EXTENSION OF S-PROCEDURE IN THE
ANALYSIS OF MULTIVARIABLE CONTROL
SYSTEMS

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Abstract: Let \( q(x) \) and \( p_i(x), i = 1, \ldots, m \) be quadratic forms of real variables \( x \in \mathbb{R}^n \). In many problems of investigation of stability and estimation of the
attraction domain and attainability sets, one faces the following question: under
what conditions, do inequalities \( p_i(x) \geq 0, i = 1, \ldots, m, x \neq 0 \) imply the inequality
\( q(x) > 0 \)? The commonly used S-procedure method (Yakubovich, 1977) consists
in checking of whether there exist values \( \tau_i \geq 0 \) such that the quadratic form
\[
q(x) - \sum_{i=1}^{m} \tau_i p_i(x)
\]
is positive definite. It is well known that, if \( m \geq 2 \), the S-procedure gives us only
sufficient conditions for positive definiteness of the quadratic form \( q(x) \) under the
constraints \( p_i(x) \geq 0, i = 1, \ldots, m \). These conditions are necessary only for \( m = 1 \).
This property is called "lossiness" of the S-procedure for multiple constraints.
The use of only sufficient conditions leads to additional conservatism of stability
criteria and attraction domains estimation. Necessary and sufficient conditions are
obtained in (Rapoport, 1989) and (Rapoport, 1996) for a special case of quadratic
constraints represented as products of two linear forms. This paper further extends
those results. The special case of \( m = 2 \), where additional conditions were imposed
on the quadratic forms \( p_1(x) \) and \( p_2(x) \) to make conditions (1) necessary and
sufficient, has been addressed in (Polyak, 1998). In this paper, the losslessness of
the S-procedure for \( m = 2 \) is proved under less restrictive additional conditions.
A case of one general-form quadratic constraint and \( m - 1 \) constraints presented
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1. CONSTRAINTS OF SPECIAL FORM

The problem of positive definiteness of quadratic forms in "sector-like" regions arises in the analysis
of stability of the Lur'e systems, estimation of

\begin{equation}
q(x) - \sum_{i=1}^{m} \tau_i p_i(x)
\end{equation}

attainability sets for the Lur'e systems, and in
other applications. The "sector-like" constraints
are inequalities imposed on the products of two linear forms:
\[
p_i(x) = (f_i^T x)(g_i^T x) \geq 0,
\]
where \( f_i, g_i \in \mathbb{R}^n, i \in 1, \ldots, m \). Let \( F \) and
\( G \) be \( R^{n \times m} \) matrices composed of the columns
\( f_i \) and \( g_i \) respectively. Results presented in the

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following subsection allow us to analyze positive definiteness of \( q(x) \) under constraints (2) using inductive dimensions reduction.

### 1.1 Preliminary results

Let us denote \( \Omega = \{ x \in R^n : p_i(x) \geq 0, i = 1, \ldots, m \} \) and \( \Omega_i = \{ x \in \Omega : p_i(x) = 0 \} \). Note that some of the sets \( \Omega_i \) may be empty. Let \( ^t \) denote transposition. The quadratic form \( q(x) = \frac{1}{2}x^t Q x \) is assumed to be not positive definite in the entire space \( R^n \); i.e., \( q(y) \leq 0 \) for some \( y \in R^n \), \( y \neq 0 \). Otherwise, there is no need to examine problem to investigate positive definiteness of \( q(x) \) in the set \( \Omega \).

**Lemma 1.** Let the quadratic form \( q(x) \) be not positive definite in the entire space \( R^n \); i.e., there exists a vector \( y \in R^n \), \( y \neq 0 \), such that \( q(y) \leq 0 \). Then, \( q(x) > 0 \) in \( \Omega \) if and only if the following conditions hold:

1. \( q(x) > 0 \) in sets \( \Omega_i, i = 1, \ldots, m \),
2. \( y \notin \Omega \).

**Remark 1.** Lemma 1 says that, if \( q(x) > 0 \) on the “faces” \( \Omega_i \) of the set \( \Omega \), then, together with a certain vector \( y \) for which \( q(y) \leq 0 \), the set \( \Omega \) does not contain any other vectors \( z \) with \( q(z) \leq 0 \). In other words, \( y \notin \Omega \) implies \( \{ z : q(z) \leq 0 \} \cap \Omega = \emptyset \).

Let \( \pi(q) \) and \( \pi(p_i) \) be the numbers of positive eigenvalues of the quadratic forms \( q(x) \) and \( p_i(x) \), respectively.

**Lemma 2.** Let \( \pi(q) + \pi(p_j) < n \) for some \( j \). Then, \( q(x) > 0 \) in \( \Omega \) if and only if condition (a) of Lemma 1 holds.

Proofs of Lemmas 1 and 2 are similar to those of two lemmas presented in (Rapoport, 1989).

### 1.2 Lossless extension of the S-procedure

Let \( s(w, u) \) be a quadratic form of \( 2m \) variables \( w, u \in R^{2m} \) of the following form

\[
s(w, u) = \frac{1}{2} w^t V w + w^t T u + \frac{1}{2} u^t U u,
\]

where \( V, T, U \) are \( m \times m \) matrices, with \( V \) and \( U \) being symmetric. The following theorem gives “lossless” extension of the S-procedure to quadratic forms \( p_i(x) \) of form (2). Let \( \Pi = \{(w, u) : w = F^t x, u = G^t x, x \in R^n \} \). The linear space \( \Pi \) coincides with \( R^{2m} \) if and only if \( rank[F \mid G] = 2m \leq n \).

**Theorem 1.** \( q(x) > 0 \) in the region \( \Omega \) defined by quadratic forms (2) if and only if there exists a quadratic form \( s(w, u) \) of form (3) such that the following conditions hold:

(a) \( q(x) - s(F^t x, G^t x) > 0 \) for \( x \neq 0 \)

(b) \( s(w, u) > 0 \) for \( (w, u) \in \Pi \) satisfying conditions \( w_i u_i \geq 0, i = 1, \ldots, m \), and \( (w, u) \neq 0 \).

**Proof.** Sufficiency is obvious. Let us prove necessity. Positive definiteness of \( q(x) \) in the region \( \Omega \) implies its positive definiteness on the subspace \( L = \{ x : F^t x = 0, G^t x = 0 \} \). The case of \( L = \{ 0 \} \) is not excluded and means positive definiteness of \( s(w, u) \) on \( L = \{ 0 \} \). Let \( (w, u) \in \Pi \). Then, the linear algebraic system \( F^t x = w, G^t x = u \) is feasible, and the function \( q(x) \) has a unique minimum under the constraints \( F^t x = w, G^t x = u \). The vector \( x^*(w, u) \) on which the minimum is attained depends linearly on \( w \) and \( u \). Then, \( s_0(w, u) = q(x^*(w, u)) \) is a quadratic form, and \( s_0(w, u) > 0 \) for nonzero \( (w, u) \) satisfying conditions \( (w, u) \in \Pi \) and \( w_i u_i \geq 0 \). Hence, for sufficiently small \( \varepsilon > 0 \), the quadratic form \( s(w, u) = s_0(w, u) - \varepsilon \| w \|^2 + \| u \|^2 \) also satisfies condition (b) of the theorem. Further,

\[
q(x) - s(F^t x, G^t x) = \varepsilon \| F^t x \|^2 + \| G^t x \|^2,
\]

from which inequality (a) follows for \( x \notin L \). If \( x \neq 0 \) and \( x \in L \), then inequality (a) follows from positive definiteness of \( q(x) \) on \( L \). Theorem is proved.

**Remark 2.** If the condition \( rank[F \mid G] = 2m \leq n \) holds, then \( \Pi = R^{2m} \), and condition \( (w, u) \in \Pi \) may be removed.

Let us consider the question of verification of condition (b) of Theorem 1. For the sake of brevity, we suppose that \( rank[F \mid G] = 2m \leq n \). In this case,

\[
\Omega = \{(w, u) \in R^{2m} : p_i(w, u) = w_i u_i \geq 0, i = 1, \ldots, m \}\]

and

\[
\pi(p_i) = 1, i = 1, \ldots, m
\]

Application of Lemma 1 and 2 requires the verification of positive definiteness of \( s(w, u) \) in the regions \( \Omega_i \). This problem has the same form as the original one and differs from it only in that it does not contain the constraint \( i \) and either \( w_i \) or \( u_i \) is equal to zero. In other words,

\[
\Omega_i = \{(w, u) \in R^{2m} : w_j u_j \geq 0, j \neq i, w_i = 0 \} \cup \{(w, u) \in R^{2m} : w_j u_j \geq 0, j \neq i, u_i = 0 \}.
\]

In order to apply Lemmas 1 and 2, let us introduce the notation

\[
M = \{1, \ldots, m\}
\]
and, for three nonintersecting subsets $N_0, N_1, N_2 \subseteq M$ satisfying the condition $N_0 \cup N_1 \cup N_2 = M$,

$$\Omega(N_0, N_1, N_2) = \left\{ (w, u) \in R^{2m} \setminus \{0\} : \begin{array}{ll} w_i u_i \geq 0 & \text{for } i \in N_0, \\
 w_i = 0 & \text{for } i \not\in N_1, \\
u_i = 0 & \text{for } i \not\in N_2. \end{array} \right\}$$

In particular,

$$\Omega = \Omega(M, \varnothing, \varnothing),$$

$$\Omega_i = \Omega(M \setminus \{i\}, \{i\}, \varnothing) \cup \Omega(M \setminus \{i\}, \varnothing, \{i\}).$$

Lemmas 1 and 2 take the following form. Suppose that the quadratic form $s(w, u)$ is not positive definite in the entire space $R^{2m}$ and $(w_0, u_0)$ is a nonzero vector for which $s(w_0, u_0) \leq 0$.

**Lemma 3.** $s(w, u) > 0$ in the region $\Omega$ of form (4) if and only if the following conditions hold:

(a) $s(w, u) > 0$ in the regions $\Omega_i = \Omega(M \setminus \{i\}, \{i\}, \varnothing)$ and $\Omega(M \setminus \{i\}, \varnothing, \{i\}), i \in M,$

(b) $(w_0, u_0) \not\in \Omega.$

**Lemma 4.** Let $\pi(s) < n - 1$. Then, $s(w, u) > 0$ in the region $\Omega$ of form (4) if and only if condition (a) of Lemma 3 holds.

The matrix of the quadratic form $s(w, u)$ is given by

$$S = \begin{bmatrix} V & T \\ T^T & U \end{bmatrix}. \tag{5}$$

For subsets of indices $N_1, N_2 \subseteq M, N_1 \cap N_2 = \varnothing$ let $s(N_1, N_2)$ be the matrix obtained from $S$ of form (5) by deleting the rows and columns with indices $i \in N_1$ and $m + i$ for $i \in N_2$. Analysis of positive definiteness of the quadratic form $s(w, u)$ in the set $\Omega_i$ is equivalent to two problems: analysis of positive definiteness in the set $\Omega(M \setminus \{i\}, \{i\}, \varnothing)$ and analysis of positive definiteness in the set $\Omega(M \setminus \{i\}, \varnothing, \{i\}).$ Both problems have one constant less than the initial problem has $(w_j u_j \geq 0, j \not= i)$ and one variable less. The matrix of the quadratic form of the first problem is obtained from $S$ by striking out the row and the column with the index $i$. The matrix of the quadratic form of the second problem is obtained from $S$ by deleting the row and the column with the index $m + i$.

When reducing the dimension of the quadratic form by one the number of positive eigenvalues either reduces by one or remains unchanged. Consider the inequality $\pi(s) < n - 1$ in the condition of Lemma 4. When eliminating one variable, the number $n - 1$ on the right-hand side of the inequality decreases by one. The number $\pi(s)$ either decreases or remains unchanged. Thus, condition of Lemma 4, which was satisfied for the initial quadratic form, may be violated after reducing the dimensions by one. Hence, together with verifying the condition (a) of Lemma 3, there may arise necessity to verify also condition (b) for some vector $(w_0, u_0)$ for which $s(w_0, u_0) \leq 0$.

On the other hand, suppose that this condition was satisfied when testing positive definiteness of the quadratic form $s(w, u)$ in the set $\Omega(M \setminus \{i\}, \{i\}, \varnothing)$ or $\Omega(M \setminus \{i\}, \varnothing, \{i\})$ by means of Lemma 3. Then, this condition will also be satisfied when verifying positive definiteness $s(w, u)$ in the original set $\Omega$.

Let us call the matrix $S(N_1, N_2)$ minimal if it has only one nonpositive eigenvalue and all matrices $S(N_1 \cup \{i\}, N_2)$ and $S(N_1, N_2 \cup \{i\}),$ obtained from $S(N_1, N_2)$ deleting the row and the column with any of indices $i$ or $m + i$, are positive definite.

For every minimal matrix, one can find a nonzero vector $y$ of the corresponding dimension such that

$$y^T S(N_1, N_2) y \leq 0;$$

i.e., there exists $(\tilde{w}, \tilde{u})$ such that

$$s(\tilde{w}, \tilde{u}) \leq 0, \quad \tilde{w}_j = 0 \text{ for } j \in N_1; \quad \tilde{u}_k = 0 \text{ for } k \in N_2. \tag{6}$$

The above discussion, together with inductive application of Lemmas 3 and 4, lead to the following result.

**Theorem 2.** Let the quadratic form $s(w, u)$ be not positive definite in the entire space $R^{2m}$. Then, $s(w, u)$ is positive definite in the region $\Omega$ of form (4) if and only if the following conditions hold:

(a) every matrix $S(N, M \setminus N)$ is positive definite, $N \subseteq M$,

(b) for every minimal matrix $S(N_1, N_2)$, the corresponding vector $(\tilde{w}, \tilde{u})$ satisfying condition (6) does not belong to the set $\Omega(M \setminus (N_1 \cup N_2), N_1, N_2)$.

**Remark 3.** Condition (a) means positive definiteness of the quadratic form $s(w, u)$ in the regions $\Omega(\varnothing, N, M \setminus N)$, which are subspaces $w_i = 0$ for $i \in N, u_i = 0$ for $i \in M \setminus N$. In other words, this condition means positive definiteness of all $2^m m \times m$ matrices obtained from $S$ given by (5) by deleting the rows and columns with indices $i \in N$ and $m + i$ for $i \in M \setminus N$ for all subsets $N \subseteq M$. This is necessary for positive definiteness of $s(w, u)$ in the region $\Omega$.

**Remark 4.** The case of positive definiteness of the quadratic form $s(w, u)$ is trivial. That is why the matrix $S$ is supposed to be not positive definite. Together with condition (a), this means the existence of minimal matrices.
Theorems 1 and 2 can be applied to the analysis of the Lur'e systems to obtain the results reported in (Rapoport, 1989), (Rapoport, 1996). Another possible application is analysis of attainable sets for linear control systems with several componentwise bounded control inputs.

2. TWO GENERAL FORM CONSTRAINTS

Let $m = 2$. Consider conditions on the quadratic forms $q(x)$, $p_1(x)$, $p_2(x)$ that guarantee losslessness of the S-procedure. This problem has been studied in (Polyak, 1998). We prove a closely related result under less restrictive assumptions using a different method. Namely, condition $n \geq 3$ is removed. Let us denote

$$\Omega_i = \{x \in R^n : p_i(x) \geq 0\} \text{ for } i = 1, 2.$$ 

Then, $\Omega = \Omega_1 \cap \Omega_2$. For any subset $X \subseteq R^n$, let $T(X)$ denote the mapping $R^n \mapsto R^2$ defined by the relations

$$T(X) = \{(\xi, \eta) \in R^2 : \xi = q(x), \eta = p_i(x), \forall x \in X\}.$$ 

The well-known Dines theorem (Yakubovich, 1977) says that $T(R^n)$ is a convex cone. Losslessness of the S-procedure for $m = 1$ follows from just this fact. Let us study the set $T(\Omega_2)$, which is obviously a closed cone.

**Lemma 5.** Let $q(x) > 0$ in the set $\Omega = \{x \in R^n : p_1(x) \geq 0, p_2(x) \geq 0\}$. Then, the cone $T(\Omega_2) \setminus \{0\}$ is linearly connected.

**Proof.** Let $y_1 = (\xi_1, \eta_1) \in T(\Omega_2) \setminus \{0\}$ and $y_2 = (\xi_2, \eta_2) \in T(\Omega_2) \setminus \{0\}$. First, let us assume that $y_1 \neq \alpha y_2$ for all real $\alpha$. Then, the path $y(\tau) = (\tau \xi_1 + (1 - \tau)\eta_1, \tau \eta_1 + (1 - \tau)\eta_2)$ for $\tau \in [0, 1]$. Then, the function $p_2(y(\tau))$ has no roots on the segment $(\bar{\tau}, 1)$. In other words, $\bar{x}(\tau) \in \Omega_2$ for $\tau \in (\bar{\tau}, 1)$. Let

$$x(\tau) = \begin{cases} x(\tau) & \tau \in [0, \bar{\tau}], \\ \bar{x}(\tau) & \tau \in (\bar{\tau}, 1]. \end{cases}$$

Then, $\bar{x}(\tau) \in \Omega_2$. Further, it is continuous at the point $\tau = \bar{\tau}$ because $T(x(\bar{\tau})) = T(\bar{x}) = T(-\bar{x}) = T(\bar{x} + 0)$. Thus, $T(x(\tau)) \in T(\Omega_2)$. Finally, it can be proved that $T(x(\tau)) \subseteq T(\Omega_2) \setminus \{0\}$ in the same way as it was proved in the first case. The lemma is proved.

**Theorem 3.** Let $q(x) > 0$ in $\Omega$. If there exist real values $\lambda$ and $\mu$ such that the quadratic form $q(x) + \lambda p_1(x)$ is positive definite or the quadratic form $q(x) + \mu p_2(x)$ is positive definite, then there exists $\tau_1 \geq 0$ and $\tau_2 \geq 0$ such that the quadratic form

$$q(x) - \tau_1 p_1(x) - \tau_2 p_2(x)$$

is positive definite.

**Proof.** Let us assume that the quadratic form $q(x) + \mu p_1(x)$ is positive definite. The case $q(x) + \mu p_2(x)$ is positive definite will follow from symmetry of the problem setup. The condition $q(x) + \mu p_1(x) > 0$ for $x \neq 0$ implies that $T(R^n) \neq R^2$. In other words, the set $T(R^n)$, which is a convex cone by virtue of the Dines theorem, does not cover the entire plane $R^2$. Being a convex cone, it belongs to some half plane of $R^2$. However, $T(\Omega_2) \subseteq T(R^n)$ and, therefore, $int(T(\Omega_2)) \subseteq T(R^n)$. In the
two-dimensional case, the linearly connected set \( \text{int}(T(\Omega_2)) \) by Lemma 5), which belongs to the half plane of \( R^2 \), is a convex set. Hence, the set \( T(\Omega_2) \) is a convex cone. Thus, the S-procedure is lossless for this case, and there exists \( \tau_1 \geq 0 \) such that

\[
q(x) - \tau_1 p_1(x) > 0 \quad \text{for} \quad x \in \Omega_2.
\]

Applying the S-procedure that is lossless for the case of single constraint \( p_1(x) \geq 0 \), we finally obtain positive definiteness of (8). \( \blacksquare \)

3. THE CASE OF ONE GENERAL FORM CONSTRAINT AND SEVERAL CONSTRAINTS OF SPECIAL FORM

Let us consider the case of \( m+1 \) constraints, where \( p_1(x) \) is a general quadratic form and \( p_i(x) \), \( i = 2, \ldots, m+1 \) are presented in form (2). The condition \( 2m < n \) is also assumed to hold. Following the scheme of the previous section, let

\[
\Omega = \{ x \in R^n : p_i(x) \geq 0, \quad i = 2, \ldots, m+1 \}. \quad (9)
\]

Lemma 6. Let \( q(x) > 0 \) in the set \( \Omega = \{ x \in R^n : p_i(x) \geq 0, \quad i = 1, \ldots, m+1 \} \). Then, the cone \( T(\Omega) \setminus \{ 0 \} \) is linearly connected.

Proof is similar to the proof of Lemma 5, but the path connecting the points \( x_1 \) and \( x_2 \) is constructed as a combination of two segments: \([x_1, x_0]\) and \([x_0, x_2]\), where \( x_0 \neq 0 \) satisfies the linear equations

\[
(f_j^i x_0) = 0 \quad \text{and} \quad (g_j^i x_0) = 0, \quad i = 2, \ldots, m+1.
\]

\( \blacksquare \)

Theorem 4. Let \( q(x) > 0 \) in \( \Omega \). If there exist real values \( \lambda \) and \( \mu \) such that the quadratic form \( \lambda q_1(x) + \mu p_1(x) \) is positive definite, then there exists \( \tau \geq 0 \) such that the quadratic form

\[
q(x) - \tau p_1(x)
\]

is positive definite in the region \( \Omega \) of form (9).

Proof is similar to the proof of Theorem 4.

4. APPLICATION TO THE ANALYSIS OF ABSOLUTE STABILITY OF LUR’E SYSTEMS

Consider the control system

\[
\begin{align*}
\dot{x} &= Ax + Bu + Dr, \\
y &= C' x, \quad z = E' x, \\
u_i(t) &= \varphi_i(u_i(t)), \quad i = 1, \ldots, m,
\end{align*}
\]

where \( x \in R^n \) is state, \( u \in R^m \) - control, \( r \in R^n \) is disturbance, \( y \in R^m \) and \( z \in R^l \) are outputs. Matrices \( A, B, C, D, \) and \( E \) have appropriate dimensions, and matrix \( A \) is Hurwitz. Let \( b_i \) be the \( i \)th column of the matrix \( B \) (\( c_i, d_i \) and \( e_i \)) are defined similarly). Nonlinear functions \( \varphi_i(y) \) satisfy the “sector-like” conditions

\[
0 \leq \eta \varphi_i(y) \leq \mu y^2, \quad y \in R. \quad (12)
\]

Here, \( 0 < \mu_i < \infty \). Absolute stability of the closed system (11), (12) means stability of the origin \( x = 0 \) for all feedback functions \( \varphi_i(y) \) satisfying (12).

Disturbances \( r(\cdot) \) are supposed absent in the absolute stability problem setup. Therefore, we start with the system (11) without variables \( r \) and \( z \). The well-known Lyapunov function

\[
V(x) = \frac{1}{2} x' P x + \sum_{i=1}^{m} \theta_i \int_0^t \varphi_i(y) dy. \quad (13)
\]

is used, where \( P \) is a positive definite matrix and \( \theta_i \) are scalars. The classical Popov criterion was obtained for the case of \( m = 1 \); let the pair \( \{ A, b \} \) be controllable, the pair \( \{ A, c \} \) be observable, and

\[
Re(\omega_0 \theta W_u(\omega) + W_u(\omega)) + \frac{1}{\mu} > 0, \quad \omega \in (-\infty, \infty),
\]

where \( W_u(\omega) = c' (A - \omega I)^{-1} b, \quad i = \sqrt{-1}, \) and \( I \) is the identity matrix. Then, the system (11), (12) is absolutely stable. It is well known that the frequency inequality (14) holds if and only if there exists Lyapunov function (13) with negative definite derivative with respect to system (11), (12):

\[
d\frac{V}{dt} = (Ax + bu)'(Px + bcu) < 0.
\]

These conditions are equivalent for \( m = 1 \). For \( m > 1 \),

\[
\frac{dV}{dt} = (Ax + Bu)'(Px + C \Theta u), \quad (15)
\]

where \( \Theta = \text{diag}(\theta_1, \ldots, \theta_m) \). Condition \( \frac{dV}{dt} < 0 \) leads to the problem described in the abstract, where

\[
q(x, u) = -(Ax + Bu)'(Px + C \Theta u), \quad p_i(x, u) = (c_i' x - \mu_i^{-1} u_i), \quad i = 1, \ldots, m.
\]

Application of the S-procedure gives only sufficient conditions of existence of matrices \( P \) and \( \Theta \) that guarantee \( q(x, u) > 0 \) in the region \( p_i(x, u) \geq 0 \). The use of only sufficient conditions leads to conservative criteria of absolute stability. Necessary and sufficient conditions of existence of the Lyapunov function (13) are obtained in (Rapoport, 1989) and (Rapoport, 1996). Here, these conditions are obtained in the form of feasibility of LMI. Suppose that \( \text{rank}(C) = m \). Then, Theorem 1 takes the following form.

Theorem 5. Let \( \text{rank}(C) = m \leq n \). Then, \( q(x, u) > 0 \) in the region \( \Omega \) defined by the conditions \( (c_i' x - \mu_i^{-1} u_i) u_i \geq 0, \quad i \in M \), if and only if there exists a quadratic form \( \varphi(u, u) \) (3) such that the following conditions hold:
(a) \( q(x, u) - s(C'x - \mu^{-1}u, u) > 0 \) for \((x, u) \neq 0 \).
(b) \( s(w, u) > 0 \) for \( w_iu_i \geq 0, i \in M, (w, u) \neq 0 \).

Let \( \mu = \text{diag}(\mu_1, \ldots, \mu_n) \). To check condition (b) of Theorem 5, Theorem 2 is used. Condition (a) of Theorem 2 requires positive definiteness of all matrices \( S(N_1, N_2) \) for \( N_1 \cap N_2 = \emptyset, N_1 \cup N_2 = M \). To check condition (b) of Theorem 2, one needs to describe all minimal matrices. For every \( i \in M \) and subsets \( N_1 \subseteq M, N_2 \subseteq M \) satisfying the conditions
\[
\{i\} \cup N_1 \cup N_2 = M, \\
N_1 \cap N_2 = \emptyset, \\
i \notin N_1, i \notin N_2 \\
\]
(17)
let us consider the matrix \( S(N_1, N_2) \) and show that, if condition (a) of Theorem 5 holds, then this matrix is minimal. Indeed, \( S(N_1, N_2) \) is the matrix of restriction of the quadratic form \( s(w, u) \) onto the subspace \( w_j = 0, j \in N_1, u_k = 0, k \in N_2 \). Let \( (x, u) \) satisfy the conditions
\[
\begin{align*}
    c_j^T x - \mu_j^{-1}u_j &= 0 \\
    u_k &= 0 \\
\end{align*}
\]
for every \( N \subseteq M \), denote
\[
\begin{align*}
    A(N) &= (A + \sum_{j \in N} \mu_j b_j c_j^T), \\
    P(N) &= (P + \sum_{j \in N} \mu_j \theta_{j} c_j e_j^T). \\
\end{align*}
\]
Then, it follows from condition (a) of Theorem 5 that
\[
\begin{align*}
    s(C'x - \mu^{-1}u, u) < -(A(N_1)x + b_1u_1)^T (P(N_1)x + c_1\theta_1u_1). \\
\end{align*}
\]
(19)
All matrices \( A(N) \) are Hurwitzian. Adding the condition
\[
\begin{align*}
    \tilde{u}_i &= 1, \\
    \tilde{x} &= -A(N_1)^{-1}b_1, \\
\end{align*}
\]
(20)
to conditions (18), we find from (18)-(20) that
\[
\begin{align*}
    \tilde{a}_j &= 0, j \in N_2, \\
    \tilde{a}_i &= 1, \\
    \tilde{a}_k &= \mu_k^{-1}c_k \tilde{x}, k \in N_1, \\
    \tilde{w} &= C'\tilde{x} - \mu^{-1}\tilde{u}, \\
    s(\tilde{a}; \tilde{u}) &< 0, \\
    -\tilde{c}_j^T A(N_1)^{-1} b_j - \mu_j^{-1} &< 0. \\
\end{align*}
\]
(21)
Let \( C(N) \) and \( B(N) \) be matrices consisting of columns \( c_j \) and \( b_j \) of the matrices \( C \) and \( B \), respectively, for \( j \in N \subseteq M \). The diagonal matrix \( \mu(N) \) is constructed similarly. By the Shur lemma,
\[
\begin{align*}
    \det \left[ \begin{array}{cc}
    -A & B(N_1 \cup \{i\}) \\
    C(N_1 \cup \{i\})^T & \mu(N_1 \cup \{i\})^{-1}
    \end{array} \right] \\
    &= \det \left[ \begin{array}{cc}
    -A & B(N_1) \\
    C(N_1)^T & \mu(N_1)^{-1}
    \end{array} \right] \\
    &\times \\
    \left( \begin{array}{cc}
    -[\tilde{c}_j]_{i \in N_1} & [-b_i]_{i \in N_1} \\
    0 & \mu_i^{-1}
    \end{array} \right) \\
    &= \det \left[ \begin{array}{cc}
    -A & B(N_1) \\
    C(N_1)^T & \mu(N_1)^{-1}
    \end{array} \right] \left( \tilde{c}_j A(N_1)^{-1} b_i + \mu_i^{-1} \right). \\
\end{align*}
\]
(22)
On the other hand,
\[
\begin{align*}
    \det \left[ \begin{array}{cc}
    -A & B(N_1) \\
    C(N_1)^T & \mu(N_1)^{-1}
    \end{array} \right] \\
    &= \det(\mu(N_1)^{-1} + C(N_1)^TA^{-1}B(N_1)) \\
    &= \det(\mu(N_1)^{-1} + C(N_1)^TA^{-1}B(N_1)) \\
    \end{align*}
\]
Combining the last identity with (21) and (22), we find that all the expressions
\[
\det(\mu(N)^{-1} + C(N)^TA^{-1}B(N)) \\
\]
must have the same sign for all \( N \subseteq M \). Thus, the following theorem is proved.

**Theorem 6.** Let \( \text{rank}(C) = m \leq n \). Then, \( q(x, u) > 0 \) in the region \( \Omega \) defined by the conditions (16) for \( (x, u) \) if and only if there exists a quadratic form \( s(w, u) \) such that the following conditions hold:
(a) \( q(x, u) - s(C'x - \mu^{-1}u, u) > 0 \) for \( (x, u) \neq 0 \),
(b) every matrix \( S(N, M \setminus N) \) is positive definite for \( N \subseteq M \),
(c) all principal minors of the matrix \( \mu^{-1} + C'A^{-1}B \) have the same sign.

The last theorem reduces the problem of existence of the Lyapunov function (13) to the problem of feasibility of the LMI.

**REFERENCES**