INFINITE EIGENVALUE ASSIGNMENT BY OUTPUT-FEEDBACKS FOR SINGULAR SYSTEMS

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Abstract. The problem of infinite eigenvalue assignment by output-feedbacks is considered. Necessary and sufficient conditions for the existence of a solution to the problem are established. A procedure for computation of the output-feedback gain matrix is given and illustrated by a numerical example.
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1. INTRODUCTION

It is well-known (Dai, 1989; Kailath, 1980; Wonham, 1979; Kučera, 1981; Kaczorek, 1993) that if a pair \((A,B)\) of standard linear system \(\dot{x} = Ax + Bu\) is controllable then there exist a state-feedback gain matrix \(K\) such that \(\det[I_2s - A + BK] = p(s)\), where 
\[ p(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_1s + a_0 \]

is a given arbitrary \(n\) degree polynomial. By changing \(K\) we may modify arbitrarily only the coefficients \(a_0, a_1, \ldots, a_{n-1}\), but we are not able to change the degree \(n\) of the polynomial which is determined by the matrix \(I_2s\). In singular linear systems we are also able to change the degree of the closed-loop characteristic polynomials by suitable choice of the state-feedback matrix \(K\). The problem of finding of a state-feedback matrix \(K\) such that \(\det[Es - A + BK] = \alpha \neq 0\) (\(\alpha\) is independent of \(s\)) has been considered in (Delin and Ho, 1999; Kaczorek 2003). The infinite eigenvalue assignment problem by feedbacks is very important problem in design of the perfect observers (Kaczorek, 2000; Kaczorek, 2002b; Kaczorek, 2003). In this paper the problem of infinite eigenvalue assignment by output-feedbacks is formulated and solved.

This is an extension of the method given in (Kaczorek, 2003) for output feedback case. Necessary and sufficient conditions for the existence of a solution to the problem will be established and a
procedure for computation of the output-feedback gain matrix will be presented.

2. PROBLEM FORMULATION

Let $R^{m \times n}$ be the set of $n \times m$ real matrices and $R^n = R^{n \times 1}$. Consider the continuous-time linear system

$$\dot{x} = Ax + Bu, \quad y = Cx$$

(1)

where $\dot{x} = \frac{dx}{dt}$, $x \in R^n$, $u \in R^m$ and $y \in R^p$ are the semistate, input and output vectors and $E, A \in R^{n \times n}, B \in R^{n \times m}, C \in R^{p \times n}$. The system (1) is called singular if $\det E = 0$ and it is called standard when $\det E \neq 0$.

It is assumed that $\text{rank} E = r < n$, $\text{rank} B = m$, $\text{rank} C = p$ and the pair $(E, A)$ is regular, i.e.

$$\det[E_s - A] \neq 0 \text{ for some } s \in C$$

(2)

(the field of complex numbers)

Let us consider the system (1) with the output-feedback

$$u = v - Fy$$

(3)

where $v \in R^n$ is a new input and $F \in R^{m \times p}$ is a gain matrix.

From (1) and (3) we have

$$\dot{x} = (A - BFC)x + Bv$$

(4)

**Problem 1.** Given matrices $E, A, B, C$ of (1) and nonzero scalar $\alpha$ (independent of $s$). Find a $F \in R^{m \times p}$ such that

$$\det[E_s - A + BFC] = \alpha$$

(5)

In this paper necessary and sufficient conditions for the existence of a solution to the problem will be established and a procedure for computation of $F$ will be proposed.

3. PROBLEM SOLUTION

From the equality

$$E_s - A + BFC = [E_s - A, B]\begin{bmatrix} I_n \\ FC \end{bmatrix} = [I_n, BF] \begin{bmatrix} E_s - A \\ C \end{bmatrix}$$

(6)

and (5) it follows that the problem has a solution only if

$$\text{rank} [E_s - A, B] = n \text{ for all finite } s \in C$$

(7)

and

$$\text{rank} \begin{bmatrix} E_s - A \\ C \end{bmatrix} = p \text{ for all finite } s \in C$$

(8)

The problem will be solved by the use of the following two steps procedure

**Step 1.** (subproblem 1). Given $E, A, B$ of (1) and a scalar $\alpha$. Find a matrix $K = FC$ such that

$$\det[E_s - A + BK] = \alpha$$

(9)

**Step 2.** (subproblem 2). Given $C$ and $K$ depending of some free parameters $k_1, k_2, ..., k_i$ (found in Step 1). Find desired $F$ satisfying the equation

$$K = FC$$

(10)

The solution of the subproblem 1 is based on the following lemma [2,7].

**Lemma 1.** If the condition (2) is satisfied then there exist orthogonal matrices $U, V$ such that

$$U[E_s - A]V = \begin{bmatrix} E_{1}\alpha - A_{1} & & * \\ 0 & & & & & & & & \end{bmatrix}$$

(11a)

$$UB = \begin{bmatrix} B_{1} & E, A_1 \in R^{n \times n}, B_1 \in R^{n \times m} \\ 0 & E_0, A_0 \in R^{n \times n} \end{bmatrix}$$

where the subsystem $(E, A, B)$ is completely controllable, the pair $(E, A_1)$ is regular, $E_1$ is upper triangular and $*$ denotes an unimportant matrix.

Moreover the matrices $E_{1}, A_{1}$ and $B_{1}$ are of the forms

$$E_{1s} - A_{1} = \begin{bmatrix} E_{11}s - A_{11} & E_{12}s - A_{12} & \cdots \\ -A_{1i} & E_{22}s - A_{22} & \cdots \\ & & \ddots & \ddots \\ 0 & 0 & \cdots & 0 & -A_{ix} & E_{xx}s - A_{xx} \\ & E_{x,1} - A_{1,1} & \cdots & \cdots & \cdots \\ & E_{x,2} - A_{2,1} & \cdots & \cdots & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots \\ & E_{x,x} - A_{x,1} & \cdots & \cdots & \cdots \\ 0 & -A_{k,1} & \cdots & \cdots & \cdots \\ \end{bmatrix}$$

(11b)

$$B_{1} = \begin{bmatrix} B_{11} & E_{j}, A_{j} \in R^{n \times n}, i, j = 1, ..., k \\ 0 & \vdots \\ B_{11} \in R^{m \times n}, \sum_{i=1}^{n} \tilde{n}_{i} = n_1 \\ 0 & \vdots \\ \end{bmatrix}$$
with $B_{i1}, A_{i1}, ..., A_{i\iota}$ of full row rank and $E_{i\iota}, ..., E_n$ nonsingular.

**Remark 1.** The matrix $C = CV$ has no special form.

**Theorem 1.** Let the condition (2) and (7) be satisfied and let the matrices $E, A, B$ of (1) be transformed to the forms (11). There exists a matrix $K$ satisfying the condition (9) if and only if

i) the subsystem $(E, A, B)$ is singular, i.e.

$$\det E_i = 0 \quad (12a)$$

ii) if $n_i > 0$ then the degree of the polynomial

$$\det[E_i s - A_i] = 0 \quad (12b)$$

**Proof. Necessity.** From (9) and (11a) we have

$$\det[E_i s - A_i + B_i K] \det[E_i s - A_i] = \alpha$$

where $K = KV \in R^{n \times n}$ and $\det[E_i s - A_i] = 1$ if $n_i = 0$.

From (13) it follows that the condition (9) holds only if the conditions (12) are satisfied.

**Sufficiency.** First let us consider the single-input ($m = 1$) case. In this case we have

$$E_i = \begin{bmatrix} e_{i1} & e_{i2} & \cdots & e_{i n_i} \\ 0 & e_{i2} & \cdots & e_{i2 n_i} \\ 0 & 0 & \cdots & e_{i n_i n_i} \end{bmatrix},$$

$$A_i = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{i n_i n_i - 1} & a_{i n_i} \\ a_{i2} & a_{i2} & \cdots & a_{i n_i n_i - 1} & a_{i n_i} \\ 0 & a_{i3} & \cdots & a_{i n_i n_i - 1} & a_{i n_i n_i} \end{bmatrix},$$

$$B_i = b_i = \begin{bmatrix} b_{i1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where $e_{i1} \neq 0, a_{i, j} \neq 0$ for $i = 2, ..., n_i$ and $b_{i1} \neq 0$.

The condition (12a) implies that $e_{i1} = 0$.

Premultiplying the matrix $[E_i s - A_i, b_i]$ by orthogonal row operations matrix $P_i$ it is possible to make zero the entries $e_{i, i}, e_{i, i}, ..., e_{i, n_i}$ of $E_i$ since $e_{i, i} \neq 0$, $i = 2, ..., n_i$. By this reduction only the entries of the first row of $A_i$ will be modified.

$$E_i = P_i E_i = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & e_{i2} & \cdots & e_{in_i} \\ 0 & 0 & \cdots & e_{in_i n_i} \end{bmatrix},$$

$$A_i = P_i A_i = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{i n_i n_i - 1} & a_{i n_i} \\ a_{i2} & a_{i2} & \cdots & a_{i n_i n_i - 1} & a_{i n_i} \\ 0 & a_{i3} & \cdots & a_{i n_i n_i - 1} & a_{i n_i n_i} \end{bmatrix},$$

$$b_i = P_i b_i = b_i$$

Let

$$\bar{K} = \frac{1}{b_{i1}} \begin{bmatrix} -a_{i1}, -a_{i2}, ..., -a_{i n_i n_i - 1}, 1 - a_{i n_i} \end{bmatrix}$$

Using (13), (15) and (16) we obtain

$$\det[E_i s - A_i + \bar{K} b_i] =$$

$$= \begin{bmatrix} 0 & 0 & \cdots & 0 \\ -a_{i1} & e_{i2} s - a_{i2} & \cdots & e_{i2 n_i s - a_{i2 n_i}} \\ 0 & -a_{i3} & \cdots & e_{i3 n_i s - a_{i3 n_i}} \\ 0 & 0 & \cdots & e_{in_i s - a_{in_i}} \end{bmatrix}$$

$$= a_{i1} a_{i2} \cdots a_{i n_i n_i - 1} = \alpha$$

where $\alpha = \alpha U V \det P_i \det[E_i s - A_i]^{-1}$. The considerations can be easily extended for multi-input systems, $m \geq 1$. In this case the matrix $P_i$ of the orthogonal row operations is chosen so that all entries of the first row of $E_i = P_i E_i$ are zero. By this reduction only the entries of $A_i$, $i = 1, ..., k$ and $B_i$ will be modified. The modified matrices will be denoted by $\bar{A}_i, i = 1, ..., k$ and $\bar{B}_i$.

Let

$$K = \bar{K} \begin{bmatrix} \bar{A}_{i1}, \bar{A}_{i2}, ..., \bar{A}_{ik} \end{bmatrix} + G$$

The matrix $G \in R^{n \times m}$ in (18) is chosen so that

$$E_i s - \bar{A}_i + \bar{B}_K =$$

$$= \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \bar{a}_{i1} & \cdots & * \\ 0 & \bar{a}_{i3} & \cdots & * \\ 0 & 0 & \cdots & \bar{a}_{i n_i} \end{bmatrix}$$

(* denotes unimportant entries)

$$h = \frac{\alpha(-1)^{i+1}}{\bar{a}_{i1} \bar{a}_{i2} \cdots \bar{a}_{i n_i} c}$$

and $c = \det U^{-1} V^{-1} \det P_i^{-1} \det[E_i s - A_i]$. Using (13), (18) and (19) it is easy to verify that

$$\det[E_i s - A_i + B_i K] = \det[E_i s - A_i + B_i K] = \alpha$$

□
Remark 2. Note that for $m > 1$ some entries of the matrix $G$ in (18) can be chosen arbitrarily.
Therefore, the matrix $K = KV^{-1}$ has a number of free parameters denoted by $k_1, k_2, ..., k_j$.
The free parameters will be chosen so that the equation (10) has a solution $F$ for given $C$ and $K$.
It is well-known that the equation (10) has a solution if and only if
\[
\text{rank } C = \text{rank } \begin{bmatrix} C \\ K \end{bmatrix}
\]  
(21a)
or equivalently
\[
\text{Im } K^T \subset \text{Im } C^T \quad (T \text{ denotes the transpose) (21b)}
\]
where $\text{Im}$ denotes the image.
The free parameters $k_1, k_2, ..., k_j$ are chosen so that (21) holds.
Therefore, the following theorem has been proved.

Theorem 2. Let the conditions (2), (7), (8) and (12) be satisfied.
The problem has a solution, i.e. there exists $F$ satisfying (5) if and only if the free parameters $k_1, k_2, ..., k_j$ of $K$ can be chosen so that the equation (10) has a solution $F$ for given $C$ and $K$.
From the condition (21) and (16) we have the following corollary.

Corollary 1. For $m = 1$ problem has a solution if and only if the row $[\overline{a}_{i1}, \overline{a}_{i2}, ..., \overline{a}_{in}, \overline{a}_{in} - 1]$ is proportional to the matrix $C$.

Remark 3. If the order of system is not high say $n \leq 5$ the elementary row and column operations instead of the orthogonal operations can be used.

4. EXAMPLE

For the singular system (1) with
\[
E = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 2 & 1 \end{bmatrix},
\]  
(22)
\[
B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 0.5 & 1 & 3 & -2 \\ 2.5 & 3 & 4 & -1 \end{bmatrix}
\]
find the gain matrix $F \in \mathbb{R}^{2 \times 2}$ such that the condition (5) is satisfied for $\alpha = 1$.
In this case the pair $(E, A)$ is regular since
\[
\text{det}[E - \lambda I] = \begin{vmatrix} -1 & 2s + 1 & s & -1 \\ 0 & s - 1 & -s - 2 & 2s \\ 0 & 1 & s - 1 & 1 - s \\ 0 & 0 & -2 & s - 1 \end{vmatrix} = (3 - s)(s - 1)^2 - (s + 2)(s - 1) + 4s
\]
The matrices (22) have already the desired forms (11) with $A_0 = 0, B_0 = 0, E_i = E, A_i = A, B_i = B$, $n_i = n = 4, \pi_1 = 2, \pi_2 = \pi_3 = \pi_4 = 1, m = 2$ and
\[
E_{11} = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}, E_{12} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, E_{13} = \begin{bmatrix} 0 \\ 2 \end{bmatrix},
\]
\[
E_{22} = [1], E_{23} = [-1], E_{13} = [1]
\]
\[
A_{11} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, A_{12} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, A_{13} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},
\]
\[
A_{21} = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}, A_{22} = [1],
\]
\[
A_{23} = [-1], A_{32} = [2], A_{13} = [1], B_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]
Using the elementary row operations (6-7) we obtain
\[
P_i = \begin{bmatrix} 1 & -2 & -3 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]
and
\[
E_{s, \overline{a}} = P_i [E - \lambda I, B] = \begin{bmatrix} -1 & 0 & 5 & -5 & 1 & -2 \\ 0 & s & -1 & 2 & 0 & 1 \\ 0 & 1 & s - 1 & 1 - s & 0 & 0 \\ 0 & 0 & -2 & s - 1 & 0 & 0 \end{bmatrix}
\]
Taking into account that in this case
\[
\overline{a}_{11}, \overline{a}_{12}, \overline{a}_{13} = \begin{bmatrix} 1 & 0 & -5 & 5 \\ 0 & 0 & 1 & -2 \end{bmatrix},
\]
\[
\overline{B}_i = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}, G = \begin{bmatrix} 0.5 & k_1 & k_2 & k_3 \end{bmatrix}
\]
and using (18) we obtain
\[
K = \overline{K} = \overline{B}_i \left[ \overline{a}_{11}, \overline{a}_{12}, \overline{a}_{13} \right] + G = \begin{bmatrix} 2 & 2k_1 & 2k_2 & -3 & 1 & 2k_3 \\ 0.5 & k_1 & k_2 + 1 & k_3 & -2 \end{bmatrix}
\]
where $k_1, k_2, k_3$ are free parameters.
The free parameters are chosen so that the condition
\[
\begin{align*}
\text{rank} & \begin{bmatrix} 0.5 & 1 & 3 & -2 \\ 2.5 & 3 & 4 & -1 \end{bmatrix} = \\
& = \text{rank} \begin{bmatrix} 0.5 & 1 & 3 & -2 \\ 2.5 & 3 & 4 & -1 \\ 2k_1 & 2k_2 & -3 & 1+2k_3 \\ 0.5 & k_1 & k_2+1 & k_3-2 \end{bmatrix}
\end{align*}
\]

is satisfied.
The condition (23) is satisfied for \( k_1 = 1, k_2 = 2, k_3 = 0 \) and the equation

\[
\begin{bmatrix} 0.5 & 1 & 3 & -2 \\ 2.5 & 3 & 4 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 & 1 \\ 0.5 & 1 & 3 & -2 \end{bmatrix}
\]

has the solution

\[
F = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}
\]

It is easy to check that

\[
\det[Es - A + BK] = \det P^{-1} \det[\overline{Es} - \overline{A} + \overline{BK}] =
\]

\[
\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0.5 & s+1 & 2 & 0 \\ 0 & 1 & s-1 & 1-s \\ 0 & 0 & -2 & s-1 \end{vmatrix} = 1
\]

5. CONCLUDING REMARKS.

The problem of infinite eigenvalue assignment by output feedbacks has been formulated and solved. Necessary and sufficient conditions for the existence of a solution to the problem have been established. Two steps procedure for computation of the output-feedback gain matrix has been derived and illustrated by a numerical example. With slight modifications the considerations can be extended for singular discrete-time linear systems. An extension of the considerations for two-dimensional linear systems (Kaczorek, 1993) is also possible but it is not trivial.

REFERENCES


