A ROBUST INVARIANCE APPROACH TO
IDLE SPEED CONTROL OF A SPARK
IGNITED ENGINE

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Abstract: Idle speed control of a spark ignition engine is studied via a control
invariant approach. A hybrid system describes torque generation at cycle–level.
The continuous–time dynamics is reset at prescribed crankshaft angles. Time–
discretization allows to handle delays in the control loop and constraints on state
and control variables in a simple form by means of a control invariant technique.
The synchronization problem is solved by a robust extension of the same technique.
Controller implements a piecewise–linear state–feedback law. Copyright© 2005
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Invariance

1. INTRODUCTION

Three main features are shared by idle speed control (ISC) of spark ignition engines: nonlinearities,
delays in the control loop and synchronization, model and disturbance uncertainty. Control ob-
jectives typically include good tracking properties, disturbance rejection, robustness. In idle speed
mode motor torque is controlled essentially by two variables, the throttle valve opening angle regu-
lying the amount of air intake into the cylinders, and the spark ignition advance. Given the narrow
range in which variables must be kept, linearized models in ISC are generally accepted and widely
used in practice [Hrovat et al., 1997, Yurkovi et al., 1997]. The open-loop behaviour can be described
in the time–domain or in the crankshaft–angle domain [Chin et al., 1986, Yurkovi et al., 1997].

Mathematically, these are equivalent descriptions, the two domains being related by
\[
\frac{dx}{dt} = \frac{dx}{d\theta_c} \frac{d\theta_c}{dt} = K_c \frac{dx}{d\theta_c}
\]
with \(\theta_c\) crankshaft angle and \(n\) angular velocity. Since crankshaft angular velocity is a natural
state variable, any model variable \(x\) undergoes a non-linear transformation going from the \(t\) to
the \(\theta_c\) domain. Newton’s law for instance, \(\frac{dn}{dt} = T\), takes on the non–linear form \(\frac{dn}{dt} = \frac{1}{nk_c}T\)
in \(\theta_c\) domain. The advantage of the \(t\)-domain is that of retaining a linear description. This is
a remarkable advantage since control techniques handling hard constraints, disturbance rejection
and model uncertainty - control invariant techniques - are much more developed in the linear
case. In the linear discrete–time case, invariant set computations are further simplified and, as linear
processing of signals can be done at quite fast a rate by modern digital technology, it becomes
possible to keep track of variables at engine–cycle level. However, control signal processing requires

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sampling at a fixed rate and a time-discretization somewhere in the plant–controller loop is necessary. This introduces synchronization difficulties since the motor torque is physically exerted only in a neighborhood of the engine top dead center angle. These difficulties in principle disappear in a neighborhood of the engine top dead center angle, and this is a measurable event, however, if speed effects are negligible in idle mode and a linear relationship for \( \dot{p} \) is justified. The second equation describes crankshaft dynamics in terms of angular velocity \( n \). The term \( a_n n \) accounts for dynamic friction of rotational mechanisms in the driveline. The term \( a_n p \) is pumping friction due to air-fuel mixture transport, opposing crankshaft rotation. The external load torque \( T_l \) is regarded as a disturbance. The variable \( T \) is a proxy of the instantaneous motor torque; in fact the real torque contributed by a single cylinder as a function of \( \theta_c \) (\( \theta_c = K_c n \)) exhibits a sharp positive peak immediately after the spark, and small negative values elsewhere as shown in Figure (1). \( T \) is an equivalent\(^3\) torque assumed positive and constant throughout the expansion stroke, and zero elsewhere.

\[
\begin{align*}
\dot{p} &= a_p p + b_p \alpha \\
n &= a_n n + a_n p \dot{p} + b_n (T - T_l).
\end{align*}
\]

The first equation describes pressure dynamics in the intake manifold in terms of the throttle opening angle \( \alpha \). Pressure dynamics depends nonlinearly on crankshaft speed. However, speed effects are negligible in idle mode and a linear relationship for \( \dot{p} \) is justified. The second equation describes crankshaft dynamics in terms of angular velocity \( n \). The term \( a_n n \) accounts for dynamic friction of rotational mechanisms in the driveline. The term \( a_n p \) is pumping friction due to air-fuel mixture transport, opposing crankshaft rotation. The external load torque \( T_l \) is regarded as a disturbance. The variable \( T \) is a proxy of the instantaneous motor torque; in fact the real torque contributed by a single cylinder as a function of \( \theta_c \) (\( \theta_c = K_c n \)) exhibits a sharp positive peak immediately after the spark, and small negative values elsewhere as shown in Figure (1). \( T \) is an equivalent\(^3\) torque assumed positive and constant throughout the expansion stroke, and zero everywhere.

Fig. 1. Torque generation for a single cylinder

In a multi-cylinder engine there is torque overlap. In a 4-cylinder 4-stroke engine\(^4\) the overall torque profile is shown in Figure (2). In this case cylinders change stroke at the same time (every 180 degrees of the crankshaft angle) and there is only one cylinder active in each stroke. We say that the en-

2. IDLE SPEED ENGINE MODEL

The hybrid engine model considered in this paper comprehends a continuous time component and an event-driven reset function. The continuous-time component is

\[\dot{p} = a_p p + b_p \alpha\]
\[\dot{n} = a_n n + a_n p \dot{p} + b_n (T - T_l).\]

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3 in terms of an energy balance.
4 in what follows we refer to this case but, the situation is similar in other engine configurations
cylinder is at a top dead center (TDC) when one of the cylinders is at a compression TDC. Combining the equivalent torques of each cylinder, the resulting profile will be a piecewise constant function with breakpoints 180 degrees apart.

Torque is a function of the manifold pressure $p^-$ and the spark advance $\theta_s^-$ decided at the last engine TDC. It can be assumed

$$ T = c_1 p^- + c_2 \theta_s^- + c_3, $$

where $p^-$, $\theta_s^-$ are values of air pressure and spark advance at the end of intake stroke.

The range in which the linear model is meaningful is described by upper and lower bounds on each of the variables $p, T, \alpha, \theta_s$ and $T_l$. The control objective is to keep engine speed in the range

$$ n_{\text{min}} \leq n \leq n_{\text{max}} $$

for all possible values of the load torque $T_l$ in its range. Assuming engine to be at a TDC at time $t = 0$, the dynamics for $0 \leq t < t_r$, where $t_r$ is the time of the next TDC when torque is reset, is

$$ \dot{x} = A_c x + B_c \alpha + C_c T_l $$

where $x = \{p \ n \ T \}^T$. At $t = t_r$ torque is reset on the basis of prior values $p^-$, $\theta_s^-$. We assume the following

**A1** The angular position of the crankshaft is known at engine TDCs

Assumption 1 captures the fact that the attainment of a TDC is a measurable event that can be profitably exploited in the controller synthesis.

The actual control algorithm is implemented in the electronic control unit (ECU) which is a digital system. This feature suggests to handle delays and controller design via a numerical control approach.

3. DISCRETIZATION AND N-STEP DYNAMICS

Consider the discrete-time version of the model. The time between two engine TDCs is subdivided into $N-1$ equal intervals of length $\delta t$ and an $N$–th interval of length $\leq \delta t$.

![Engine TDC at t = 0 and at t_r, between (N - 1)\delta t and N\delta t](image)

The discrete–time evolution within one engine cycle is

$$ x(k + 1) = A_x x(k) + B_x \alpha(k) + C_x T_l(k) $$

if $k = 0, 1 \ldots N - 2$,

$$ x^-(N) = A_r x(N - 1) + B_r \alpha(N - 1) + C_r T_l(N - 1) $$

before reset, and

$$ x(N) = \overline{A}_1 x(0) + \overline{B}_\alpha(0) + \overline{C}_c x^-(N) $$

after reset, where subscript $d$ denotes discretization with a fixed sample time $\delta t$, subscript $\tau$ discretization with a variable sample time $0 \leq \tau \leq \delta t$ and $\overline{A}_{1,2}, \overline{B}, \overline{C}$ are reset matrices. In the third equation notice the delayed effect of pressure and spark advance, as per eq. (2).

4. N-STEP BOX INVARIANCE

We are interested in the $N$–step dynamics, e.g. the evolution of state control and disturbance variables every $N$ time–steps. Thus labelling

$$ t_k \quad \text{the time-instant } k\delta t $$

$$ t_k + 1 \quad \text{the time-instant } (k + N)\delta t $$

$$ t_k + 2 \quad \text{the time-instant } (k + 2N)\delta t $$

the discrete–time description of the $N$–step dynamics is

$$ x(t_k + 1) = Ax(t_k) + Bu(t_k) + Cv(t_k) $$

where $t_k \epsilon I$, the set of non-negative integers, $x \epsilon \mathbb{R}^3$ and $u, v \epsilon \mathbb{R}^{N+1}$ with

$$ u(t_k) = [\theta_s(k) \ \alpha(k) \ldots \alpha(k + N - 1)]' $$

$$ v(t_k) = [c_3 \ T_l(k) \ldots T_l(k + N - 1)]' $$

It is easily checked that matrix $B$ is full rank. Notice if a feedback law $u = F(x)$ is found, this means that the state $x(t_k)$ at time $k\delta t$ determines the control $\theta_s(k\delta t)$, plus the $N$ controls $\alpha(k\delta t) \ldots \alpha((k + N - 1)\delta t)$. The set of all these controls determines the state at time $(k + N)\delta t$ which is $x(t_k + 1)$. In other words, the system evolves closed loop for $t$ in the set $\{ (k + mN)\delta t, \ m = 0, 1, \ldots \}$ and open loop for $t$ not in this set. State constraints can be represented as a box

$$ X = \{ x : \underline{x} \leq x \leq \bar{x} \} $$

where $\underline{x}_1 = p_{\text{min}}, \ \bar{x}_1 = p_{\text{max}}, \ \underline{x}_2 = \theta_s_{\text{min}}, \ \bar{x}_2 = \theta_s_{\text{max}}, \ \underline{x}_3 = 0 \ \text{and} \ \bar{x}_3 = T_{\text{max}}$.

Disturbance constraints are represented by a box
where \( \mathcal{R}(X) = \{ x : \exists \sigma \in Q : x - w - Ax \leq \sigma \leq x - w - \bar{A}x \} \) with \( \bar{w}, \bar{w} \) as defined in the statement. Notice that \( \mathcal{R}(X) \neq \emptyset \quad \Rightarrow \quad \bar{w} - w \leq \bar{x} - \bar{x} \) (21)
which is the last row of the matrix inequality in \( i \). Assuming this holds
\[
\mathcal{R}(X) = \{ x : \exists \sigma \in Q : x - w - Ax \leq \sigma \leq x - w - \bar{A}x \} \leq \mathcal{R}(X)
\]
Next, we impose invariance
\[
X \subset \mathcal{R}(X) \iff \max_{i} A_i x \leq [\bar{x} - \bar{w} - \bar{a}]
\]
or
\[
\max_{i} -A_i x \leq [-\bar{x} + \bar{w} + \bar{a}]
\]
Inequalities (25,26) are the first two rows of the matrix inequality in \( i \). Under condition \( i \), there exists a feasible \( \sigma \) satisfying the inequality in (22), which proves \( ii \).

We remark that if the origin is contained in the interior of \( X \) and \( V \), then inclusion (13) in absence of noise implies contractivity as well as invariance of \( X \), e.g. there exists \( \lambda \in (0,1) \) such that under a feasible control \( X \) maps into \( \lambda X \) at each step. This ensures asymptotic stabilizability of the equilibrium point. Notice also that from the inequalities in \( ii \), the choice of \( \sigma \) given \( x \) is not unique. For instance, a possible control law is the midpoint control law, where \( \sigma \) is chosen as the arithmetic mean of its bounds.

Theorem 2. i. Set \( X \) is invariant for (7) under (10,15) if and only if

\[
\begin{bmatrix}
A^+ - I & A^- \\
-A^- & A^+ + I
\end{bmatrix}
\begin{bmatrix}
x \\
\bar{w}
\end{bmatrix}
\leq
\begin{bmatrix}
-\bar{w} - \bar{a} \\
\bar{w} + \bar{a}
\end{bmatrix}
\]
where
\[
\bar{w} = \min_{\bar{w} \leq \bar{v} \leq \bar{w}} C_i v = C_i^+ \bar{v} + C_i^- \bar{v} \\
\bar{w} = \max_{\bar{w} \leq \bar{v} \leq \bar{w}} C_i v = C_i^+ \bar{v} + C_i^- \bar{v}
\]
ii. Under this condition invariance is achieved by controls satisfying
\[
\max(\bar{w}, \bar{w} - \bar{A}x) \leq \sigma \leq \min(\bar{w}, \bar{x} - \bar{w} - \bar{A}x)
\]
with \( \min, \max \) taken componentwise.

Proof We first calculate \( \mathcal{R}(X) \) for system (16)
\[
\mathcal{R}(X) = \{ x : \exists \sigma \in Q : x - w - Ax \leq \sigma \leq x - w - \bar{A}x \} \leq \mathcal{R}(X)
\]
with \( \bar{w}, \bar{w} \) as defined in the statement. Notice that
\[
\mathcal{R}(X) \neq \emptyset \quad \Rightarrow \quad \bar{w} - w \leq \bar{x} - \bar{x}
\]
ii. Under this condition invariance is achieved by controls satisfying
\[
\max(\sigma, \bar{\lambda} - \bar{w} - \bar{A}x) \leq Bu \\
\leq \min(\sigma, \bar{\lambda} - \bar{w} - \bar{A}x) \quad (27)
\]

**Proof** Directly from Thm 1.

Notice that the bounds for \(Bu\) are piecewise linear-affine functions of \(x\). This makes the control law simple to implement. Notice also that these bounds do not determine \(u\) uniquely. Such a degree of freedom can be usefully exploited to limit, e.g., minimize, departure of the state from \(X\) in the time intervals comprised between two consecutive TDCs. Before undertaking this minimization, we establish the following robustness result.

**Theorem 3.** Given (7,9,10)

i. If the conditions of Thm 1 are satisfied for \(A^i, B^i, C^i\) with \(\bar{w}^i, \bar{\lambda}^i, \bar{\sigma}^i\) \(i = 1 \ldots p\) then \(X\) is invariant for any system with \(A, B, C \in \text{Conv}(A^1, B^1, C^1 \ldots A^p, B^p, C^p)\).

ii. The control law satisfying inequalities (27) with
\[
\bar{w} = \min\bar{w}^i, \quad \bar{\lambda} = \max\bar{\lambda}^i \quad (28)
\]
and \(A, B\) replaced by \(A^i, B^i, i = 1 \ldots p\) makes \(X\) invariant for any of the above systems.

**Proof** Let \(\mathcal{R}_\alpha(X)\) be the reach set of \(X\) under system \((A^i, B^i, C^i)\) and \(\mathcal{R}_\alpha(X)\) the same under \(\sum\alpha_i(A^i, B^i, C^i)\) (with \(\alpha\) in the unit simplex of \(\mathbb{R}^p\)). If conditions of Thm 1 hold, \(\cap_i \mathcal{R}_\alpha(X)\) is non-void and satisfies \(X \subset \cap_i \mathcal{R}_\alpha(X)\). Due to convexity, \(\cap_i \mathcal{R}_\alpha(X) \subset \mathcal{R}_\alpha(X)\) \(\forall\alpha\) and we conclude \(X \subset \mathcal{R}_\alpha(X)\) \(\forall\alpha\), which proves i.

Part ii. follows from the fact that if \(X\) is invariant under any system with \(A, B, C \in \text{Conv}(A^1, B^1, C^1 \ldots A^p, B^p, C^p)\) with constraints \(U, V\), it must keep the property with constraints \(U' \supset U, V' \subset V\). Now if \(U_i, V_i\) are the boxes defined by \(\bar{w}^i, \bar{\lambda}^i, \bar{\sigma}^i\) then \(\bar{\sigma}, \bar{\lambda}\) define the minimal box containing all \(U_i\)’s and \(\bar{w}, \bar{\lambda}\) the minimal box containing all \(V_i\)’s.

This result permits to handle uncertainty over \(\tau\); in fact, the matrices \(A, B\) and \(C\) are dependent on the uncertain parameter \(\tau\) (see Appendix 7.1) and the \(p, A^i, B^i\) and \(C^i\) with \(i = 1 \ldots p\) in Thm 3 are chosen such that \((A, B, C) \in \text{Conv}(A^1, B^1, C^1)\).

5. **INTRACYCLE DYNAMICS: OPTIMAL CHOICE OF U**

If \(X\) is \(N\)-step invariant trajectories originating in \(X\) return to \(X\) at most every \(N\) time steps.

As \(X\) need not be invariant (e.g. 1-step invariant) it is of interest to estimate the smallest box that contains all trajectories originating in \(X\). We term intracycle the dynamics occurring between two TDCs. Let \(x_c \in X\). Since trajectories have finite amplitude over finite time, there exists a \(\lambda < \infty\) such that if \(x(0) \in X\) (see fig 4)
\[
x(m) \in \lambda(X - x_c) + x_c \quad \forall m \in (0, N) \quad (29)
\]

![Fig. 4. Departure from \(N\)-step invariant box](image)

with \(0 < m < N\), \(u^m = (u(0) \ldots u(m))^T\) and likewise for \(v^m\) (intracycle matrices \(A_m, B_m, C_m\) defined in the Appendix). As this must hold for all \(v \in V\), we re-write the above as
\[
x_c - \bar{w}^m + \lambda(x - x_c) \leq A_m x + B_m u^m + C_m v^m \leq x_c + \lambda(x - x_c),
\]
\[
\frac{w^m}{\bar{w}^m} = \frac{C_m + v^m}{C_m + v^m} \leq \frac{w}{\bar{w}}, \quad \frac{\bar{w}^m}{\bar{w}^m} = \frac{C_m + \bar{v}^m}{C_m + \bar{v}^m}.
\]

Finally, we must ensure \(N\)-step invariance of \(X\)
\[
x - \bar{w} \leq A x + B u \leq \bar{x} - \bar{w} \quad (31)
\]
and satisfaction of control constraints
\[
u \leq u \leq \bar{u}. \quad (32)
\]

For given \(x = x(0)\) we can compute \(\hat{\lambda}(x) = \min_{u} \lambda\) subject to (30,31,32) – a LP problem. The solution to this problem yields a control law of the form
\[
x(0) \mapsto u(0) \ldots u(N - 1)
\]

that guarantees \(N\)-step invariance of \(X\), and minimal departure from \(X\) of system trajectories originating in \(X\). An a-priori estimate of the maximal departure from \(X\) can be obtained by noting that \(\lambda - 1\) is the Minkowski functional of \(X\), a well known convex function yielding the bound.
\[ \lambda \leq \max_k \hat{\lambda}(\xi_k) \] where \( \xi_k \) is the \( k \)–th vertex of \( X \). We conclude that \( X \) is invariant for (4-6) if and only if \( \max_k \hat{\lambda}(\xi_k) = 1 \); otherwise, \( X \) is safe wrt \((X - x_c)\lambda + x_c \) for \( \lambda = \max_k \hat{\lambda}(\xi_k) \).

6. SIMULATED RESULTS AND CONCLUSION

The graphs below illustrate the performance of the \( N \)-step controller on a simulated model, in correspondence of highly random changes in the load torque.

Fig. 5. Simulated result: state variables

Fig. 6. Simulated result: control and disturbance variables

In conclusion, we presented an innovative technique for ISC of a spark ignition engine.

With respect to typical control tools (PID, LQ, \( H^\infty \), \( l^1 \), etc) used in literature, see [Carnevale et al., 1993, Jayasuriya et al., 1994, Butts et al., 1982, Williams et al., 1989], our approach has the advantage of handling hard constraints on state and control variables by means of invariant techniques, see [Berardi et al., 2001]. The use of invariants of prescribed shape (boxes) while appealing in engineering practice, permits drastic simplification in the computation.

7. APPENDIX

7.1 \( N \)-step dynamics

Iterating (4) for \( N - 1 \) steps and grouping control and disturbance terms as in (8) we get (7) with

\[ A = \overline{A_1} + \overline{A_2} A_r A_d^{N-1} \] (33)
\[ B = [B; \overline{A_2} A_r A_d^{N-2} B_d; \ldots; \overline{A_2} A_r B_d; \overline{A_2} B_r] \] (34)
\[ C = [C; \overline{A_2} A_r A_d^{N-2} C_d; \ldots; \overline{A_2} A_r C_d; \overline{A_2} C_r] \] (35)

7.2 Intracycle dynamics

\( A_m, B_m, C_m \) are defined as in (33–35) with \( N \) replaced by \( m \).

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