STATE-SPACE SOLUTION TO STOCHASTIC
$H_{\infty}$-OPTIMIZATION PROBLEM WITH
UNCERTAINTY

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Abstract: Robust stochastic anisotropy-based $H_{\infty}$-optimization problem for discrete linear time-invariant (LTI) systems with structured parametric uncertainty is considered. It is shown that the problem can be reduced to mixed $H_2/H_\infty$-like problem. The resulting control problem involves the minimization of anisotropic and $H_\infty$ norms of the system. Explicit state-space formulas are also obtained for the optimal controller. The problem covers the standard $H_2/H_\infty$-optimization problem and $H_\infty$-optimization problem as two limiting cases. Copyright ©2005 IFAC.

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1. INTRODUCTION

Well-known $H_2$- and $H_\infty$-theories of optimization for LTI systems are based on the $H_2$- and $H_\infty$-norms as performance criteria. The $H_2$-control theory assumes that the input disturbance is the Gaussian white noise. The $H_\infty$-control theory assumes it to be a square-summarable signal. As a consequence, use of $H_2$-optimal controllers in feedback loop leads to poor functioning of a closed-loop system if strongly colored random noise is fed to the input. On the other hand, $H_\infty$-optimal controllers are conservative if the input disturbance is white or slightly colored noise.

Stochastic approach to $H_\infty$-optimization for discrete LTI systems was proposed in (Semyonov et al., 1994). This approach exploits an input signal "colourness" characteristic introduced in (Vladimirov et al., 1995) and called mean anisotropy. Anisotropy norm (Diamond P. et al., 2001) of the closed-loop transfer function is proposed to be the performance criterion. The controller design problem with such performance criterion was solved in (Vladimirov et al., 1996). Stochastic (anisotropy-based) $H_\infty$-optimal controllers are located "between" $H_2$-optimal and $H_\infty$-optimal controllers. Moreover, $H_2$- and $H_\infty$-optimal controllers are the limiting cases of anisotropy-based controllers when mean anisotropy of input signal tends to zero or to infinity, respectively. The anisotropy-based optimization problem with mean anisotropy level $\alpha$ will be referred to $AB_\alpha$-problem.

The problem of robust state feedback $H_\infty$-control design for class of LTI systems with parametric uncertainty was solved in (Xie et al., 1991).

In this paper we formulate and solve the robust anisotropy-based stochastic $H_\infty$ optimization problem for discrete LTI systems with parametric uncertainty. It is shown that the problem can be reduced to mixed $H_2/H_\infty$-like problem (Doyle et al., 1994). The resulting control problem involves the minimization of anisotropic and $H_\infty$-
unknown parameters, which satisfies are unknown matrix functions corresponding to matrices of appropriate dimensions; $\Omega$ is the control, $z$ is the observation; $x$ is the controlled signal, $w$ is the disturbance, $u$ is the input, and the mean anisotropy of $W$ is defined as

$$\bar{A}(G) = -\frac{1}{4\pi} \int_{-\pi}^{\pi} \ln \det \left( \frac{m}{\|G\|^2} \tilde{G}(\omega)(\tilde{G}(\omega))^* \right) d\omega.$$  

Let us consider the corresponding class of signals $W_a = \{ w_k \in l_2 : w_k = Gv_k, \text{ where } V \in G(0, I), G \in G_a \}$. 

We will use the power norm of a signal $u$: 

$$||u||_p = \left( \lim_{N \to \infty} \frac{1}{2N} \sum_{k=-N}^{N} ||u_k||^2 \right)^{1/2} = \sqrt{\text{Trace} R_{uu}(0)},$$

where $R_{uu}(n)$ denotes the auto-correlation function of sequence $U$. The Fourier transform of $R_{uu}(n)$ is called the spectral density and is denoted by $S_{uu}$. The signals having bounded power norm and bounded spectrum are referred to as BP-signals and BS-signals, respectively.

The uncertainty $\Delta_k$ is called admissible if $\Delta_k \in RH_\infty$ and the system $U\Psi_k \Delta_k$ is internally stable. A controller $K$ is called strictly causal if control input $u_k$ depends on any instant $k$ only on the preceding observations $y_j, j < k$. A controller $K$ is called admissible if it is strictly causal and it internally stabilizes the closed-loop system on figure 1. The set of all admissible controllers for the given system $F$ is denoted by $K$ and the set of all admissible uncertainties $\Delta_k$ for the system $F$ is denoted by $\mathcal{D}$.

The anisotropic norm of an arbitrary causal system $F \in H_\infty$ is

$$\|F\|_a = \sup_{G \in G_a} \frac{\|FG\|_p}{\|G\|_p}. \quad (2.3)$$

The robust anisotropy-based stochastic $H_\infty$-optimization problem is formulated as follows:

**Problem 1.** For the given system (2.1) and input mean anisotropy level $a \geq 0$, find an admissible controller $K \in K$ that minimizes the maximal value of $\alpha$-anisotropic norm of the closed-loop transfer function $L(F, K)$

$$\sup_{\Delta_k \in \mathcal{D}} \|L(F, K)\|_a \ \inf. \quad (2.4)$$

The last criterion is identical to

$$\sup_{\Delta_k \in \mathcal{D}} \sup_{G \in G_a} \|z\|_p^2 \ \inf, \ K \in K. \quad (2.5)$$

The problem described above will be referred to as RASHO problem.

The following standard assumptions on system (2.1) will be used throughout the text.

**Assumptions**
respectively. Original system (2.1) coincides with one, and control signal \( w \) consequently, \( \eta \) transfer function the same as in (2.1). System (3.6) has three inputs. Let us find the controller value of \( K \) controller mean anisotropy level \( M \). The equality (3.7) means that \( \eta \) is fed for the third one. By let \( \tilde{w} \) be an admissible controller, which minimizes the cost function \( J(K, \gamma) \) for some fixed \( \gamma \neq 0 \). Then \( K_3 \) minimizes the cost \( J_1(K, \gamma) \equiv \sup_{\eta \in \mathcal{D}_{\eta}} \| \tilde{z} \|_p \) and \( \inf_{K \in \mathcal{K}} \sup_{\gamma \in \mathcal{G}} J_1(K, \gamma) \), where \( \mathcal{D}_{\eta}, \mathcal{G} \) are some scalar values and let \( \gamma = \text{col}\{\gamma_1, \gamma_2, \gamma_3\} \). Define \( \Theta \equiv \sum_{k = -\infty}^{\infty} \langle \xi_k, Q \xi_k - \eta_k^T \Gamma \eta_k + w_k^T S_0 w_k \rangle \). This means that we can reduce the RASHO problem to a new problem which are called the mixed \( AB_a/H_\infty \)-problem. The criterion for the new "generalized" problem is (4.11)

To solve problem 3, new mixed \( H_2/H_\infty \)-like method is proposed.

4. SADDLE-POINT TYPE CONDITION OF OPTIMALITY IN THE MIXED \( AB_a/H_\infty \)-PROBLEM

The problem (4.11) is a minimax problem; hence game-theory approach may be appropriate. Saddle point of the game is a triplet \( (K^*, G_0^*, G_1^*) \) such that the following inequality holds:

\[
J(K^*, G_0^*, G_1^*) \leq J(K, G_0^*, G_1^*) \leq J(K, G_0^*, G_1^*).
\]

Let us consider the following sets:

\[
K_* \equiv \text{Arg min}_{K \in \mathcal{K}} T, \quad G_0* \equiv \text{Arg max}_{G_0 \in \mathcal{G}_0, \|G_0\|_2 = 1} T, \quad G_1* \equiv \text{Arg max}_{G_1 \in \mathcal{G}_1, \|G_1\| = 1} T.
\]
\[ T = \left\| L(U(\tilde{F}, \Delta_k), K) \begin{bmatrix} G_0 & 0 \\ 0 & G_1 \end{bmatrix} \right\|_2. \]

The set (4.12) is formed by the controllers which are solutions of the mixed backwards-forward Riccati optimization problem corresponding to the assertion that the input \( W \) of the closed-loop system \( L(\tilde{F}, K) \) is generated by known generating filter \( G_0 \in \mathcal{G}_a \), i.e. \( W = G_0 \otimes V \). The input \( \eta \) is generated by a known generating filter \( G_1 \in RH_\infty, \) i.e. \( \text{col}(\eta_k) = G_1 \otimes W \). Appropriately, the set (4.13) is formed by the worst-case input generating filters with bounded anisotropy for a given controller \( K \in \mathcal{K} \) and filter \( G_1 \in RH_\infty^{m \times m} \). Similarly, the set (4.14) is formed by the worst-case input generating filter with unbounded anisotropy for given controller \( K \in \mathcal{K} \) and filter \( G_0 \in \mathcal{G}_a \).

If the assumption holds, the set (4.12) consists of a unique I/O-operator.

**Lemma 1.** If the controller \( K \) is a fixed point of the following map

\[ K \in \mathcal{K} \begin{bmatrix} \tilde{F}, G_{0*}, G_{1*} \end{bmatrix}, \]

(4.15)

then it is a solution to problem (2.4).

5. WORST-CASE BP-INPUT DISTURBANCE SCENARIO VS FINITE-DIMENSIONAL CONTROLLER IN THE PRESENCE OF ARBITRARY BS-INPUT

The closed-loop system \( L(\tilde{F}, K) \) has the following state-space realization:

\[ L(\tilde{F}, K) = \begin{bmatrix} A & B_2 \tilde{C} & B_0 & B_1 \\ \tilde{B} C_2 & \tilde{A} & \tilde{B} D_{21} & 0 \\ C_1 & D_{12} \tilde{C} & 0 & 0 \\ \tilde{C}_1 & 0 & 0 & D_{21} & 0 \end{bmatrix}. \]

(5.16)

**Theorem 2.** Let \( \gamma > \| L(\tilde{F}, K) \|_\infty \). Then

\[ \sup \theta \leq \text{Trace} \left\{ B_2^T (I + 2 A_k) (P + P F_k P F_k \Pi - 1 F_k^T P) B_1 + P F_k \Pi - 1 F_k^T P B_1 + S_0 \right\}, \]

where \( \Pi = \Gamma - F_t^T P F_t \geq 0 \) and \( P \in \mathbb{R}^{n \times n} \) is an admissible solution of the discrete algebraic Riccati equation

\[ A_t^T P A_t - P + A_t^T F_t \Pi - 1 F_t^T P A_t + Q = 0. \]

(5.17)

The worst case input scenario is

\[ \tilde{\eta}_k = \Pi - 1 F_t^T \left( A_t x_k + B_t w_k \right) \]

(5.18)

and the matrix \( A_t + F_t \Pi - 1 F_t^T P A_t \) is stable.

The worst case disturbance scenario \( \tilde{\eta}_k \) can be generated from the BS-signal \( w_k \) by the shaping filter \( G_1 \), whose internal state is a copy of the system state \( \xi_k \), i.e. its realization is

\[ \tilde{G}_1 = \begin{bmatrix} \tilde{P} A_t & \tilde{P} B_t \\ \Pi - 1 F_t P A_t & \Pi - 1 F_t P B_t \end{bmatrix}, \]

(5.19)

where \( \tilde{P} = I + F_t \Pi - 1 F_t P \).

6. WORST-CASE BP-INPUT DISTURBANCE SCENARIO VS FINITE-DIMENSIONAL CONTROLLER FOR THE WORST-CASE BP-INPUT

In this section our goal is to find worst-case shaping filter generating a signal \( w \) that maximizes BS-disturbance gain over all Gaussian sequences, from Gaussian white noise \( \mathbf{G}(0, I) \). Direct calculation gives

\[ L(\tilde{F}, K) \begin{bmatrix} I & 0 \\ 0 & \tilde{G}_1 \end{bmatrix} = \begin{bmatrix} A_w & B_w \\ C_w & D_w \end{bmatrix}, \]

(6.20)

where \( A_w = \tilde{P} A_t, \quad B_w = \Xi B_t, \quad C_w = \begin{bmatrix} C_1 & D_{12} \tilde{C} \end{bmatrix}, \quad D_w = 0. \)

It is clear that

\[ J = \sup_{w_k \in \mathcal{W}_a} \left\| L(\tilde{F}, K) \begin{bmatrix} I & 0 \\ 0 & \tilde{G}_1 \end{bmatrix} \right\|_p^2, \]

where \( \tilde{G}_1 \) satisfies (5.19). The problem at the right-hand side of the last equality can be solved by using anisotropic technique developed in (Vladimirov et al., 1996). The frequency description of the worst shaping filter \( \tilde{G}_1 \) is given by the following proposition.

**Theorem 3.** (Diamond et al. 2001). Let the system \( F \in H_\infty^{m \times m} \) and the filter \( G \in H_\infty^{m \times n} \) satisfy

\[ \tilde{G}(\omega) (\tilde{G}(\omega))^* = \left( I_m - q F^*(\omega) \tilde{F}(\omega) \right)^{-1}, \]

(6.21)

for \( q = \overline{A}^{-1}(G) \). Then \( G(s) \) belongs to the set of worst-case input generating filters (4.13).

Let \( L = [L_1 \; L_2] \in \mathbb{R}^{m \times 2n} \) be a matrix such that \( A + B L \) is asymptotically stable, and let \( \Sigma \in \mathbb{R}^{m \times m} \) be a positive definite symmetric matrix. Consider the generating filter \( G_0 \) with the input \( V \) and output \( W \) governed by the equations (3.6), combined with

\[ w_k = L_1 x_k + L_2 \xi_k + \Sigma^{1/2} v_k = L_\xi + \Sigma^{1/2} v_k. \]

(6.22)
It is straightforward to verify that
\[ G(s) = \left[ \frac{A_w + B_w L}{L} \frac{B_w \Sigma^{1/2}}{\Sigma^{1/2}} \right]. \]  
(6.23)

**Lemma 2.** (Diamond et al. 2001). For given asymptotically stable system
\[ \begin{bmatrix} A & B \\ * & * \end{bmatrix}, \]
the mean anisotropy of the sequence \( W = G \otimes V \) generated by asymptotically stable filter (6.23) is equal to
\[ \bar{A}(G) = -\frac{1}{2} \ln \det \left( \frac{m_1 \Sigma}{\text{Trace} (LYLT + \Sigma)} \right), \]
where \( \bar{Y} \in \mathbb{R}^{n \times n} \) is the controllability gramian of \( G \) satisfying the Lyapunov equation
\[ \bar{Y} = (A + BL)\bar{Y}(A + BL)^T + B\Sigma B^T. \]  
(6.24)

We consider the following Riccati equation for the matrix \( R \in \mathbb{R}^{n \times 2n} \)
\[ R = A^T R A_w + q C_w^T C_w + L_T \Sigma^{-1} L_w, \]  
(6.25)
\[ L = (\Sigma B_w^T R A_w + q D_w^T C_w), \]  
(6.26)
\[ \Sigma = (I_m - B_w^T R B_w)^{-1}. \]  
(6.27)

A solution \( R \) of (6.25)–(6.27) is called admissible if \( R \) is symmetric, \( \Sigma \) is positive-definite and \( A + BL \) is asymptotically stable. Note that for any \( q \in [0, \|F\|_{\infty}^2] \), the equation above has a unique admissible solution, which is positive and semidefinite.

The formulas for the worst-case shaping filter are based on the following theorem.

**Theorem 4.** (Diamond et al. 2001). Let the system (6.20) be asymptotically stable, and the matrices \( L \) and \( \Sigma \) correspond to the admissible solution \( R \) of Riccati equation (6.25)–(6.27), where parameter \( q \in [0, \|F\|_{\infty}^2] \) is the solution of equation
\[ a = -\frac{1}{2} \ln \det \left( \frac{m_1 \Sigma}{\text{Trace} (LYLT + \Sigma)} \right), \]  
(6.28)
and \( \bar{Y} \) satisfies (6.24). Then generating filter (6.23) satisfies (6.21). In that case, \( a \)-anisotropic norm (2.3) of the system \( F \) is given by
\[ \|F\|_a = \left( \frac{1}{q} \left( 1 - \frac{m_1}{\text{Trace} (LYLT + \Sigma)} \right) \right)^{1/2}. \]

### 7. STATE ESTIMATING FORMULAS

Denote by \( \mathcal{F}_k^Y \) a \( \sigma \)-algebra of random events induced by the history \( \{y_j\}_{j \leq k} \) of the observation signal \( Y \) at the instant \( k \). In other words, \( \mathcal{F}_k^Y \) is the flow of \( \sigma \)-algebras in \( \mathcal{F} \) generated by the sequence \( Y \).

Admissible controller (3.10) is called state-estimating, if its \( n \)-dimensional internal state \( \Xi \) coincides with the sequence of one-step predictors for the internal state \( X \) of the system \( F \) via the observation signal \( Y \) under the worst-case input disturbance \( W \), i.e. if
\[ \xi_k = \mathbb{E} \left( x_k \mid \mathcal{F}_k^Y \right), \quad -\infty < k < +\infty, \]
where \( W = G \otimes V \) with the worst-case generating filter \( G \) (here, \( \mathbb{E} (\cdot | \cdot) \) stands for the conditional expectation).

The system with the worst BP- and BS-inputs has the following state-space realization:
\[ L(\bar{F}, K) \begin{bmatrix} I & 0 \\ 0 & \bar{G}_1 \end{bmatrix} \bar{G}_0 = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{B}_1 \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{B}_2 \\ \bar{C}_{21} & \bar{C}_{22} & \bar{D} \end{bmatrix}, \]
where \( \tilde{A} \) is the matrix divided into uniform blocks corresponding with \( x \) and \( \xi \).

Prior to formulate a criterion for the state-estimating property, we consider the Riccati equation
\[ S = \tilde{A}_{11} S \tilde{A}_{11}^T + \tilde{B}_1 \tilde{B}_1^T - \Lambda \Theta \Lambda^T, \]  
(7.29)
\[ \Theta = \tilde{C}_{21} S \tilde{C}_{21}^T + \bar{D} \bar{D}^T, \]  
(7.30)
\[ \Lambda = (\tilde{A}_{11} \tilde{C}_{21} + \tilde{B}_1 \bar{D}) \Theta^{-1}, \]  
(7.31)
where the matrices \( \Sigma \) and \( L \) are defined in Theorem 4.

A solution \( S = ST \in \mathbb{R}^{n \times n} \) of equation (7.29)–(7.31) is called admissible if the matrix \( S \) is positive semidefinite and \( A_{11} - \Lambda C_{21} \) is asymptotically stable.

**Theorem 5.** Let system (3.6) satisfy Assumption 1, and let the state-space realization matrices of admissible controller (3.10) obey the relations
\[ \tilde{A} = \tilde{A}_{11} + \tilde{A}_{12} - \Lambda (\tilde{C}_{21} + \bar{C}_{22}), \]
\[ \bar{B} = \Lambda, \]  
(7.32)
where the matrix \( \Lambda \) is expressed through the admissible solution of Riccati equation (7.29)–(7.31), where, in turn, the matrices \( \Sigma \) and \( L \) determine the worst-case generating filter as described in Theorem 4. Then controller (3.10) is state-estimating.

### 8. STATE-SPACE FORMULAS FOR THE OPTIMAL CONTROLLER

In order to formulate the final result, we consider the following Riccati equation
\[ T = A_u^T T A_u + C_u^T C_u - N^T Y N, \]  
\[ Y = B_u^T T B_u + D_{12}^T D_{12}, \]
\[ N \equiv \begin{bmatrix} N_1 & N_2 \end{bmatrix} = -Y^{-1}(B_u^T T A_u + D_{12}^T C_u), \]

where the matrix \( N \) is partitioned into two blocks \( N_1, N_2 \in \mathbb{R}^{n_2 \times n} \) and the matrices \( A_u \in \mathbb{R}^{2n \times 2n} \), \( B_u \in \mathbb{R}^{2n \times n_2} \) and \( C_u \in \mathbb{R}^{p_1 \times 2n} \) are given by

\[
A_u = \begin{bmatrix} \tilde{A}_{11} & B_2 \hat{C} \\ 0 & \tilde{A}_{12} - A\hat{C}_{21} + \hat{C}_{22} \end{bmatrix},
\]

\[
B_u = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, C_u = \begin{bmatrix} C_1 & 0 \end{bmatrix}.
\]

A solution \( T = T^T \in \mathbb{R}^{2n \times 2n} \) of equation (8.33)–(8.34) is called admissible if the matrix \( T \) is non-negative definite and \( A_u + B_u N \) is asymptotically stable.

**Theorem 6.** Let system (2.1) satisfy assumptions (A)-(E) and let the state-space realization matrices of state-estimating controller (3.10) obey (7.32) in combination with the following equation:

\[ \hat{C} = N_1 + N_2, \]

where the matrices \( N_1, N_2 \) are expressed via the admissible solution of the Riccati equation (8.33)–(8.34). Then the controller is a solution to Problem 3.

9. **EXPLICIT FORMULAS FOR STATE ESTIMATING CONTROLLER**

Now we can collect the results derived above. The solution of the RSAHO problem can be divided into several steps. First, we fix values \( \gamma_i \neq 0 \). Then, cross-coupled Riccati equations (5.17), (6.25)-(6.26), (7.29)-(7.31), (8.33)-(8.35), the Lyapunov equation (6.24) and the equation of special type (6.28) should be solved. These equations can be solved numerically using homotopic methods (Diamond et al., 1997). A solution of these equations gives the controller \( K_u(s) \), which is suboptimal solution for the original problem. To obtain the optimal solution, it is necessary to find the minimal value \( \gamma_{min} \) such that the Riccati equations above have admissible solutions. This can be accomplished by gradually decreasing the parameters \( \gamma_i \). The controller \( K_{\gamma_{min}}(s) \) tends to be the optimal one in the sense (2.4).

10. **CONCLUSION**

In this paper a state-space solution to the robust anisotropy-based stochastic \( H_\infty \) optimization problem for discrete finite-dimensional LTI systems was proposed. It is shown that solving the problem for uncertain system can be replaced by solving mixed \( H_2/H_\infty \)-problem. The solution of the last problem is reduced to solving four cross-coupled algebraic Riccati equations, Lyapunov equation and one equation of a special type.

**REFERENCES**


