quad \text{if } \alpha^2 > 4\beta,
\]

where \( \lambda_1 = \frac{\alpha - 2\gamma}{2\sqrt{\beta - \alpha^2} + \gamma^2} \), \( \lambda_2 = \frac{\alpha - \sqrt{\alpha^2 - 4\beta}}{2} \).

Note that \( \eta \) is well-defined under (9).

We are now ready to give an explicit necessary and sufficient stability condition in terms of poles and zeros of \( T_i(s) \).

**Theorem 7.** Consider the system (1). Suppose that the system is well-posed, the state space is proper and \( \det C_i \neq 0 \) for all \( i \). Then the following two statements hold true.

(i) Suppose (9) holds for all \( i \). Then the origin is globally asymptotically stable, if and only if it holds that

\[
\prod_{i=1}^m \left( \eta_i ||z_{iq_i}|| \right) < \prod_{i=1}^m ||z_{ip_i}||. \tag{12}
\]

(ii) Suppose that there exists a mode \( i \) such that (9) does not hold. Then the origin is globally asymptotically stable, if and only if (10) holds for all \( i \).

The following algorithm based on Theorem 7 leads to an exact stability test for the system (1).

**Algorithm 1.**

\[ \text{Arccos}(x) \] expresses the principal value of the inverse cosine of \( x \), i.e. \( \text{Arccos}(x) \in [0, \pi] \).
Step 1) Check if the state space of a given system is proper and the system is well-posed (see (Imura, 2002; Imura, 2003; Imura and van der Schaft, 2000) for well-posedness conditions). If yes, go to Step 2. If not, the algorithm ends.

Step 2) Check if \( \det C_i \neq 0 \) for all \( i \). If not, partition the state space with the \( x_1 \)-axis and the \( x_2 \)-axis.

Step 3) Check if (9) holds for all \( i \). If yes, then go to Step 4-i. If not, then go to Step 4-ii.

Step 4)

(i) Check if (12) is satisfied. If yes, the origin is globally asymptotically stable. If not, the origin is not globally asymptotically stable.

(ii) Check if (10) holds for all \( i \). If yes, the origin is globally asymptotically stable. If not, the origin is not globally asymptotically stable.

Remark 8. The bimodal system (2) is globally asymptotically stable, if and only if all the following inequalities holds (Iwatani and Hara, 2004b):

\[
\begin{align*}
\beta_i &> \max(0, -|\alpha_i|/4), \quad i = 1, 2, \\
\frac{|\alpha_1\alpha_1|}{\beta_1} + \frac{|\alpha_2\alpha_2|}{\beta_2} &> 0.
\end{align*}
\]

Algorithm 1 may lead to the same stability condition, which implies that the result in this paper is an extension of the result in (Iwatani and Hara, 2004b) when \( x \in \mathbb{R}^2 \). However, it requires further partition of the state space, and hence we need a lot of work to derive the stability condition from Algorithm 1. On the other hand, another approach provided in (Iwatani and Hara, 2004a) leads to the stability condition more easily than Algorithm 1.

6. NUMERICAL EXAMPLES

This section illustrates three typical numerical examples in order to clarify differences between trajectories of linear time invariant systems and PLSs from a view point of stability.

Example 9. Consider a four-modal system with

\[
\begin{align*}
A_1 = A_3 &= \begin{bmatrix} -\sigma & 1 \\ -\omega^2 & \sigma \end{bmatrix}, & A_2 = A_4 &= \begin{bmatrix} 1 & \pi \\ -\pi & 1 \end{bmatrix}, \\
C_1 = -C_4 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & C_2 = -C_3 &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (13)
\end{align*}
\]

Each set \( S_i \) expresses the \( i \)-th quadrant. The aim here is to derive a necessary and sufficient stability condition of the system in terms of \( \sigma \in \mathbb{R} \) and \( \omega \in \mathbb{R}_+ \) via Algorithm 1. As a result, Algorithm 1 shows that the origin is asymptotically stable, if and only if \( e^{1+\frac{\sigma}{\omega}} < \omega^\sigma \) holds.

Example 10. Consider a four-modal system with

\[
\begin{align*}
A_1 &= A_3 = \begin{bmatrix} \lambda + 1 & -\lambda - 1 \\ \lambda - 1 & \lambda + 1 \end{bmatrix}, \\
A_2 &= A_4 = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix}, \\
C_i \ (i=1,2,3,4) &\text{ defined by (13). Assume } \lambda < 1, \text{ which guarantees well-posedness of the system. It is seen that (9) does not hold for any } i. \text{ Thus, let us investigate (10). Consequently, Algorithm 1 shows that the system is stable, if and only if } \lambda < 0. \text{ Typical trajectories can be seen in Figure 6. We have a remark for the unstable case, i.e. } 0 < \lambda < 1. \text{ Every trajectory converges to the origin, if the initial state is not on the line through } [1,1]^T \text{ and } [-1,1]^T \text{ (Iwatani and Hara, 2004a). In other words, } \lim_{t \to \infty} x(t,x_0) = 0, \text{ a.e. } x_0 \in \mathbb{R}^2, \text{ even if the origin is unstable as illustrated in Figure 6-(ii).}
\end{align*}
\]

Example 11. Consider a three-modal system with
In this paper, we have derived a necessary and sufficient condition for planar and multi-modal PLSs to be stable. The condition is given in terms of poles and zeros of subsystems, and it is computationally tractable. Also, we have shown three numerical examples which provide typical trajectories of PLSs. They clarify differences between a class of linear time invariant systems and a class of PLSs from a viewpoint of stability.

There still remain several open problems on stability of PLSs to be addressed in the future, although we have established some basic tools for stability analysis in this paper. In particular, we need to discuss stability analysis for the higher-order case.

A necessary condition and a sufficient condition for stability of higher-order and bimodal systems are addressed in (Iwatani and Hara, 2004b).

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