RANK-CONSTRAINED LMI APPROACH TO MIXED $H_2/H_\infty$ STATIC OUTPUT FEEDBACK CONTROLLERS

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Abstract:
This paper deals with a numerical method for the design of mixed $H_2/H_\infty$ static output feedback controllers. We first formulate the problem as a new type of rank-constrained linear matrix inequalities (LMIs). Then, the LMI optimization problem subject to a rank condition is tackled by the recently developed penalty function method, where a linear penalty function is introduced for the nonconvex rank constraint. The overall procedure results in solving a series of convex optimization problems. With an increasing sequence of the penalty parameter, the solution of the penalized optimization problem moves towards the feasible region of the original nonconvex problem. Comparisons with previous research are performed to illustrate the proposed method. Copyright© 2005 IFAC

Keywords: Linear matrix inequality (LMI), penalty method, rank constraint, static output feedback (SOF)

1. INTRODUCTION

Recently, a semidefinite program formulation applicable to static output feedback (SOF) stabilization has been proposed (Mesbahi, 1999); however, most SOF control problems including mixed $H_2/H_\infty$ control still remain open.

The purpose of mixed $H_2/H_\infty$ control guarantees optimal closed-loop performance while maintaining a prescribed level of robustness (Bernstein and Hadad, 1989; Khargonekar and Rotea, 1991). Considering performance and robustness simultaneously often arises in many control fields, but no analytic solution exists to date.

The conventional representation of $H_2/H_\infty$ SOF problems based on the celebrated elimination lemma (Boyd et al., 1994; Gahinet and Apkarian, 1994; Skelton et al., 1997) leads to a linear matrix inequality (LMI) optimization problem subject to a nonconvex algebraic or rank constraint on the Lyapunov variables (Leibfritz, 2001). The use of a single Lyapunov matrix in multiobjective control is known to produce conservative results. To reduce the degree of conservatism, several methods have been proposed in the LMI framework (Arzelier and Peaucelle, 2002; Halder and Kailath, 1999; Shimomura and Fujii, 1999).

Meanwhile, to solve nonconvex rank-constrained LMI problems, several global and local methods have been presented during the last decade (Goh et al., 1994; Grigoriadis and Skelton, 1996; Ghaoui et al., 1997; Fazel et al., 2003).
More recently, a partially augmented Lagrangian (PAL) method (Apkarian et al., 2003) has been developed. This second-order method has a superior convergence property over the local methods, but the implementation of the algorithm is not easy since the gradient and Hessian of the objective function must be derived for the Newton-type method.

In this paper, mixed $H_2/H_\infty$ SOF problems are converted to a new type of rank-constrained LMI problems. The rank condition here is not imposed on the Lyapunov matrix but imposed on the slack matrix; thus the proposed method can be applied to simultaneous stabilization, polytopic uncertain plant models and multi-objective control problems. For the SOF stabilization problem, a similar method was addressed by (Peaucelle et al., 2002). After formulating the problem, we seek a static control law $u$ where

$$u = K y$$

such that the performance condition (2) holds if and only if there exist matrices $P > 0, W \succeq 0$ satisfying the LMI subject to the rank condition,

$$\begin{bmatrix}
\star \\
\star \\
\star \\
\star \\
\end{bmatrix}^T \begin{bmatrix}
P & 0 & 0 & 0 \\
0 & I & 0 & 0 \\
Q & R & S & 0 \\
0 & 0 & 0 & I \\
\end{bmatrix} \begin{bmatrix}
I & 0 & 0 & 0 \\
A & B_1 & B_2 & 0 \\
0 & C_1 & D_{11} & D_{12} \\
0 & I & 0 & 0 \\
\end{bmatrix} \prec 
\begin{bmatrix}
\star \\
\star \\
\star \\
\star \\
\end{bmatrix} \begin{bmatrix}
C_2 & D_{21} & 0 & 0 \\
0 & 0 & I & 0 \\
\end{bmatrix}$$

$$\text{rank}(W) = n_u.$$  \hfill (3)

If $W$ satisfying (3) and (4) is found, $K$ can be computed by solving the LMI in the variable $K$,

$$\begin{bmatrix} W_1 + W_2 K + K^T W_2^T & K^T W_3 \\ W_3 K & -W_3 \end{bmatrix} \succeq 0,$$  \hfill (5)

where

$$W = \begin{bmatrix} W_1 & W_2 \\ W_2 & W_3 \end{bmatrix}.$$  \hfill (6)

**Proof.** Define $\Omega$ as

$$\Omega = \begin{bmatrix} \star^T & P \end{bmatrix} \begin{bmatrix} I & 0 & 0 & 0 \\ A & B_1 & B_2 & 0 \\ 0 & C_1 & D_{11} & D_{12} \\ 0 & I & 0 & 0 \end{bmatrix} + \begin{bmatrix} \star^T & Q \end{bmatrix} \begin{bmatrix} R^T & S \end{bmatrix} \begin{bmatrix} C_2 & D_{21} & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix}.$$  \hfill (7)

Then, by virtue of Lyapunov stability theory and Finsler’s lemma, the existence condition of an SOF controller can be expressed as the following LMI in $P > 0$ and $K$ (Arzelier and Peaucelle, 2002)

$$\begin{bmatrix} \star^T & \Omega \end{bmatrix} \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I \end{bmatrix} \prec 0.$$  \hfill (8)

which is equivalent to

$$\begin{bmatrix} \Omega & \mu \begin{bmatrix} C_2^T K & D_{21}^T K^T \\ 0 & -I \end{bmatrix} \\ \mu \begin{bmatrix} C_2^T K & D_{21}^T K^T \\ 0 & -I \end{bmatrix} & K C_2 & K D_{21} & -I \end{bmatrix} \prec 0.$$  \hfill (9)

where $\mu > 0$. From (7), we can easily obtain (3), (4) and (5).

In the case of $H_\infty$ controllers, the performance matrices are given by $Q = I, S = -\gamma^2 I, R = 0$. Also,
$H_2$ optimal controllers can be described in a similar manner.

Lemma 2. For system (1) with $D_{11} = D_{21} = 0$, we can find a static control law such that the $H_2$ performance of the closed-loop system is $||T_{wz}||_2 < \gamma_2$ if and only if there exist matrices $P_2 > 0, W \succeq 0$ satisfying,

$$\text{tr}(B_1^T P_2 B_1) \leq \gamma_2^2,$$

$$\left( \begin{array}{c}
\phi^T \\
0
\end{array} \right) \left( \begin{array}{ccc}
0 & P_2 & 0 \\
0 & 0 & 0
\end{array} \right) \left( \begin{array}{c}
I \\
A
\end{array} \right) \left( \begin{array}{c}
B_1 \\
B_2
\end{array} \right) \succeq 0,$$

$$\left( \begin{array}{c}
\phi^T \\
0
\end{array} \right) W \left( \begin{array}{ccc}
C_2 & 0 & 0 \\
0 & I
\end{array} \right),$$

where all notations have the same meaning as in (1).

Remark 3. The advantage of Lemmas 1 and 2 lies in that no constraints are imposed on the Lyapunov matrix $P$. Thus, we can use separate Lyapunov matrices for polytopic plants or multiobjective control syntheses to reduce conservatism.

3. MIXED $H_2/H_\infty$ STATIC OUTPUT CONTROL

In this section, we present a rank-constrained LMI approach to the mixed $H_2/H_\infty$ static control problem shown in Fig. 2, whose state-space representation is

$$\dot{x} = \left( \begin{array}{ccc}
A & B_0 & B_1 \\
C_0 & D_{00} & 0 \\
C_1 & 0 & D_{12}
\end{array} \right) x + \left( \begin{array}{ccc}
B_2 & 0 \\
D_{20} & 0 \\
0 & 0
\end{array} \right) u,$$

$$y = C x + D w,$$

where all notations have the same meaning as in (1). The channel $(w_{\infty}, z_{\infty})$ is for the robustness condition of the system, and the channel $(w_2, z_2)$ for the optimal $H_2$ performance of the closed-loop system. The mixed $H_2/H_\infty$ SOF problem for system (11) can be written as follows.

**Problem 4.** For a given $\gamma_\infty > 0$, find a static control law $u = Ky$ that minimizes $||T_{z_2 w_{\infty}}||_2$ subject to $||T_{z_2 w_{\infty}}|| < \gamma_\infty$.

Based on the formulation of the previous section, we use two Lyapunov matrices for $H_2$ and $H_\infty$ channels. To find a single control gain, a common $W$ matrix is chosen at the expense of some conservatism. The resulting problem to be solved reduces to

$$\min_{P_2 > 0, P_\infty > 0, W \succeq 0} \text{tr}(B_1^T P_2 B_1)$$

subject to

$$\left( \begin{array}{ccc}
0 & P_\infty & 0 \\
P_\infty & 0 & 0 \\
0 & 0 & I
\end{array} \right) \left( \begin{array}{ccc}
I & 0 & 0 \\
A & B_0 & B_2 \\
C_0 & D_{00} & D_{02}
\end{array} \right) \succeq 0,$$

$$\text{rank}(W) = n_u.$$
Algorithm 1. The PFM for rank-constrained LMI problems

(1) Initialization. Set the penalty parameter $\mu = 0, \rho_0 \gg 1$ and find an initial feasible point $x_0$ by solving the LMI optimization problem:

$$x_0 = \min_x \{ \rho c^T x + \text{tr}(X) : x \in C \}.$$ 

Set $x_k = x_0$. Choose $\mu_k = \mu_0 > 1, \rho_k = \rho_0, \alpha \in (0, 1), \beta \ll 1, \gamma > 1, \xi > 1, \epsilon_1 \ll 1, \epsilon_2 \ll 1$.

(2) Computation of $V$. Compute $V_k$ from $X(x_k)$ by eigenvalue decomposition.

(3) Convex optimization. Compute $x_{k+1}$ by solving the convex LMI optimization problem,

$$x_{k+1} = \min_x \{ \varphi(x; \rho_k, \mu_k, V_k) : x \in C \}.$$ 

(4) Feasibility test. If $p(x_{k+1}; V_k) \leq \epsilon_1$, then $x_{k+1}$ is feasible and stop when computing a feasible solution.

(5) Optimality test. If $x_{k+1}$ is feasible and $|c^T x_{k+1} - c^T x_k| \leq \epsilon_2$ then a locally optimal solution $x_{k+1}$ is obtained. Stop.

(6) Penalty parameter update. If $x_{k+1}$ is not feasible and $p(x_{k+1}; V_k) > \alpha p(x_k; V_{k-1})$, then increase the penalty parameter by $\mu_{k+1} = \tau \mu_k$.

(7) Optimization weight update. If $x_{k+1}$ is feasible and $|c^T x_{k+1} - c^T x_k| < \beta$, then increase the optimization weight by $\rho_{k+1} = \xi \rho_k$.

(8) Next step. Set $k = k + 1$ and go to step (2).

The implementation code of the PFM is almost the same as that of the cone complementarity linearization algorithm (Ghaoui et al., 1997) except for eigenvalue decomposition. Though the PFM is similar to the first-order method, it can be applied to optimization problems, and it shows good convergence characteristics attributed to the tuning factors $\mu$ and $\rho$.

6. NUMERICAL EXAMPLES

We selected some $H_2/H_{\infty}$ static output feedback control examples to evaluate the performance of the proposed algorithm. Throughout the simulation, we have used the SeDuMi package as an LMI solver and the YALMIP for a SeDuMi interface (Sturm, 2001; Löfberg, 2004). The computation parameters used were

$$\mu_0 = 5000, \rho_0 = 1000, \alpha = 0.99, \gamma = 1.05,$$

which were selected by a trial-and-error approach. Thus further work on initial values, computation parameters, and convergence properties is needed.

Example 1. This is a classical example taken from (Levine and Athans, 1970). The state-space matrices of the system are given by

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C_0 = \begin{pmatrix} 0 & 1 \end{pmatrix}, \quad D_{00} = 0,$$
$D_{02} = 0$, $C_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $D_{12} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$,

$C_2 = \begin{pmatrix} 0 & 1 \end{pmatrix}$, $D_{20} = 0$.

The analytical solution to the mixed $H_2/H_\infty$ static control for this system is completely known (Arzelier and Peaucelle, 2002). Table 1 shows the computation results for the robustness condition $\gamma_\infty \leq 1.2$, and Fig. 3 shows the computational behavior of the PFM. In the table, $\gamma_2$-bound means the solution to Problem 4, and $\gamma_2$-actual is computed from the closed-loop system with the obtained static gain $K$. The optimal gain is $-0.9458$, and the computed gain by the PFM is $-0.9735$. As is shown in Fig. 3, the penalty function of the PFM is always decreasing and tends to zero in 20 iterations. We can see that the obtained solution is not overly conservative.

**Table 1. Results for example 1**

<table>
<thead>
<tr>
<th></th>
<th>$\gamma_2$-bound</th>
<th>$\gamma_2$-actual</th>
<th>$\gamma_\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Optimal</td>
<td>-</td>
<td>1.5735</td>
<td>1.2000</td>
</tr>
<tr>
<td>Arzelier(2002)</td>
<td>1.6825</td>
<td>1.5778</td>
<td>1.1706</td>
</tr>
<tr>
<td>PFM</td>
<td>1.5838</td>
<td>1.5772</td>
<td>1.1746</td>
</tr>
</tbody>
</table>

![Fig. 3. Behavior of penalty function and $H_2$-norm upper bound for example 1.](image)

**Example 2.** As a second example, we choose a mass-spring system described in (Shimomura and Fujii, 1999) with data matrices

$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -1.25 & 1.25 & 0 & 0 \\ 1.25 & -1.25 & 0 & 0 \end{pmatrix}$, $B_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$,

$B_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $B_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $C_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$,

$C_1 = \begin{pmatrix} 0.2 \\ 0 \\ 0 \end{pmatrix}$, $D_{12} = \begin{pmatrix} 0 \\ 0.2 \end{pmatrix}$,

$D_{02} = \begin{pmatrix} 0 \\ 0.2 \end{pmatrix}$, $C_2 = I_{4\times4}$.

We design a static controller with the $H_\infty$ specification, $\gamma_\infty \leq 1$. Computation results are displayed in Table 2, and Fig. 4. Calculated control gain is

$K = \begin{pmatrix} -1.2127 \\ -0.2828 \\ -1.4208 \\ -0.6675 \end{pmatrix}$.

In this numerical experiment, our result is less conservative.

**Table 2. Results for example 2**

<table>
<thead>
<tr>
<th></th>
<th>$\gamma_2$-bound</th>
<th>$\gamma_2$-actual</th>
<th>$\gamma_\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shimomura(1999)</td>
<td>1.7827</td>
<td>1.7223</td>
<td>0.3979</td>
</tr>
<tr>
<td>PFM</td>
<td>1.5114</td>
<td>1.5111</td>
<td>0.9416</td>
</tr>
</tbody>
</table>

![Fig. 4. Behavior of penalty function and $H_2$-norm upper bound for example 2.](image)

**Example 3.** This is the longitudinal motion of a VTOL helicopter (Leibfritz, 2001). The system data matrices are given by

$A = \begin{pmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.0100 & 0.0024 & -0.4028 \\ 0.1002 & 0.3681 & -0.7070 & 1.4200 \end{pmatrix}$,

$B_1 = \begin{pmatrix} 0.0468 \\ 0.0457 \\ 0.0099 \\ 0.0437 \end{pmatrix}$, $C_1 = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$,

$B_2 = \begin{pmatrix} 0.4422 \\ 0.1761 \\ -5.52 \\ 4.49 \end{pmatrix}$, $C_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$,

$D_{20} = \begin{pmatrix} 0.0039 & 0.00174 \end{pmatrix}$,

$D_{02} = D_{12} = I_{2\times2}/\sqrt{2}$,

$B_0 = B_1$, $C_0 = C_1$.

Table 3 and Fig. 5 show the results with the constraint $\gamma_\infty \leq 0.423722$. The computed gain is

$K = \begin{pmatrix} 1.0792 \\ 11.8505 \end{pmatrix}$.
Table 3. Results for example 3

<table>
<thead>
<tr>
<th></th>
<th>$\gamma_2$-bound</th>
<th>$\gamma_2$-actual</th>
<th>$\gamma_\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Leibfritz(2001)</td>
<td>0.4687</td>
<td>0.1033</td>
<td>0.2943</td>
</tr>
<tr>
<td>PFM</td>
<td>0.1050</td>
<td>0.1002</td>
<td>0.1778</td>
</tr>
</tbody>
</table>

Fig. 5. Behavior of penalty function and $H_2$-norm upper bound for example 3.

From the results above, we can see that the PFM can efficiently solve mixed $H_2/H_\infty$ static output control problems with a new rank-constrained LMI representation, and that our results are less conservative than those of the previous research.

7. CONCLUDING REMARKS

We have addressed a simple iterative algorithm for mixed $H_2/H_\infty$ static output control problems. The mixed $H_2/H_\infty$ problem was transformed to a new type of rank-constrained LMI optimization problem, which were solved iteratively by the recently developed penalty function method. Numerical experiments showed promising results.

REFERENCES


