GENERALIZATION OF CAYLEY-HAMILTON THEOREM FOR N-D POLYNOMIAL MATRICES

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Abstract: The Cayley-Hamilton theorem is extended for real polynomial matrices in n variables. The known extensions of the classical Cayley-Hamilton theorem are particular cases of the proposed extension.

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1. INTRODUCTION

The classical Cayley-Hamilton theorem (Gantmacher, 1974; Lancaster, 1969) says that every square matrix satisfies its own characteristic equation. The Cayley-Hamilton theorem has been extended for rectangle matrices (Kaczorek, 1995c), block matrices (Kaczorek, 1995b; Victoria, 1982) pairs of matrices (Chang and Chen, 1992; Lewis, 1982; Livsic, 1983; Lewis, 1986; Mertzios and Christodoulous, 1986), pairs of block matrices (Kaczorek, 1998) and standard and singular two-dimensional linear systems (Kaczorek, 1992/93; Kaczorek, 1994; Kaczorek, 1995a; Smart and Barnett, 1989; Theodoru, 1989).

The Cayley-Hamilton theorem and its generalizations have been used in control systems, electrical circuits, systems with delays, singular systems, 2D linear systems, etc., (Kaczorek, 1992/93; Kaczorek, 1995c; Lancaster, 1969).

In this note the Cayley-Hamilton theorem will be extended for n-dimensional (n-D) real polynomial matrices. The known extensions of the classical Cayley-Hamilton theorem are particular cases of the proposed extension.

2. PRELIMINARIES

Let \( R^{m \times n}[s_1, s_2, \ldots, s_n] \) be the set of \( m \times n \) real polynomial matrices in \( n \) variables \( s_1, s_2, \ldots, s_n \). Consider an \( n \)-dimensional \((n-D)\) polynomial matrix of the form

\[
A(s_1, s_2, \ldots, s_n) = \sum_{i_1=0}^{q_1} \sum_{i_2=0}^{q_2} \cdots \sum_{i_n=0}^{q_n} A_{i_1,i_2,\ldots,i_n} s_1^{i_1} s_2^{i_2} \cdots s_n^{i_n} \quad (1)
\]

where \( A_{i_1,i_2,\ldots,i_n} \in R^{m \times n} \) (the set of \( m \times n \) real matrices) and \( q_k, k = 1, \ldots, n \) are nonnegative integers.

It is assumed that the matrix (1) is invertible, i.e. the n-D polynomial

\[
\det A(s_1, s_2, \ldots, s_n) = \sum_{j_1=0}^{N_1} \sum_{j_2=0}^{N_2} \cdots \sum_{j_n=0}^{N_n} a_{j_1,j_2,\ldots,j_n} s_1^{j_1} s_2^{j_2} \cdots s_n^{j_n} \quad (2)
\]
is nonidentically vanishing.

Let

\[ A^{-1}(s_1, s_2, \ldots, s_n) = \sum_{i_1=-\mu_1}^{\infty} \cdots \sum_{i_n=-\mu_n}^{\infty} \Phi_{i_1i_2\cdots i_n} s_1^{-i_1} s_2^{-i_2} \cdots s_n^{-i_n} \]  

where \( \Phi_{i_1i_2\cdots i_n} \in \mathbb{R}^{m \times m} \) and \( (\mu_1, \mu_2, \ldots, \mu_n) \) is the n-D index of the matrix (1).

If the coefficient \( a_{N_1N_2\ldots N_n} \neq 0 \) then all \( \mu_k, k = 1, \ldots, n \) are finite otherwise some of them may be infinite.

**Lemma** The matrices \( \Phi_{i_1i_2\cdots i_n} \) satisfy the equalities

\[ \sum_{i_1=0}^{q_1} \cdots \sum_{i_n=0}^{q_n} A_{i_1i_2\cdots i_n} \Phi_{i_1+1,i_2+1,\ldots,i_n+1} \]

\[ = \begin{cases} I_n & \text{for } k_1 = k_2 = \cdots = k_n = 0 \\ 0 & \text{for } k_1 \neq 0 \text{ or/and } k_2 \neq 0 \end{cases} \]

where \( M_k(k=1,\ldots,n) \) are nonnegative integers.

Comparison of the matrix coefficients at the same powers of \( s_k^k \) for \( i_k = -1, -2, \ldots; k = 1, \ldots, n \) in (7) yields (6).

**Remark** If instead of (3) we use the expansion

\[ A^{-1}(s_1, s_2, \ldots, s_n) = \sum_{i_1=-\mu_1}^{\infty} \cdots \sum_{i_n=-\mu_n}^{\infty} \Phi_{i_1i_2\cdots i_n} \times \]

\[ s_1^{-(i_1+1)} s_2^{-(i_2+1)} \cdots s_n^{-(i_n+1)} \]

then the equalities (6) take the form

\[ \sum_{i_1=0}^{N_1} \cdots \sum_{i_n=0}^{N_n} a_{i_1i_2\cdots i_n} \Phi_{i_1+1,i_2+1,\ldots,i_n+1} = 0 \]

for \( k_j = 0, 1, 2, \ldots; j = 1, \ldots, n \)

Note that the well-known extensions of the classical Cayley-Hamilton theorem (Chang and Chen, 1992; Kaczorek, 1994; Kaczorek, 1995a; Lewis, 1982; Livsic, 1983; Lewis, 1986; Mertzios and Christodoulous, 1986; Smart and Barnett, 1989) are particular cases of the proposed extension. For \( A(s) = [I_s - A] \) the proposed theorem is equivalent to the classical Cayley-Hamilton theorem for the matrix \( A \). If \( A(s) = [E_s - A] \) \((E, A \in \mathbb{R}^{n \times n})\) we obtain the generalization of the theorem for singular systems and for

\[ A(z_1, z_2) = \begin{bmatrix} I_{n_1} z_1 - A_1 & -A_2 \\ -A_3 & I_{n_2} z_2 - A_4 \end{bmatrix} \]

\[ A_1 \in \mathbb{R}^{n_1 \times n_1}, A_4 \in \mathbb{R}^{n_2 \times n_2} \]

we obtain the generalization of the theorem for 2D linear systems described by the Roesser model.

**Proof.** Taking into account that \( \text{Adj}A(s_1, s_2, \ldots, s_n) = A^{-1}(s_1, s_2, \ldots, s_n) \) det \( A(s_1, s_2, \ldots, s_n) \) (\( \text{Adj}A \) denotes the adjoint matrix) and using (2) and (3) we may write

\[ \text{Adj}(s_1, s_2, \ldots, s_n) = \]

\[ = \sum_{i_1=0}^{M_1} \cdots \sum_{i_n=0}^{M_n} \Phi_{i_1i_2\cdots i_n} s_1^{-i_1} s_2^{-i_2} \cdots s_n^{-i_n} \]

(7)

where \( M_k(k=1,\ldots,n) \) are nonnegative integers.
Examples. Three examples of n-D invertible polynomial matrices for $n = 1, 2, 3$ will be considered.

The inverse matrix of the 1-D polynomial matrix

$$A(s) = \begin{bmatrix} 1 + s & -s \\ s^2 & 2s \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} s^2 + \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} s + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= A_2s^2 + A_1s + A_0$$

has the form

$$A^{-1}(s) = \begin{bmatrix} 2s^{-2} - 2s^{-3} + 2s^{-4} - 4s^{-6} + ... \\ -2s^{-1} + 4s^{-2} - 4s^{-3} + 8s^{-5} - 16s^{-6} + ... \end{bmatrix} = \Phi_1s^{-1} + \Phi_2s^{-2} + \Phi_3s^{-3} + \Phi_4s^{-4} + \Phi_5s^{-5} + \Phi_6s^{-6} + ...$$

where

$$\Phi_1 = \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix}, \Phi_2 = \begin{bmatrix} 2 & 1 \\ 4 & 1 \end{bmatrix}, \Phi_3 = \begin{bmatrix} -4 & -2 \\ -4 & -1 \end{bmatrix}, \Phi_4 = \begin{bmatrix} 4 & 2 \\ 0 & 0 \end{bmatrix}, \Phi_5 = \begin{bmatrix} 0 & 0 \\ 8 & 2 \end{bmatrix}, \Phi_6 = \begin{bmatrix} -8 & -4 \\ -16 & -4 \end{bmatrix}$$

Taking into account that det $A(s) = s^3 + 2s^2 + 2s$ ($a_0 = 0, a_1 = a_2 = 2, a_3 = 1$) and using (6) and (11) we obtain:

for $k_1 = 1$

$$a_0\Phi_1 + a_1\Phi_2 + a_2\Phi_3 + a_3\Phi_4 =$$

$$= 2 \begin{bmatrix} 2 & 1 \\ 4 & 1 \end{bmatrix} + 2 \begin{bmatrix} -4 & -2 \\ -4 & -1 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 \\ 0 & 0 \end{bmatrix}$$

for $k_1 = 2$

$$a_0\Phi_2 + a_1\Phi_3 + a_2\Phi_4 + a_3\Phi_5 =$$

$$= 2 \begin{bmatrix} -4 & -2 \\ -4 & -1 \end{bmatrix} + 2 \begin{bmatrix} 4 & 2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 8 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and for $k_1 = 3$

$$a_0\Phi_3 + a_1\Phi_4 + a_2\Phi_5 + a_3\Phi_6 =$$

$$= 2 \begin{bmatrix} 4 & 2 \\ 0 & 0 \end{bmatrix} + 2 \begin{bmatrix} 0 & 0 \\ 8 & 2 \end{bmatrix} + \begin{bmatrix} -8 & -4 \\ -16 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The inverse matrix of the 2-D polynomial matrix

$$A(s_1, s_2) = \begin{bmatrix} 1 + s_1s_2 & s_2 \\ -s_2 & s_1s_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} s_1s_2 + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} s_2 + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= A_{11}s_1s_2 + A_{01}s_2 + A_{00}$$

has the form

$$A^{-1}(s_1, s_2) = \begin{bmatrix} s_1^{-1} s_2^{-1} - s_1^{-2} s_2^{-2} + s_1^{-3} s_2^{-3} - s_1^{-4} s_2^{-4} + ... \\ s_1^{-2} s_2^{-1} - s_1^{-3} s_2^{-2} - s_1^{-4} s_2^{-3} + s_1^{-5} s_2^{-4} + ... \\ -s_1^{-1} s_2^{-2} + s_1^{-2} s_2^{-1} - s_1^{-3} s_2^{-2} + s_1^{-4} s_2^{-3} - s_1^{-5} s_2^{-4} + ... \\ -s_1^{-2} s_2^{-1} + s_1^{-3} s_2^{-2} + s_1^{-4} s_2^{-3} - s_1^{-5} s_2^{-4} + ... \end{bmatrix} = \Phi_{11}s_1^{-1} s_2^{-1} + \Phi_{12}s_1^{-2} s_2^{-1} + \Phi_{21}s_1^{-1} s_2^{-2} + \Phi_{22}s_1^{-2} s_2^{-2} + \Phi_{31}s_1^{-1} s_2^{-3} + \Phi_{32}s_1^{-2} s_2^{-3} + \Phi_{41}s_1^{-3} s_2^{-1} + \Phi_{42}s_1^{-4} s_2^{-1} + \Phi_{51}s_1^{-3} s_2^{-2} + \Phi_{52}s_1^{-4} s_2^{-2} + ...$$

where

$$\Phi_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \Phi_{12} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \Phi_{21} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \Phi_{22} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \Phi_{31} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \Phi_{32} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \Phi_{41} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \Phi_{42} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \Phi_{51} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \Phi_{52} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$$

Taking into account that det $A(s_1, s_2) = s_1^2 s_2^2 + s_1 s_2 + s_2$

$$(a_{22} = a_{11} = a_{02} = 1)$$

and using (6) and (13) we obtain for $k_1 = k_2 = 1$

$$a_{02}\Phi_{13} + a_{11}\Phi_{22} + a_{22}\Phi_{31} =$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and for $k_1 = k_2 = 2$

$$a_{02}\Phi_{24} + a_{11}\Phi_{33} + a_{22}\Phi_{44} =$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
The inverse matrix of 3-D polynomial matrix
\[ A^{-1}(s_1, s_2, s_3) = \begin{bmatrix} 1 + s_1 s_2 s_3, s_1 s_2 s_3 \end{bmatrix} = \]
\[ = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} s_1 s_2 s_3 + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = A_{111} s_1 s_2 s_3 + A_{000} \]

has the form
\[ A^{-1}(s_1, s_2, s_3) = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \]
\[ = \Phi_{111} s_1^{-1} s_2^{-1} s_3^{-1} + \Phi_{000} \]

where
\[ \Phi_{111} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \Phi_{000} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \]

Taking into account that
\[ \det A(s_1, s_2, s_3) = s_1 s_2 s_3 \]
\[ a_{ijk} = \begin{cases} 1 & \text{for } i = j = k = 1 \\ 0 & \text{otherwise} \end{cases} \]

and using (6) and (15) we obtain
\[ a_{ijk} \Phi_{1+k,1+j+k,k+k} = 0 \]

for \( i, j, k \in \{0, 1, \ldots\} \)

and \( k_1 + k_2 + k_3 \geq 1 \)

4. APPLICATION

Application of the Theorem will be illustrated by the following two examples.
Consider the discrete-time linear system with \( h \geq 1 \) delays described by the equation
\[ x_{t+1} = A_0 x_t + A_1 x_{t-1} + A_2 x_{t-2} + \ldots + A_h x_{t-h} + B u_t \]

where \( x_t \in R^n \) is the state vector, \( u_t \in R^m \) is the input vector, and \( A_k \in R^{n \times n}, k = 0, 1, 2, \ldots \)
\( B \in R^{n \times m} \).

The characteristic equation of (16) can be written in the form
\[ \det[I z - A_0 - A_1 z^{-1} - A_2 z^{-2} - \ldots - A_h z^{-h}] = z^h \det[I z^{h+1} - A_0 z^{-h} - A_1 z^{h-1} - \ldots - A_h] = z^{-h}(z^N - a_{N-1} z^{N-1} - a_{N-2} z^{N-2} + \ldots - a_1 z - a_0) \]
\[ N = n(h + 1) \]

The matrix
\[ A(z) = [I z^{h+1} - A_0 z^{-h} - A_1 z^{h-1} - \ldots - A_h] \]

is invertible.
Let
\[ A^{-1}(z) = \sum_{i=0}^{\infty} \Phi_i z^{-i} \]

From Theorem and (17), (18) we obtain
\[ \sum_{i=0}^{N} a_{i} \Phi_{i+k} \text{ for } k = 0, 1, 2, \ldots \]

Example. Consider the 2D linear system described by the equation
\[ x_{t+1,j+1} = A_0 x_{i,j} + A_1 x_{i+1,j} + A_2 x_{i,j+1} + B u_{i,j} \]

where \( x_{i,j} \in R^n \) is the state vector, \( u_{i,j} \in R^m \) is the input vector and \( A_k \in R^{n \times n}, k = 0, 1, 2, \ldots \)
\( B \in R^{n \times m} \).

The characteristic equation of (20) has the form
\[ \det[I z_1 z_2 - A_0 - A_1 z_1 - A_2 z_2] = \]
\[ = \sum_{i=0}^{n} \sum_{j=0}^{n} a_{ij} z_1^i z_2^j \]

The matrix
\[ A(z_1, z_2) = [I z_1 z_2 - A_0 - A_1 z_1 - A_2 z_2] \]

is invertible. Let
\[ A^{-1}(z_1, z_2) = \sum_{i=0}^{\infty} \sum_{j=0}^{n} \Phi_{ij} z_1^{-i} z_2^{-j} \]

From Theorem and (21), (22) we obtain
\[ \sum_{i=0}^{n} \sum_{j=0}^{n} a_{ij} \Phi_{i+k,j+k} = 0 \]

5. CONCLUSION

The classical Cayley-Hamilton theorem has been extended for real polynomial matrices in \( n \) variables. It has been shown that the known extensions of the Cayley-Hamilton theorem are particular cases of the proposed extension. Application of the proposed extension has been illustrated by examples.
REFERENCES


