Abstract: To describe the dynamic behavior of a fire rescue turntable ladder the flexibility of the ladder set has to be analyzed. It turned out, that it can be modelled as a flexible manipulator consisting of an arbitrary number of links governed by the Euler-Bernoulli beam equation. The first link is fixed at one end and is driven by a control torque. The last link carries a payload. A peculiarity of the model is that the density and flexural rigidity parameters are allowed to be discontinuous but piecewise constant functions of the spatial coordinate. For the Galerkin approximation of the control system considered, we derive a result on stabilization of the equilibrium. The controller design scheme is based on explicit construction of a Lyapunov function and application of the invariance principle. A simulation of the closed-loop dynamics is carried out in order to show the efficiency of the controller proposed.

Keywords: Stabilizing feedback, Lyapunov function, Modal control, Flexible arms, Galerkin approximation

1. INTRODUCTION

Fire turntable ladders are manipulators with a fairly large workspace. As these manipulators are mobile vehicles and the area for strutting the vehicle is limited, the aim is to achieve the maximum of outreach. Therefore, this can be only achieved by reducing the weight of the ladder set itself. The result is a lightweight construction of the ladder set, which is then characterized by a significant flexibility. In order to develop an oscillation damping control for these systems, the modelling of the ladder set as a flexible beam plays a crucial role.

Figure 1 shows a typical fire rescue turntable ladder. The manipulator is equipped with four active axes: the turning axis \( \phi_T \); the raising of the ladder set \( \phi_R \); another rotating joint \( \phi_J \) for the upper ladder part 6; and the telescoping of the ladder parts 1 to 5, described by the variable \( l \).

Figure 1: Fire turntable ladder.

The bearings of the ladder parts can be assumed to be passive joints of the system.

The dynamical behavior of the distributed parameter vibration systems, such as beams and flexible-
link manipulators, is generally described by partial differential equations; see (Chen et al., 1987; Bloch and Titi, 1991; Xu and Baillieul, 1993; Coron and d’Andrea Novel, 1998; Luo et al., 1999) and references therein. However, finite dimensional approximate models obtained by the assumed modes and finite elements methods are used more frequently for solving the motion planning and stabilization problems (Talebi et al., 2001; Sawodny et al., 2002). The goal of this paper is to derive both complete and approximate dynamical models for the multi-link flexible manipulator representing a fire-rescue turntable ladder. In our case, mathematical modelling of the ladder dynamics as a homogeneous beam is not adequate since the real object consists of several segments having different mechanical parameters.

2. MOTION EQUATIONS

Consider a flexible manipulator performing planar motion under the action of a control torque \( M \) (see Figure 2). The manipulator is fixed at one end (hub at the point \( O \)), while the other end carries a payload of mass \( m \).

![Figure 2: A multi-link flexible manipulator.](image)

We assume that the manipulator consists of \( n \) flexible links connected via elastic joints at \( O_1, ..., O_{n-1} \). It is assumed also that the manipulator length, \( l \), is much greater than the width of each link, thus we shall use the Euler-Bernoulli beam equation for modelling the links oscillations. For each time \( t \geq 0 \), the links configuration is described by the graph of \( u(x,t), x \in [0,l] \) in the Cartesian frame \( Oxy \). Given a partition \( 0 = t_0 < t_1 < ... < t_n = l \) of \([0,l]\), the deflection \( u(t) \) is assumed to be of class \( C^0[0,l] \cap C^4((0,l) \setminus \{t_1, ..., t_{n-1}\}) \) for all \( t \geq 0 \). For a function \( f(x,t) \), we denote by \( f_{a=0}, f_{a-0} \), and \( f_a \) the values \( \lim_{x \to a+0} f(x,t), \lim_{x \to a-0} f(x,t), \) and \( f(a,t) \), respectively. Thus, \( u(x,t) \) satisfies the following geometric boundary condition:

\[
(u)_0 = \left( \frac{\partial u}{\partial x} \right)_0 = 0.
\]

Neglecting the effects of shear deformation and rotary inertia of the beams, the kinetic energy \( T \) is given by the following expression:

\[
2T = \int_0^l \left( (u\phi')^2 + (x\phi + \frac{\partial u}{\partial t})^2 \right) \rho(x) dx + \frac{m}{2} \left( (u\phi')^2 + (l\phi + \frac{\partial u}{\partial t})^2 \right) + J \left( \phi + \left( \frac{\partial^2 u}{\partial t^2} \right)_l \right)^2,
\]

where \( \phi(t) \) is the angle between \( Ox \) and the fixed axis \( O\xi \), \( \rho(x) \) is the mass per unit length of the beam, and \( J \) is the payload moment of inertia.

The potential energy \( U \) of the mechanical system considered takes the form:

\[
2U = \int_0^l \left( \frac{\partial^2 u}{\partial x^2} \right)^2 \rho(x) dx + \sum_{j=1}^{n-1} \int_{l_{j-1}}^{l_j} \left( \frac{\partial u}{\partial x} \right)_{l_{j+1}}^2 - \frac{\partial u}{\partial x} \right)_{l_{j-1}}^2 dx,
\]

where \( c^2(x) = E(x)I(x)/\rho(x) \) is the flexural rigidity per unit length of the beam, \( \gamma_j^2 \) is the stiffness coefficient of the torsion spring at \( O_j \). In this paper, \( \rho(x) \) and \( c(x) \) are assumed to be piecewise-constant functions, i.e., \( \rho(x) = \rho_j \) and \( c(x) = c_j \) for \( x \in [l_{j-1}, l_j] \), \( j = 1, 2, ..., n \).

Suppose that \( \phi(t) \) and \( u(x,t) \) define motion of the system for given \( t \in [t_1, t_2] \) and control torque \( M(t) \).

Then Hamilton’s principle (Goldstein, 1980) yields

\[
\delta \left( \int_{t_1}^{t_2} L dt \right) + \int_{t_1}^{t_2} M(t) \delta \phi(t) dt = 0,
\]

for each variation \( \delta \phi(t), \delta \phi(x,t) \) such that

\[
\begin{align*}
\delta \phi \in C^2[t_1, t_2], & \quad \delta \phi(t_1) = \delta \phi(t_2) = 0, \\
\delta u \in C^2((0,l) \setminus \{t_1, ..., t_{n-1}\}) \times [t_1, t_2], & \quad \delta u(x, t_1) = \delta u(x, t_2) = 0, \quad \forall x \in [0,l], \\
(\delta u)_0 = \left( \frac{\partial \delta u}{\partial x} \right)_0 = 0, & \quad \delta u(\cdot, t) \in C^0[0,l], \quad \forall t \in [t_1, t_2],
\end{align*}
\]

where \( L = T - U \) is the Lagrangian of the system.

The variation of \( \int_{t_1}^{t_2} L dt \) in (1) is obtained by performing several integrations by parts 2:

\[
\begin{align*}
-\delta \left( \int_{t_1}^{t_2} L dt \right) & = \int_{t_1}^{t_2} \left[ \frac{d}{dt} \left( m(\dot{u}^2 + \dot{l}^2) + m \frac{\partial u}{\partial t} + J \phi + J \frac{\partial^2 u}{\partial t^2} \right) \right] dt + \\
& + \int_{t_1}^{t_2} \left( \frac{d}{dt} + J \frac{\partial u}{\partial t} \right) \rho dx - \int_{t_1}^{t_2} \left( \frac{\partial^2 u}{\partial t^2} \right)_l J \delta \phi dt + \\
& + \int_{t_1}^{t_2} \mu \left( \frac{\partial^2 u}{\partial t^2} - \phi, \phi, \delta \phi \right) dt = 0,
\end{align*}
\]

2 We omit cumbersome computations here.
where
\[
\mu = \int_0^1 \left( \frac{\partial^2 u}{\partial t^2} + c^2 \frac{\partial^4 u}{\partial x^4} + x\ddot{\phi} - \phi^2 u \right) \delta u \rho \, dx + \\
+ m \left[ \frac{\partial^2 u}{\partial t^2} - \frac{c^2 \rho}{m} \frac{\partial^3 u}{\partial x^3} + i\ddot{\phi} - \phi^2 u \right] \delta t_j + \\
+ J \left( \frac{\partial^3 u}{\partial t \partial x^2} + \frac{c^2 \rho^2}{J} \frac{\partial^2 u}{\partial x^2} + \phi \right) \frac{\partial^2 u}{\partial x} \right] \delta t_j + \\
+ \sum_{j=1}^{n-1} \left\{ \left[ c^2 \rho \frac{\partial^3 u}{\partial x^3} \right]_{j=0} - \left[ c^2 \rho \frac{\partial^3 u}{\partial x^3} \right]_{j=0} \right\} (\delta u)_{j=0} + \\
+ \left[ \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} \right) \right]_{j=0} - \left[ \frac{\partial u}{\partial x} \right]_{j=0} \right\} \frac{\partial^2 u}{\partial x} \right\} \delta t_j.
\]

To simplify (3), we introduce new control \( v \) by means of the following feedback transformation:
\[
v = \left( m u^2 + \int_0^1 \left( 2x^2 + u^2 \right) \rho \, dx \right)^{-1} \times \\
\times \left( M + \left( \frac{c^2 \rho^2}{J} \frac{\partial^2 u}{\partial x^2} \right) \right) - m \phi \left( 2u \frac{\partial u}{\partial t} + i \phi u \right) + \\
+ \phi \int_0^1 \left( \ddot{x} - 2 \frac{\partial u}{\partial x} \right) \rho u \, dx.
(4)
\]

As (3) should vanish for each variation \( \delta \phi, \delta u \) satisfying (2), we have
\[
\phi = v, \; v \in \mathbb{R},
\]
\[
\frac{\partial^2 u}{\partial t^2} + c^2 \frac{\partial^4 u}{\partial x^4} = \phi^2 u - xv, \; x \in (0, l) \setminus \{ l_1, \ldots, l_{n-1} \},
\]
\[
\left( \frac{\partial^2 u}{\partial t^2} - \phi^2 u + i \nu \frac{\partial^4 u}{m \partial x^4} \right) \bigg|_{t=x_l} = 0,
\]
\[
\left( \frac{\partial^2 u}{\partial t \partial x} + v + \frac{c^2 \rho \phi^2}{J} \frac{\partial^2 u}{\partial x^2} \right) \bigg|_{t=x_l} = 0,
\]
\[
u_{l=0} = \frac{\partial u}{\partial x} \bigg|_{x=l_0} = 0, \; \nu_{l=0} = \frac{\partial u}{\partial x} \bigg|_{x=l_{j+1}}, \; j = 0, \ldots, \frac{k}{2} - 3,
\]
\[
\frac{\partial^2 u}{\partial x^2} \bigg|_{x=l_{j+1}} = \frac{c^2 \rho_j \phi^2}{J} \frac{\partial^2 u}{\partial x^2} \bigg|_{x=l_{j+1}}, \; k = 2, 3,
\]
\[
\frac{\partial u}{\partial x} \bigg|_{x=l_{j+1}} = \frac{\partial u}{\partial x} \bigg|_{x=l_{j+1}}, \; j = 1, n - 1.
\]

For the case without a payload, a similar system describing the dynamics of serially connected beams was considered in (Chen et al., 1987).

Our goal is to stabilize the equilibrium \( \phi = \dot{\phi} = 0 \), \( u = \frac{\partial u}{\partial x} = 0 \) by means of a state feedback law \( v = \gamma (\phi, \dot{\phi}, u, \frac{\partial u}{\partial x}) \). This problem will be solved in Section 5 for a finite dimensional approximation of the boundary value problem (5).

### 3. EIGENFUNCTIONS OF THE HOMOGENEOUS PROBLEM

To derive an approximate dynamical model, we first apply separation of variables by substituting
\[
u(x, t) = \psi(x) \theta(t), \; \phi(t) = \text{const}, \; \psi(t) = 0
\]
into (5). This yields
\[
\theta(t) = A \cos(\omega t) + B \sin(\omega t)
\]
and
\[
\psi(x) = C_{j1} \sin(\eta_j x) + C_{j2} \cos(\eta_j x) + C_{j3} \sinh(\eta_j x) + \\
+ C_{j4} \cosh(\eta_j x)
\]
provided that \( C_{j1} \) satisfy the following system of linear algebraic equations
\[
M(\omega, P) \cdot \left( C_{11}, \ldots, C_{14}, \ldots, C_{n1}, \ldots, C_{n4} \right)^T = 0.
(8)
\]
Here \( M(\omega, P) \) is a \( 4n \times 4n \)-matrix whose coefficients depend on parameters \( \omega \) and \( P = (m, J, n, l_1, \rho_j, \eta_j, \chi_j) \).

For fixed \( P \), the system (8) admits a non-trivial invariant subspace if and only if \( \omega \) is a solution of the transcendental equation \( \det M(\omega, P) = 0 \). If \( \omega \) is such a solution, we will refer to \( \psi(x) \) as a form corresponding to \( \omega \) if its coefficients \( (C_{j1}) \neq 0 \) satisfy (8).

**Proposition 1.** Let \( \psi_1 \) and \( \psi_2 \) be forms corresponding to frequencies \( \omega_1 \neq \omega_2 \). Then \( \psi_1 \) and \( \psi_2 \) are orthogonal with respect to the following bilinear form
\[
\langle \psi_1, \psi_2 \rangle_x = \int_0^l \psi_1(x) \psi_2(x) \rho \, dx + m \psi_1(l) \psi_2(l) + J \psi_1^2(l) \psi_2^2(l).
\]

The **proof** exploits integration by parts with regard to the boundary conditions at \( x = l_j \).

In order to illustrate particular solutions (6) of the boundary value problem (5), we compute some number of first frequencies \( \omega_k \) for the following (dimensionless) values of parameters:
\[
n = 2, \; l_1 = \frac{l_2}{2} = c_1 = c_2 = \rho_1 = \rho_2 = \frac{\chi_2^2}{2} = m = J = 1.
(9)
\]

We have: \( \omega_1 = 0.3875771806; \; \omega_2 = 1.455554174; \; \omega_3 = 5.629214811; \; \omega_4 = 16.22789077 \). The corresponding forms \( \psi \) are shown in Figure 3. The forms are normalized there so that \( \langle \psi, \psi \rangle_x = \delta_x \) for each \( 1 \leq i \leq j \leq 4 \).

**Remark.** We see that the derivatives \( \psi_x^i(x) \) are discontinuous at \( x = l_j \). Thus, \( \psi_x^i(\cdot) \) do not belong to the Sobolev space \( H^2(0, l) \) in contrast to the case of a beam having continuous density function.

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3 The expression for \( \det M(\omega, P) \) has been obtained via the Maple software. Computational details are omitted in this paper.
Let \( (\psi_1(\cdot), \ldots, \psi_N(\cdot)) \) be forms corresponding to frequencies \( \omega_1 < \omega_2 < \ldots < \omega_N \). To derive a Galerkin approximation of the boundary value problem (5), we restrict \( u(\cdot,t) \) and \( \delta u(\cdot,t) \) to the finite dimensional space

\[
\mathcal{N} = \text{span}\{ \psi_1(\cdot), \ldots, \psi_N(\cdot) \}
\]

in the variational form (3). Our goal is to find \( \delta \phi(t) \) and \( \delta u_N(\cdot,t) \in \mathcal{N} \) satisfying (3) for each admissible variation \( \delta \phi(t) \) and \( \delta u_N(\cdot,t) \in \mathcal{N} \); cf. (Donea and Huerta, 2003). Let \( (q_1(t), \ldots, q_N(t)) \) be the coordinates of \( u_N(\cdot,t) \) with respect to \( (\psi_1(\cdot), \ldots, \psi_N(\cdot)) \):

\[
u_N(x,t) = \sum_{k=1}^N \psi_k(x)q_k(t).
\]

By substituting \( u_N \) into (3) and exploiting Proposition 1, we get the Galerkin approximation of (5) as follows:

\[
\dot{\phi} = v, \quad v \in \mathbb{R},
\]

\[
\dot{q}_k = -\omega_k^2q_k + \left( q_k - \sum_{p=1}^N a_{kp}q_p \right) \phi^2 - b_k v, \quad k = 1, 2, \ldots, N,
\]

where \( v \) is the control, \( (\phi, \dot{\phi}, q_1, \dot{q}_1, \ldots, q_N, \dot{q}_N) \) is the state,

\[
a_{kp} = \frac{\langle x, \psi_k \rangle_X}{\| \psi_k \|_X^2}, \quad b_k = \frac{\langle x, \psi_k \rangle_X}{\| \psi_k \|_X^2} \| \psi_k \|_X^2 = (\psi_k, \psi_k)_X.
\]

The control torque \( M \) is related to \( v \) in (11) by means of the feedback transformation (4) with \( u = u_N(x,t) \).

5. STABILIZATION OF THE EQUILIBRIUM

The main result we shall prove is the following

**Proposition 2.** Let \( (\psi_1, \ldots, \psi_N) \) be forms corresponding to frequencies \( \omega_1 < \omega_2 < \ldots < \omega_N \), and let, moreover, \( \omega_1 > 0 \). Then there exists a smooth feedback control \( v = \gamma(\phi, \dot{\phi}, q_1, \dot{q}_1, \ldots, q_N, \dot{q}_N) \) ensuring global asymptotic stability of the origin for (11).

**Proof.** Consider the following energy-based Lyapunov function candidate for (11):

\[
2V_N(\phi, \dot{\phi}, q_1, \dot{q}_1, \ldots, q_N) = k_1\phi^2 + k_2\dot{\phi}^2 + \sum_{j=1}^N \omega_j^2\| \psi_j \|_X^2 q_j^2 + 2\phi \sum_{j=1}^N \langle x, \psi_j \rangle_X q_j + \left( \| x \|_X^2 + \sum_{j=1}^N \| \psi_j \|_X^2 q_j^2 \right) \left( \sum_{j=1}^N \psi_j(l)q_j \right)^2 \phi^2,
\]

\[ (k_1 > 0, k_2 > 0). \]

To show that \( V_N \) is positive definite, let us first write its quadratic part \( Q_N \) as follows:

\[
2Q_N = k_1\phi^2 + k_2\dot{\phi}^2 + \| \frac{\partial u_N}{\partial t} \|_X^2 + \phi x \|_X^2 + \sum_{j=1}^N \omega_j^2\| \psi_j \|_X^2 q_j^2,
\]

where \( u_N \) is given by (10). We have

\[
V_N = Q_N + \frac{\phi^2 \gamma}{2} \left[ \int_0^t \sum_{j=1}^N \psi_j(x)q_j \right] \rho dx + m \left( \sum_{j=1}^N \psi_j(l)q_j \right)^2.
\]

(12)

It is easy to see that \( Q_N \geq 0 \), \( Q_N \) vanishes only if \( \phi = \dot{\phi} = q_1 = \ldots = q_N = 0 \) and

\[
\| \sum_{j=1}^N \psi_j q_j \|_X = 0.
\]

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Huerta, 2003). Let \( (q_1(t), \ldots, q_N(t)) \) be the coordinates of \( u_N(\cdot,t) \) with respect to \( (\psi_1(\cdot), \ldots, \psi_N(\cdot)) \):
But the above expression implies $q_1 = \ldots = q_N = 0$ since the forms $\langle \psi_1, \ldots, \psi_N \rangle$ are linearly independent on $[0, l]$. Therefore, the quadratic form $Q_N$ is positive definite. This fact together with (12) implies positive definiteness of $V_N$ and compactness of the level sets

$$L_c = \{(\phi, \phi, q_1, q_1, \ldots, q_N, q_N) \in \mathbb{R}^{2N^2+2} | V_N \leq c\}$$

for each constant $c > 0$. The time-derivative of $V_N$ with respect to the open-loop system (11) takes the form

$$\dot{V}_N = (\alpha_N + \beta_N \nu) \phi,$$

where

$$\alpha_N = k_1 \phi + \sum_{j=1}^{N} \left( 2 \parallel \psi_j \parallel^2 \phi^2 + \langle x, \psi_j \rangle \phi \right) q_j - J \phi \left( \sum_{j=1}^{N} \psi_j(l) q_j \right) \left( \sum_{j=1}^{N} \psi_j(l) \left( 2 q_j + \frac{\langle x, \psi_j \rangle X \phi}{\| \psi_j \|^2} \right) \right)$$

and

$$\beta_N = k_2 + ||x||^2 - J \left( \sum_{j=1}^{N} \psi_j(l) q_j \right)^2 - \sum_{j=1}^{N} \left( \| \psi_j \|^2 q_j^2 - \frac{\langle x, \psi_j \rangle^2 X}{\| \psi_j \|^4} \right) > 0.$$ 

We define a feedback control $v = \gamma(\phi, \phi, q_1, \ldots, q_N, \dot{q}_N)$ by the following expression

$$\gamma(\phi, \phi, q_1, \ldots, q_N, \dot{q}_N) = -\frac{\alpha_N + h \phi}{\beta_N}, \quad (13)$$

where $h > 0$ is a constant. This yields $\dot{V}_N = -h \phi^2 \leq 0$ with respect to the closed-loop system (11), (13). It is easy to show that each positive semi-trajectory of the closed-loop system, restricted to the set

$$Z_0 = \{(\phi, \phi, q_1, q_1, \ldots, q_N, \dot{q}_N) \in \mathbb{R}^{2N^2+2} | V_N = 0\},$$

satisfies the following relations:

$$\phi(t) = \phi_0 = \text{const},$$

$$q_k(t) = A_k \cos(\omega_k t) + B_k \sin(\omega_k t), \quad k = 1, 2, \ldots, N,$$

$$k_1 \phi_0 = \sum_{j=1}^{N} \langle x, \psi_j \rangle \omega_j^2 q_j(t), \quad \text{for all } t \geq 0.$$ 

The above relations imply $\phi_0 = A_1 = B_1 = \ldots = B_N = 0$ since the functions $\cos(\omega_1 t), \ldots, \sin(\omega_0 t)$ are linearly independent on $\mathbb{R}^+$, and $\langle x, \psi_j \rangle X \omega_j \neq 0$ under our assumptions. Therefore, the only solution of the closed-loop system, restricted to $Z_0$ for all $t \geq 0$, is the trivial one. Now global asymptotic stability of the closed-loop system (11), (13) follows from the Barbashin - Krasovskii and LaSalle invariance principle.

6. OBSERVABILITY PROBLEM

In order to implement the feedback law (13) in practice, one has to reconstruct the complete state vector of (11) from the outputs which can be measured.

An observability problem for a rigid body endowed with two elastic beams has been studied in (Kovalev et al., 2002). It has been proved that the finite dimensional body-beams system (without a payload) is observable, provided that one measures the relative displacements of certain points at the beams. However, the values of displacements $u(x, t)$ cannot be directly measured in a real flexible ladder. Instead, there is a sensor located at a certain point $x = \Delta, 0 \leq \Delta \leq l_1$ that allows measurement of $\frac{\partial u}{\partial x} |_{x=\Delta}$.

In the case of a single flexible beam without a payload, an output feedback controller was considered in (Luo and Guo, 1997) with $\Delta = 0$. That approach is not applicable in our case, as one should take into account the motion of a payload and assume $\Delta > 0$ for a turntable ladder.

By replacing $u(x, t)$ with $u_N(x, t)$, we assume that the following output signal is available for the finite dimensional approximation (11):

$$y_1(t) = \frac{\partial^2 u_N(x, t)}{\partial x^2} |_{x=\Delta} = \sum_{k=1}^{N} \psi''_k(\Delta) q_k(t), \quad y_2(t) = \varphi(t). \quad (14)$$

**Proposition 3** Assume that $\psi''_k(\Delta) \neq 0$ for all $k = 1, N$, and $\omega_1 < \omega_2 < \ldots < \omega_N$. Then the linear approximation of (11) around zero is observable with respect to the output (14).

The assertion of Proposition 3 follows from the rank observability condition (Wonham, 1985).

The above result justifies a possibility of applying the feedback control (13) at least for the linearized system. Indeed, if the assumptions of Propositions 2 and 3 are satisfied, there is a dynamic state observer for the linear approximation of (11). Then, by substituting the state estimate from the above observer into the linearized formulae (13) and (4), we can compute the control torque $M$ and conclude about stability of the observer-based linear dynamics; see (Wonham, 1985). Certainly, this does not give a rigorous answer to the question of output stabilizability in the nonlinear case.

7. SIMULATION RESULTS

A simulation is carried out for the closed-loop approximation (11), (13) with $N = 2$ and the following initial conditions:

$$\varphi(0) = \frac{\pi}{2}, \quad \phi(0) = q_1(0) = q_2(0) = q_1(0) = q_2(0) = 0.$$ 

The mechanical parameters are chosen here as in (9), and the control parameters are: $h = 10, k_1 = k_2 = 1$. Figure 4 shows that the feedback proposed is able to steer the approximate system to the origin.

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5 Such a sensor is implemented by a piezoelectric film patch attached to the beam around $x = \Delta$. Neglecting the small thickness of the piezoelectric film, the strain on the film can be considered the same as the train on the surface of the beam. Thus, the charge generated in the film is (approximately) proportional to $\frac{\partial u}{\partial x} |_{x=\Delta}$. 

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In this paper, dynamical equations describing planar motion of a manipulator with an arbitrary number of flexible links, interconnected via elastic joints, have been derived. This model is motivated by the aim to improve dynamical properties of a controlled fire rescue turntable ladder. For an arbitrary order of elastic coordinates, we have proposed a feedback controller that stabilizes the origin of the Galerkin approximation (Proposition 2). We do not study the stabilization problem in infinite dimensions, partial asymptotic stability in the resonance case (Zuyev, 2005), and convergence issues in this paper as they require subtle estimates of the spectrum and nonlinear terms for the evolution equation. For the same reason we do not analyze the control spillover (Balas, 1978), leaving it for future work.

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