ROBUST DESIGN OF SMITH PREDICTIVE CONTROLLER FOR MOMENT MODEL SET

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Abstract: The paper presents a simple and systematic procedure for automatic tuning of dead time compensating controllers (DTC). It integrates a simple identification experiment providing process characteristic numbers and a robust design method for an exactly defined model family. This family contains all transfer functions having (a) the given a priori form of lag/dead time transfer function (b) the experimentally obtained moment characteristic numbers. The independently interesting result of this paper is the explicit description of the value set for the model family. Copyright © 2005 IFAC

Keywords: Control system design, Control oriented models, Uncertain linear systems, Moment method, Process control, Delay compensation, Robustness, Regions.

1. INTRODUCTION

Dead times between inputs and outputs are common phenomena in many industrial processes and cause considerable difficulties in effective control of them. For more details see for example (Richard, 2003). Smith (1958) suggested a dead-time compensation scheme called Dead Time Compensator (DTC) or Smith predictor. The Smith predictor whose parameters are tuned using common frequency based design criteria gives an excellent performance when an exact process model is available but yields robustness troubles when even small mismatches occur because of the more complicated Nyquist curve shape at high frequencies (Palmor, 1980). Nevertheless, a properly tuned DTC can outperform conventional controllers while achieving the same robustness. It is shown in many simulations and experimental studies (Lee et al., 1996; Åström and Hägglund, 2001; Mataušek and Kvaščev, 2003) but the range of their applicability is usually not well defined.

This paper presents a simple and systematic DTC tuning procedure which never fails under given assumptions. Furthermore, the procedure can be easily automated. The key concepts introduced are process characteristic numbers (the first three moments of the process transfer function) and the exactly defined model set containing all transfer functions that are consistent with the a priori form of the lag/dead time transfer function and with the experimentally obtained characteristic numbers. Using these concepts an exact robust design problem can be formulated and solved.

The paper is organized as follows: In Section 2 the process characteristic numbers, their properties and a simple identification experiment are introduced. In Section 3 the model set is defined and several associated concepts are presented. The parametrization of all ultimate transfer functions is given in Section 4 while Section 5 describes basic concepts of the robustness regions method for DTC design. Section 6 introduces the DTC robust design problem involving

1 This work was partially supported by the Ministry of Education of the Czech Republic – Project No. MSM 2352 00004 and Grant Agency of the Czech Republic – Project No. 102/02/0425.
the robustness region method and the ultimate transfer function parametrization as its basic concepts. Finally, an example can be found in Section 7 and conclusions are summarized in Section 8.

2. PROCESS CHARACTERISTIC NUMBERS

The process transfer function \( P(s) \) can be characterized by its moment sequence

\[
m_i = \int_0^\infty t^i h(t) dt, \quad i = 1, 2, \ldots,
\]

where \( h(t) \) is the corresponding process impulse response. The first few moments describe the low frequency properties of the process well because of the fact that the first elements of the Taylor series

\[
F(s) = f_0 + f_1 s + f_2 s^2 + \ldots
\]

are determined by

\[
f_i = \frac{1}{i!} P^{(i)}(0) = (-1)^i \frac{1}{i!} m_i.
\]

For processes with the monotonous step response it turns out that the only first three moments may be sufficient for a rough low frequency process model. Further, the numbers \( m_0, m_1, m_2 \) can be converted into another triplet of characteristic numbers

\[
\kappa = m_0, \quad \mu = \frac{m_1}{m_0}, \quad \sigma^2 = \frac{m_2}{m_0} - \frac{m_1^2}{m_0^2}
\]

with the following meanings: \( \kappa \) is the static gain of the process, \( \mu \) and \( \sigma \) are the mean and variance of the ‘density function’ \( h(t)/\kappa \), respectively. In our context, \( \mu \) is usually called the resident time (Åström and Hägglund, 1995) and \( \sigma^2 \) is some measure for the length of the process response. It is illustrated by the following three examples.

Example 1. The characteristic numbers of the first order system \( 1/(\tau s + 1) \) are \( \kappa = 1, \mu = \tau, \sigma^2 = \tau^2 \).

Example 2. The characteristic numbers of the pure dead time \( e^{-Ds} \) are \( \kappa = 1, \mu = D, \sigma^2 = 0 \).

Example 3. The characteristic numbers of the zero-order hold (ZOH) system

\[
F_{\text{ZOH}}(s) = \frac{1}{s} (1 - e^{-Ls})
\]

are \( \kappa = L, \mu = L/2, \sigma^2 = L^2/12 \). Note that the ZOH system (5) converts the input Dirac pulse into the unit rectangle pulse with the length \( L \).

It is easy to prove the following lemma.

### Lemma 1

Let the transfer function \( P_i(s) \) has characteristic numbers \( \kappa_i, \mu_i, \sigma_i^2 \), \( i = 1, 2, \ldots, m \), given according to (1) and (4), then for the characteristic numbers \( \kappa, \mu, \sigma^2 \) of the transfer function

\[
P(s) = P_1(s) P_2(s) \cdots P_m(s)
\]

it holds

\[
\kappa = \kappa_1 \kappa_2 \cdots \kappa_m, \quad \mu = \mu_1 + \mu_2 + \ldots + \mu_m, \quad \sigma^2 = \sigma_1^2 + \sigma_2^2 + \ldots + \sigma_m^2.
\]

From Lemma 1 and Examples 1 and 2 it emerges that the transfer function of the monotonous process

\[
P(s) = \frac{K_p e^{-Ds}}{(\tau_1 s + 1) \cdots (\tau_n s + 1)}
\]

has the following characteristic numbers

\[
\kappa = K_p, \quad \mu = D + \tau_1 + \tau_2 + \ldots + \tau_n, \quad \sigma^2 = \tau_1^2 + \tau_2^2 + \ldots + \tau_n^2.
\]

Now, the way how the characteristic numbers \( \kappa, \mu, \sigma^2 \) can be obtained from a real identification experiment will be described. For this purpose, consider the hypothetical series connection \( H(s) = F_{\text{ZOH}}(s) P(s) \), depicted in Fig. 1, where \( F_{\text{ZOH}} \) is given by (5) and \( P(s) \) is a process transfer function.

The impulse response \( h(t) \) of this series connection is clearly identical with the response of the process \( P(s) \) to the rectangle pulse

\[
u(t) = \begin{cases} 1, & \text{for } t \in [0, L] \\ 0, & \text{elsewhere} \end{cases}
\]

as it follows from Fig. 1. Thus, the characteristic numbers \( \kappa_H, \mu_H, \sigma_H^2 \) of the transfer function \( H(s) \) can be computed from the response of the process \( P(s) \) to the rectangle pulse (10) according to (1) and (4). Now, Lemma 1 and Example 3 give the following expressions for the process characteristic numbers

\[
\kappa = \frac{\kappa_H L}{L}, \quad \mu = \mu_H - \frac{L}{2}, \quad \sigma^2 = \sigma_H^2 - \frac{L^2}{12}.
\]

3. MODEL SET

In this section the model set of all lag/dead time transfer functions with the order \( n \) and the given characteristic numbers \( \kappa, \mu \) and \( \sigma^2 \) is defined.
**Definition 1.** (Model Set). Let a fixed $n$ and the characteristic numbers $\kappa, \mu, \sigma^2$ be given. A process transfer function $P(s)$ is called unfalsified (or an element of the model set $S^n(\kappa, \mu, \sigma^2)$) if it is consistent with the two following conditions:

(i) (A priori Hypothesis)

\[ P(s) = \frac{K_p}{(\tau_1 s + 1) \cdots (\tau_n s + 1)}, \quad (12) \]

where $K_p > 0, \tau_i \geq 0, i = 1, 2, \ldots, n$.

(ii) (Interpolation Conditions) The transfer functions $P(s)$ has characteristic numbers $\kappa, \mu, \sigma^2$.

**Remark 1.** The condition (i) of Definition 1 expresses the fact that the whole set of all real poles stable systems of the order at most $n$ is a priori admissible. It means that all systems (8) with the pure dead time are included for the case $n \to \infty$.

**Lemma 2.** The model set $S^n(\kappa, \mu, \sigma^2)$ is not empty iff

\[ \frac{1}{n} \leq \frac{\sigma^2}{\mu^2} \leq 1. \quad (13) \]

Moreover, there exist infinitely many members of the model set $S^n(\kappa, \mu, \sigma^2)$ if the both strict inequalities hold in (13).

The proof is given in (Večerek, 2004).

4. PARAMETRIZATION OF ALL ULTIMATE TRANSFER FUNCTIONS

**Definition 2.** (Value Set) Let $\omega$ is a given frequency, then the set

\[ F^n(\kappa, \mu, \sigma^2; \omega) = \{ P(j\omega) : P(s) \in S^n(\kappa, \mu, \sigma^2) \} \]

called is the value set of the model set $S^n(\kappa, \mu, \sigma^2)$ at frequency $\omega$. The symbol $\partial F^n(\kappa, \mu, \sigma^2; \omega)$ denotes the boundary of the value set $F^n(\kappa, \mu, \sigma^2; \omega)$.

**Definition 3.** (Ultimate Transfer Function) An unfalsified transfer function $P(s)$ in $S^n(\kappa, \mu, \sigma^2)$ is said to be ultimate if there exist at least one frequency $\omega > 0$, such that

\[ P(j\omega) \in \partial F^n(\kappa, \mu, \sigma^2; \omega). \quad (14) \]

Without loss of generality the normalized case of $\kappa = 1$ and $\mu = 1$ (obtained by gain and time normalization) can be considered. Note, that the model set $S^n(1, 1, \sigma^2)$ contains more than one element iff

\[ \frac{1}{n} < \sigma^2 < 1 \quad (15) \]

as it follows from Lemma 2.

**Theorem 1.** Let (15) holds and $k$ is maximal integer less than $\frac{1}{n} + 1$, then the unfalsified transfer function $P(s)$ is ultimate iff it can be expressed in the form

\[ P^\nu_\omega(s) = \frac{1}{(\tau_\nu(s) + 1)^{\nu_\omega}} \quad (16) \]

where $\nu = (n_1, n_2, n_3)$ is a multiindex ranging over the list which depends on $k$:

(i) If $k = 2$ then the respective list is the following:

\[ (1, 1, 1), (1, 2, 1), \ldots, (1, n - 2, 1), \quad (a) \]

\[ (n - 2, 1, 1). \quad (b) \]

(ii) If $k \in \{3, \ldots, n - 1\}$ then the respective list is:

\[ (1, k - 1, 1), (1, k, 1), \ldots, (1, n - 2, 1), \quad (a) \]

\[ (n - 2, 1, 1), (n - 3, 1, 2), \ldots \]

\[ , (n - k + 1, 1, k - 2), \quad (b) \]

\[ (n - k, 1 - k - 1), \quad (c) \]

\[ (1, k - 2, 1). \quad (d) \]

(iii) If $k = n$ then the respective list is the following:

\[ (n - 2, 1, 1), (n - 3, 1, 2), \ldots \]

\[ , (1, 1, n - 2), \quad (b) \]

\[ (1, n - 2, 1). \quad (d) \]

Moreover, the parameters $\tau_\nu(\alpha), \vartheta_\nu(\alpha)$ and $\zeta_\nu(\alpha)$ are given by

\[ \tau_\nu(\alpha) = \alpha, \quad \vartheta_\nu(\alpha) = \frac{1 - n_1 \alpha}{n_2 + n_3} - \frac{\sqrt{n_3}}{\sqrt{n_2}} \cdot \frac{\sqrt{\sigma^2(n_2 + n_3) - (1 - n_1 \alpha)^2 - n_1(n_2 + n_3)\alpha^2}}{\sqrt{n_2(n_2 + n_3)}}, \]

\[ \zeta_\nu(\alpha) = \frac{1 - n_1 \alpha}{n_2 + n_3} + \frac{\sqrt{n_3}}{\sqrt{n_2}} \cdot \frac{\sqrt{\sigma^2(n_2 + n_3) - (1 - n_1 \alpha)^2 - n_1(n_2 + n_3)\alpha^2}}{\sqrt{n_2(n_2 + n_3)}}, \]

where $\alpha$ ranges over the interval $I_\nu = [a_\nu, b_\nu]$. The expression for the end point $b_\nu$ is

\[ b_\nu = \frac{1}{n_1 + n_2 + n_3} - \frac{\sqrt{n_3} \sqrt{\sigma^2(n_1 + n_2 + n_3)}}{\sqrt{n_1 + n_2(n_1 + n_2 + n_3)}} - \frac{1}{\sqrt{n_1(n_1 + n_2 + n_3)}}, \]

and the expression for the end point $a_\nu$ depends on the type of a row to which $\nu$ belongs: If $\nu$ is in the row (a) or (c) then $a_\nu = 0$, if $\nu$ is in the row (b) or (d) then

\[ a_\nu = \frac{1}{n_1 + n_2 + n_3} - \frac{\sqrt{n_3} \sqrt{\sigma^2(n_1 + n_2 + n_3)}}{\sqrt{n_1 + n_2(n_1 + n_2 + n_3)}} - \frac{1}{\sqrt{n_1(n_1 + n_2 + n_3)}}, \]

The proof is given in (Schlegel, 2000).

Now, some nearly evident consequences of Theorem 1 are briefly stated. Let $\nu = (n_1, n_2, n_3)$ belongs to the list of multindexes from Theorem 1, then the value set of the set $\{ P^\nu_\omega(j\omega) : \alpha \in I_\nu \}$ for fixed frequency $\omega$ is clearly a smooth curve called $\nu$-arc. For each point of this $\nu$-arc there exists just one corresponding ultimate transfer function in the form (16) and vice versa. The endpoints of the $\nu$-arcs correspond with the ultimate transfer functions in the form
For simplicity, ultimate transfer functions in the form
$k \frac{1}{\sigma^2}$
will be called extreme.

Furthermore, it follows from Theorem 1 that the value
$k \frac{1}{\sigma^2}$
in the following and respective open loop system in the complex plain.

Almost arbitrary shaping of the Nyquist curve can be performed by involving more such points to the design procedure covering all usual frequency-domain design specifications (gain and phase margins, constraints on sensitivity functions peak values…).

Denote for simplicity
$P(j\omega) \triangleq a + jb, \quad Z(j\omega) \triangleq q + jr, \quad d \triangleq \frac{k_i}{k}$
Then, for open loop transfer function (Nyquist curve) $L(j\omega)$ of the system from Fig. 3 it must hold
corresponding closed loop systems are stable and all the optimal point which minimizes the disturbance dened by (22) divides the parametric plane. Now, from the 'satisfactory' region where all the corresponding Nyquist curves are properly shaped, points and then a 'satisfactory' region can be found in the complex plane. It is evident that for all points on the regions boundaries the corresponding Nyquist curve \( L(j\omega) \) must pass through the point \( c \) in the complex plane. In other words it must hold

\[
L(j\omega) = c \triangleq u + jv
\]

at some frequency \( \omega \) where \( L(j\omega) \) is given by (20). The equation (21) has a unique solution for unknown controller parameters \( k \) and \( \delta \):

\[
k = \frac{\Upsilon}{\Psi}, \quad d = \frac{\Xi}{\Upsilon}, \quad (22)
\]

where

\[
\Upsilon = au + qv^2 + qu^2 + bu,
\]

\[
\Psi = a^2 + 2aau - 2aev + q^2v^2 + q^2u^2 + 2bvq + 2bru + b^2 + v^2r^2 + u^2r^2,
\]

\[
\Xi = (ru^2 + bu - av + vr^2)\omega.
\]

The parametric curve \((k(\omega), k_i(\omega) = (k(\omega))d(\omega))\) defined by (22) divides the parametric plane \( k - k_i \) into several regions. All points of the given region fulfill the property that the point \( c \) has the same index to all corresponding Nyquist plots \( L(j\omega) \). If such regions are plotted for several different points \( c \) a region can be isolated with \( L(j\omega) \) properly shaped. This procedure can be performed for finite number of processes and points and then a 'satisfactory' region can be found where Nyquist plots \( L(j\omega) \) of all systems are properly shaped.

Now, from the 'satisfactory' region where all the corresponding closed loop systems are stable and all the corresponding Nyquist curves are properly shaped, the optimal point which minimizes the disturbance rejection performance index

\[
J = \frac{1}{k_i}
\]

is chosen according to (Åström et al., 1998).

6. ROBUST DTC DESIGN FOR THE MODEL SET

This section describes the design procedure of DTC for the model set \( S(n, 1, \sigma^2) \). Let \( n \), the normalized \( \sigma \), \( 0 < \sigma < 1 \), and several frequency design specifications in the form of points \( c \) are given. The objective is to design a fixed DTC which fulfills the given specifications for all unfalsified transfer functions from the model set \( S(n, 1, \sigma^2) \). It is easy to prove that instead of \( S(n, 1, \sigma^2) \) it is sufficient to consider only its small subset containing all the ultimate transfer functions. In this way, much more easier and equivalent robust design problem is obtained because of Theorem 1. Since the set of all the ultimate transfer functions is infinite, it is necessary to use some its finite approximation for the computation e.g. set of all the extreme transfer functions (17) associated with the model set \( S(n, 1, \sigma^2) \). Though such approximation does not lead generally to the exact solution of the above problem, it turns out that the controller obtained in this way is at least very close to the exact solution.

7. EXAMPLE

In this example the model set \( S(n, 1, \sigma^2) \), \( n = 100 \), \( \sigma \in [0, 0.8] \) is considered. The frequency domain robust specifications are following:

(a) For the maximum of sensitivity function

\[
M_s \triangleq \max_{\omega \in [0, +\infty]} \frac{1}{1 + L(j\omega)}
\]

it holds \( M_s \leq 1.7 \)

(b) The open loop Nyquist plot \( L(j\omega) \), \( \omega > 0 \), does not intersect the real axis in the interval \([0.6, +\infty)\).

Both of these specifications can be (approximatively) transformed to the form of four points \( c_i \) in the complex plane: \( c_1 = -0.41, c_2 = -0.43 - 0.15j, c_3 = -0.49 - 0.29j, c_4 = 0.6 \) as can be seen in Fig. 6. Solutions obtained for all extreme transfer functions of the model set for different values \( \sigma \) are depicted in Fig. 4 and approximated by

\[
f(\sigma) = a_0e^{a_1\sigma} + a_2\sigma^2 + a_3\sigma^3 + a_4\sigma^4,
\]

where corresponding coefficients \( a_0, a_1, \ldots, a_4 \) are given in Tab. 1.

Table 1. Coefficients of the DTC parameters

<table>
<thead>
<tr>
<th>approximation (24)</th>
<th>( a_0 )</th>
<th>( a_1 )</th>
<th>( a_2 )</th>
<th>( a_3 )</th>
<th>( a_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k )</td>
<td>0.213</td>
<td>-0.947</td>
<td>34.2</td>
<td>-65.0</td>
<td>37.7</td>
</tr>
<tr>
<td>( k_i )</td>
<td>2.22</td>
<td>-0.179</td>
<td>8.09</td>
<td>-16.2</td>
<td>13.1</td>
</tr>
</tbody>
</table>

Fig. 4. Gains \( k \) and \( k_i \) of the controller (19).

Figs. 5 – 7 treat the special case \( \sigma = 0.4 \). Fig. 5 presents the corresponding robustness regions. The optimal point used for controller design according to
Fig. 5. Robustness regions in the parameter plain. The criterion (23) is emphasized by an arrow. Fig. 6 depicts the open loop Nyquist plots and Fig. 7 shows closed loop set-point and load disturbance step responses for all the extreme transfer functions.

8. CONCLUSIONS

This paper describes a new systematic tuning procedure for DTC which guarantees the fulfillment of all design specifications for arbitrary order lag/dead time process transfer functions. The procedure integrates all necessary steps from the simple identification experiment which provides just the three process characteristic numbers to the tuning formulae by which the robust parameters of the controller are computed. Notice, that the same tuning procedure can be used for tuning of DTCs for integrating processes (Vecerek, 2004) and also for conventional PI(D) controllers.

REFERENCES


Fig. 6. Nyquist curves $L(j\omega)$ in the complex plain.

Fig. 7. Set point and load disturbance step responses.


