DISCRETE TAKAGI-SUGENO FUZZY MODELS: REDUCED NUMBER OF STABILIZATION CONDITIONS

François Delmotte, Thierry Marie Guerra

Abstract – Takagi-Sugeno’s fuzzy models enable to represent a wide class of non linear models in a compact set of the state variables. According to this representation stabilization conditions can be obtained and are usually written as Linear Matrix Inequalities. Since the obtained conditions are only sufficient, current researches try to lower the conservatism of the results. In this paper several matrix properties are used with the help of the elimination lemma for discrete TS models. Copyright © 2005 IFAC

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1. INTRODUCTION

Since nearly twenty years, Takagi-Sugeno’s fuzzy models (Takagi and Sugeno, 1985) have been used to model and control non linear systems. Stability and stabilization are mainly based on Lyapunov functions (Wang, et al., 1996; Ma et al, 1998; Tanaka, et al, 1998; Yoneyama et al, 2000). These latter are usually quadratic. Sometimes piecewise quadratic functions are used (Johansson et al 1999; Feng and Wang, 2001). There are also some results using non linear functions, in the continuous case (Blanco et al, 2001; Tanaka et al, 2001) and in the discrete case (Guerra and Vermeiren, 2004). Nevertheless in this case, the complexity of the LMI problem has been seriously increased.

In every case, the number of conditions put in the form of LMI increases highly as the number of models increases. Usually the number of LMI is about \( r(r+1)/2 \) with \( r \) the number of linear models of the TS fuzzy model.

Several approaches have been developed to lower the conservatism of the conditions. One approach is based on reducing the number of models (Lauber, 2003; Taniguchi et al, 2001), another one uses matrix properties to reduce the conservatism of the conditions themselves (Guerra et al, 2003). Results presented in this paper follow the latter idea.

The paper is organized as follows. The second part recalls useful mathematical tools. The third part presents the new conditions and part fourth compares various conditions on an example.

2. TOOLS

2.1 Basic conditions

Let be a Takagi-Sugeno’s fuzzy model (Takagi and Sugeno, 1985) with \( r \) the number of rules, \( x(t)=[x_1(t), x_2(t), \ldots, x_r(t)]^T \) the state vector, \( u(t) \) the control signal, \( y(t) \) the output, and \( z(t) \) the premises variables. The fuzzy model is given by:

\[
\begin{align*}
    x(t+1) &= \sum_{i=1}^{r} h_i(z(t))(A_ix(t)+Bu(t)) \\
    y(t) &= \sum_{i=1}^{r} h_i(z(t))C_ix(t)
\end{align*}
\]

(1)

The non-linearities of the global model are due to the terms \( h_i(z(t)) \geq 0 \), with the convex sum property, i.e. \( \sum_{i=1}^{r} h_i(z(t))=1 \). In this paper is assumed that, \( \forall i \) pairs \( (A_i, B_i) \) are controllable.
The usual control law used to stabilize model (1) is called a PDC (Parallel Distributed Compensation) and is given by (Wang et al, 1996):

\[ u(t) = -\sum_{i=1}^{r} h_i(z(t)) F_i x(t) \]  

(2)

The design of \( u(t) \) requires to obtain the feedback gains \( F_i \). Several problems can be addressed, robustness, performances and so on (Tanaka et al, 1998) (Liu and Zhang 2003). LMI tools (Boyd et al, 1994) are often a very convenient way to solve these problems. Let be a quadratic Lyapunov function \( V = x^T P x \) with \( P > 0 \). Then with \( X = P^{-1} \) and \( N_j = F_j P^{-1} \) is defined:

\[ Y_j = \begin{bmatrix} -X \\ AX - B_j N_j \\ -X \end{bmatrix} \]  

(3)

The most basic conditions of stabilization are presented theorem 1.

**Theorem 1** (Wang et al, 1996) : Model (1) is globally asymptotically stable in closed loop with the control law (2) and the \( Y_j \) defined in (3) if there exist:

\[ X > 0, \ N_j, \ i, j \in \{1, \ldots, r\} \] such that:

\[ \forall i \ Y_{i, i} < 0 \]  

(4)

\[ \forall i, j \ i < j \ Y_{i, j} + Y_{j, i} < 0 \]  

(5)

According to the work of (Guerra and Vermeiren, 2004, Guerra et al, 2003) the following notation is defined. For scalar functions \( h_i(z) \geq 0 \) and \( U_i \), \( i \in \{1, \ldots, r\} \) matrices of the same dimension, we note: \( U_z = \sum_{i=1}^{r} h_i(z) U_i \).

Similarly \( U_{zz} = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(z) h_j(z) U_{ij} \)  

(6)

### 2.2 Properties

The following properties are useful to establish the main result.

**Lemma 1 (Congruence):** Let \( X \) be a full rank matrix. If \( Y > 0 \) then:

\[ X Y X^T > 0 \]  

(7)

**Lemma 2:** (Shur’s complement Boyd et al, 1994) Matrices \( X \), \( Y \) and \( R \) being of appropriate sizes, we have:

\[ \begin{bmatrix} Y - X R^{-1} X^T \ & > 0 \\ R \ & > 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} Y \ & (\ast) \\ X^T \ & R \end{bmatrix} > 0 \]  

(8)

\( \ast \) represents all terms induced by symmetry in a symmetric matrix.

**Lemma 3:** The two next problems are equivalent:

(i) Find \( P = P^T \) such that \( T + A^T P A < 0 \)  

(9)

(ii) Find \( P = P^T \), \( L \), \( G \) such that:

\[ \begin{bmatrix} T + A^T L^T + L A \ & (\ast) \\ -L^T + G^T A \ & P - G - G^T \end{bmatrix} < 0 \]  

(10)

It is a generalization of a lemma proposed in (Peaucelle et al, 2000) that generalizes (Oliverira et al, 1999).

**Proof:**

(ii) implies (i): \[ \begin{bmatrix} I \ & A^T \end{bmatrix} \] being a full row matrix using the congruence, lemma 1, gives the result.

(i) implies (ii): consider \( L = 0.5 A^T P \) and \( G = 0.5 (P + P') \) with \( P' > 0 \) an unspecified matrix. Thus the condition (i) becomes:

\[ \begin{bmatrix} -T - A^T P A \ & (\ast) \\ -0.5 P'A \ & P' \end{bmatrix} > 0 \]  

(11)

Applying the Shur’s complement (8) gives: (11) is equivalent to: \( -T - A^T P A - 0.25 A^T P'A > 0 \). Since \( -T - A^T P A > 0 \) by hypothesis, an enough small \( P' > 0 \) such that (11) is satisfied can always be defined.

This lemma can be extended to matrices defined by blocks. For example:

(i) Find \( P = P^T \) such that:

\[ \begin{bmatrix} T_1 + A^T P A \ & (\ast) \\ T_2 \ & T_3 \end{bmatrix} < 0 \]  

(12)

(ii) Find \( P = P^T \), \( L_1 \), \( L_2 \) and \( G \) such that:

\[ \begin{bmatrix} T_1 + A^T L_1^T + L_1 A \ & (\ast) \\ T_2 + L_2 A \ & T_3 \end{bmatrix} < 0 \]  

(13)

**Remark 1:** (12) can be recovered from (13) using the congruence with the row full rank matrix

\[ \begin{bmatrix} I \ & 0 \ & A^T \\ 0 \ & I \ & 0 \end{bmatrix} \]

**Lemma 4.** (Peaucelle et al, 2000) The two next problems are equivalent:

(i) Find \( P > 0 \), such that: \( T + A^T P + PA < 0 \)  

(14)

(ii) Find \( P > 0 \), \( L \), \( G \) such that:

\[ \begin{bmatrix} T + A^T L^T + LA \ & (\ast) \\ P - L^T + G^T A \ & -G - G^T \end{bmatrix} < 0 \]  

(15)

This lemma is the pending of lemma 3 for the continuous case. Similarly, it can be extended to matrices defined by blocks, for example there is equivalence between:
(i) Find $P > 0$ such that
\[
\begin{bmatrix}
T_1 + A^T P + PA \\
T_2 \\
T_3
\end{bmatrix} < 0
\] (16)

(ii) Find $P > 0$, $L_1$, $L_2$ and $G$ such that
\[
\begin{bmatrix}
T_1 + A^T L_1^T + L_1 A \\
T_2 + L_2 A \\
T_3
\end{bmatrix} < 0
\] (17)

Several relaxations of conditions (4) and (5) have been defined in the literature. The main idea is to relax the crossed terms $Y_{ij} + Y_{ji}$ by introducing a new LMI depending on the whole terms $Y_{ij} + Y_{ji}$ and $Y_{ii}$. First results were proposed by Kim and Lee (Kim and Lee, 2000). They were extended in (Liu and Zhang, 2004), and we will use this latter approach. The work presented in (Teixeira et al., 2003) can also be quoted, but it implies a serious increase of the number of variables involved in the problem.

**Lemma 5.** (Liu and Zhang, 2004) Consider matrices $Y_{ii}$, the condition:
\[
\sum_{i=1}^{n} h_i^2(z) + \sum_{i=1}^{n} h_j^2(z)(Y_{ij} + Y_{ji}) < 0
\] (18)
is true if there exists $Q_i$ and $Q_j$ such as the following conditions are satisfied:
\[
\forall i, j \quad Y_{ij} + Y_{ji} < 0
\] (19)
\[
\forall i, j < k \quad Y_{ij} + Y_{ji} + Q_i + Q_j \leq 0
\] (20)
\[
\begin{bmatrix}
Q_i & (*) \\
\vdots & \vdots \\
Q_n & (*)
\end{bmatrix} > 0
\] (21)

**Lemma 6.** (Boyd et al., 1994). Consider the following condition:
\[
G(z)U(z)XV(z) + V(z)X^T U^T(z) > 0
\] (22)
with $z$ and $X$ two variables. $U$ and $V$ do not depend on $X$. Moreover $X$ must be an unspecified matrix with no constraint. Then (22) is equivalent to:
\[
\begin{align*}
G(z) - \sigma U(z) & > 0 \\
G(z) - \sigma V(z) & > 0
\end{align*}
\] (23)
with $z$ the first variable, and $\sigma \in \mathbb{R}$.

This result is based on the Finsler’s lemma and enables to obtain an equivalent problem with a reduced complexity, since we replace an unknown matrix by an unknown scalar.

This lemma has two simplified versions:
If either $U$ or $V$ is the Identity matrix, then its corresponding condition can be removed in (23).
If we have the simplified problem:
\[
\begin{bmatrix}
G_{11} & (*) \\
G_{21} & G_{22}
\end{bmatrix} + U X \begin{bmatrix} I^T \\
0
\end{bmatrix} + \begin{bmatrix} I \\
0
\end{bmatrix} X^T U^T > 0 , \quad \text{then (23)}
\] reduces to:
\[
G(z) - \sigma U(z)U^T(z) > 0 \quad \text{and} \quad G_{22}(z) > 0 .
\]

**Lemma 7 (Inversion matrix lemma).** Let be $A, B, C, D$ matrices of appropriate dimension. Then:
\[
(A + BCD)^{-1} = A^{-1} - A^{-1}B \left[ C^{-1} + DA^{-1}B \right]^{-1} DA^{-1}
\] (24)

The best previous conditions to guarantee the stability of the closed loop for discrete fuzzy models are recalled in the next theorem. With the same notations as previously for theorem 1:

**Theorem 2** (Liu and Zhang, 2003): Fuzzy model (1) is globally asymptotically stable in closed loop with control law (2) and the $Y_{ij}$ defined in (3) if there exists matrices: $X > 0$, $N_i$, $Q_i > 0$, $Q_{ij} = Q_{ji}^T$ ($j > i$), $i, j \in \{1, \ldots , r\}$ such that: (19), (20) and (21) hold.

**Remark 2:** Theorem 2 includes conditions of theorem 1.

**Remark 3:** The number of LMI to check with theorem 2 (excepted condition (21)) is equal to $r(r+1)/2$.

3. **MAIN RESULT**

**Theorem 3:** The fuzzy model (1) is globally asymptotically stable in closed loop with control law (2) if there exists matrices: $X > 0$, $U$, $T$, $L_1$ and $L_2$ such that:
\[
\beta = T + T^T - X > 0
\] (25)
and for $i \in \{1, \ldots , r\}$:
\[
\begin{bmatrix}
X & (*) \\
-AT - BL_1^T & \beta - BL_1^T - L_2B_1^T
\end{bmatrix} > 0
\] (26)
Moreover, the expression of the control law is:
\[
u = -\left[ B_1^T \beta^{-1} B_1 \right]^{-1} B_1^T \beta^{-1} A_1 x
\] (27)

**Proof:** The variation of the quadratic Lyapunov function along the trajectories of the closed loop model, i.e. $\Delta V(k) = V(k+1) - V(k) < 0$ gives:
\[
(A_1 x + B_1 u)^T P(A_1 x + B_1 u) - P < 0
\] (28)
By applying lemma 3 with $L = 0$, (28) is equivalent to:
If the theorem 3 conditions (25) and (26) are verified, it ensures that (35) holds. We need now to prove that (34) also holds. The proof is based on the inversion matrix lemma (24). Applied to (35) we obtain:

$$X - T^*A_{f}^*\Psi A_{f}^*T > 0$$

Then (34) holds if it exists $N_\varepsilon$ satisfying:

$$N_\varepsilon T^*B_\varepsilon^{-1}A_{f}^*T + T^*A_{f}^*B_\varepsilon^{-1}A_{f}T - N_\varepsilon T^*B_\varepsilon^{-1}B_\varepsilon T^*B_\varepsilon^{-1}B_\varepsilon N_\varepsilon > 0$$

or equivalently if it exists $F_\varepsilon$ satisfying:

$$F_\varepsilon T^*B_\varepsilon^{-1}A_{f}^*T + A_{f}^*B_\varepsilon^{-1}A_{f}T - F_\varepsilon T^*B_\varepsilon^{-1}B_\varepsilon F_\varepsilon > 0$$

Introducing the control law (27) gives:

$$A_{f}^*B_\varepsilon^{-1}A_{f} - F_\varepsilon T^*B_\varepsilon^{-1}B_\varepsilon F_\varepsilon > 0$$

Since it exists $\sigma < 0$, (38) holds.

Remark 4: The number of LMI has been reduced from $r(r+1)/2$ to $r+1$. We have to stress that no relaxation principle, such as with lemma 5, is required anymore. Indeed there is no double sum in (33).

Remark 5: Due to the expression of the control law (27) it becomes impossible to use a pole placement approach to obtain feedback gains and to search after a $P > 0$ for the Lyapunov function. This is done for example in (Teixeira et al, 2003) where the stabilization problem is replaced by a stability problem.

4. REGULATOR PROBLEM

We want to minimize the following criterion:

$$u = \arg \min \left\{ \sum_{i} (x^TQx + u^TRu) \right\}$$

An upper bound of this criterion is given solving the following problem.

Theorem 4: The fuzzy model (1) is globally asymptotically stable in closed loop with control law (2) and an upper bound of (39) is guaranteed if there exists matrices: $X > 0$, $U$, $T$, $L_1$ and $L_k$ such that:

$$\min : \gamma, \ \text{subject to} \ \begin{bmatrix} T + T^*X & x_0^T \end{bmatrix} \begin{bmatrix} x_0 & \gamma \end{bmatrix} > 0$$

for $i \in \{1, \ldots, r\}$.
With $P = X^{-1}$, the control law is given by:

$$u = -(B_z^TPB_z + R)^{-1}B_z^TPA_x$$  \hspace{1cm} (41)

\textbf{Proof} The inequality:

$$(x^TQx + u^TRu < 0)$$

(\text{42})

gives

$$(A_x - B_ux)^TP(A_x - B_ux) - P + x^TQx + u^TRu < 0$$

(\text{43})

Minimizing with the variable $u$ leads to:

$$-B_z^TPA_x + B_z^TPA_x + 2(B_z^TPB_z + R)u = 0$$

so once the matrix $P$ is known, the control law is given by:

$$u = -(B_z^TPB_z + R)^{-1}B_z^TPA_x$$

(\text{44})

Now by summing equation (42) between 0 and $k$, it is obvious that $V(0)$ is an upper bound of (39).

The same usual operations applied to (43) gives, with the new variables $G^{-1}PG^{-1} = X > 0$, $G^{-1} = T$ and $F_zG^{-1} = N_z$:

$$X - T^TQT - N_z^TRN_z - T^TA_z^T + N_z^TB_z > 0$$

Then the Shur’s complement is applied to get:

$$X - A_z^TT^TX_z + T^TX_z > 0$$

Then by elimination lemma gives two conditions. The first is $T + T^T - X > 0$. The second is:

$$X - A_z^TT^TX_z + T^TX_z > 0$$

(\text{45})

Lemma 2 is applied with $L = 0$ to get:

$$[G + G^T - X - GT^{-1}x_0] > 0$$

With $T^{-1} = G$ we obtain the criteria to minimize.

\section{5. Results}

We will take the example 2 of (Guerra and Vermeiren, 2004) for which there is no solution with theorem 2 (with a quadratic Lyapunov function). The system is described by two rules defined by the matrices:

$$A_1 = \begin{bmatrix} -0.5 & 2 \\ -0.1 & 0.5 \end{bmatrix}, A_2 = \begin{bmatrix} -0.9 & 0.5 \\ -1.0 & -1.7 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 4.1 \\ 4.8 \end{bmatrix}, B_2 = \begin{bmatrix} 3 \\ 0.1 \end{bmatrix}$$

Previous theorems do not rely on functions $h$, which play no role in the comparison. For the sake of plotting results for a dynamical model, we have chosen:

$$h_1 = \frac{1}{2\pi}(\pi/2 - \text{Arc} \tan(x_1)),$$

$$h_2 = \frac{1}{2\pi}(\pi/2 + \text{Arc} \tan(x_2))$$

A solution with the new conditions is found with the matrix $P$ defined by: $P = \begin{bmatrix} 0.32 & 2.65 \\ 2.65 & 27.2 \end{bmatrix}$. The next figures show the evolutions of the states $x_1$, $x_2$ and the control signal $u$ when $x_0 = \begin{bmatrix} 10 & 2 \end{bmatrix}^T$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{Fig. 1. evolution of $x_1(t)$}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig2.png}
\caption{Fig. 2. evolution of $x_2(t)$}
\end{figure}
6. CONCLUSION

The study presented in this paper tries to reduce the conservatism of the conditions by lowering the number of conditions while still keeping all the degrees of freedom. The number of decision variables has been reduced, and thus the complexity of the LMI problem is smaller. Results show that we were able to obtain such results without raising the conservatism of our conditions. The elimination lemma and several matrix transformations were used for this purpose.

REFERENCES


