ON THE STABILITY IN ALMOST PERIODIC DISCRETE SYSTEMS

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Abstract: In this paper, the discrete system which is described by difference equations
\[ x_{n+1} = f_n(x_n), \quad f_n(0) = 0, \quad n = 0, 1, 2, \ldots \]
is considered. This system has the trivial (zero) solution \( x_n = 0 \). Sufficient conditions of its asymptotic stability are obtained in the cases when functions \( f_n(x) \) are almost periodic in \( n \).

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1. INTRODUCTION

Difference equations have been studied in various branches of mathematics for a long time. First results in qualitative theory of such systems were obtained by Poincaré and Perron in the end of nineteenth and the beginning of twentieth centuries. The systematic description of the theory of difference equations one can find in (Agarwal, 1992; Elaydi, 1996; Lakshmikantham & Trigiante, 1998). Difference equations are a convenient model for discrete dynamic systems description and for mathematical simulation of systems with impulse effect (Gladilina & Ignatyev, 2003; Halanay & Wexler, 1971; Ignatyev, 2003; Lakshmikantham et al., 1989; Samoilenko & Perestyuk, 1995). One of directions arising from applications of difference equations is linked with qualitative investigation of their solutions (stability, boundedness, controllability, observability, oscillation, robustness...) (Abu-Saris et al., 2002; Agarwal et al., 2003; Bacciotti & Bigio, 2001; Corduneanu, 1995; Győri et al., 1991; Győri & Pituk, 2001; Hooker et al., 1987; Marzulli & Trigiante, 1995).

Consider the discrete system of the form
\[ x_{n+1} = f_n(x_n), \quad f_n(0) = 0 \quad (1.1) \]
where \( n = 0, 1, 2, \ldots \) is the discrete time, \( x_n = (x^n_1, x^n_2, \ldots, x^n_p) \in \mathbb{R}^p, \ f_n = (f^n_1, f^n_2, \ldots, f^n_p) \in \mathbb{R}^p, \ f_n \) satisfy Lipschitz conditions uniformly in \( n \) : \( \|f_n(x) - f_n(y)\| \leq L_r \|x - y\| \) for \( \|x\| \leq r, \ |y| \leq r \).

System (1.1) has the trivial (zero) solution \( x_n \equiv 0 \). \hspace{1cm} (1.2)

Denote \( x_n(n_0, u) \) the solution of system (1.1) coinciding with \( u \) under \( n = n_0 \). We also denote \( B_r = \{x \in \mathbb{R}^p : \|x\| \leq r\} \). Suppose functions \( f_n(x) \) to be defined in \( B_H \) where \( H > 0 \) is some fixed number. According to (Lakshmikantham & Trigiante, 1998) we denote \( \mathbb{Z}_+ \) the set of nonnegative integers.

The sufficient conditions of the stability of solution (1.2) of system (1.1) are obtained in this paper provided that sequences \( \{f_n(x)\} \) are almost periodic for all \( x \in B_H \).

2. MAIN DEFINITIONS AND PRELIMINARIES

By analogy to ordinary differential equations (Hahn, 1967; Rouche et al., 1977; Savchenko & Ignatyev, 1989), let us introduce the following definitions.
Definition 2.1. Solution (1.2) of system (1.1) is said to be stable if for any $\varepsilon > 0$, $n_0 \in \mathbb{Z}_+$ there exists $\delta = \delta(\varepsilon, n_0) > 0$ such that $\|x(n_0)\| \leq \delta$ implies $\|x(n)\| \leq \varepsilon$ for each $n > n_0$.

Definition 2.2. Solution (1.2) of system (1.1) is called attractively if for every $n_0 \in \mathbb{Z}_+$ there exists $\eta = \eta(n_0) > 0$ and for every $\varepsilon > 0$ and $x_{n_0} \in B_\eta$ there exists $\sigma = \sigma(\varepsilon, n_0, x_{n_0}) \in \mathbb{N}$ such that $\|x(n)\| < \varepsilon$ for any $n \geq n_0 + \sigma$. Here $\mathbb{N}$ is the set of natural numbers.

In other words, solution (1.2) of system (1.1) is attractive if
\[ \lim_{n \to \infty} \|x(n(n_0, x_{n_0}))\| = 0. \] (2.1)
for all $n_0 \in \mathbb{Z}_+, x_{n_0} \in B_\eta$.

Definition 2.3. The zero solution of system (1.1) is called equiattractive if for every $n_0 \in \mathbb{Z}_+$ there exists $\eta = \eta(n_0) > 0$ and for every $\varepsilon > 0$ there is $\sigma = \sigma(\varepsilon, n_0) \in \mathbb{N}$ such that $\|x(n_0, x_{n_0})\| < \varepsilon$ for all $x_{n_0} \in B_\eta$ and $n \geq n_0 + \sigma$.

Definition 2.4. The trivial solution (1.2) of system (1.1) is called:
- asymptotically stable if it is stable and attractive;
- equiasymptotically stable if it is stable and equiatractive;

Definition 2.5. (Hahn, 1967; Rouche et al., 1977) Function $r : \mathcal{R}_+ \to \mathcal{R}_+$ belongs to the class of Hahn functions $\mathcal{K}$ ($\sigma \in \mathcal{K}$) if $r$ is continuous increasing function, and $r(0) = 0$.

Definition 2.6. A sequence $\{u_n\}_{n=1}^{\infty}$ is said to be almost periodic if for every $\varepsilon > 0$ there exists $l = l(\varepsilon) \in \mathbb{N}$ such that each segment $[sl, (s + 1)l]$, $s \in \mathbb{Z}$ contains an integer $m$ such that $\|u_{n+m} - u_n\| < \varepsilon$ for all $n \in \mathbb{Z}$. Here $\mathbb{Z}$ is the set of integers. Numbers $m$ with such properties are called $\varepsilon$-almost periods of the sequence $\{u_n\}$.

Definition 2.7. A sequence of functions $\{f_n(x)\}$ is called uniformly almost periodic if for every $\varepsilon > 0$ there exists $l = l(\varepsilon) \in \mathbb{N}$ such that each segment of the form $[sl, (s + 1)l]$, $s \in \mathbb{Z}$ contains an integer $m$ such that $\|f_{n+m}(x) - f_n(x)\| < \varepsilon$ for all $n \in \mathbb{Z}$, $\|x\| < r$.

A.Halanay and D.Wexler (Halanay & Wexler, 1971) proved the following theorems:

Theorem 2.1. Solution (1.2) of system (1.1) is uniformly stable if there exists a sequence of functions $\{V_n(x)\}$, with the next properties:
\[ a(\|x\|) \leq V_n(x) \leq b(\|x\|), \quad a \in \mathcal{K}, \quad b \in \mathcal{K}, \quad n \in \mathbb{Z}_+, \] (2.2)
\[ V_n(x) \geq V_{n+1}(x_{n+1}) \] for every solution $x_n$.

Theorem 2.2. Suppose that there exists a sequence of functions $\{V_n(x)\}$, with properties (2.2) and
\[ V_n+1(x_{n+1}) - V_n(x_n) \leq -c(\|x_n\|), \quad c \in \mathcal{K}, \] (2.4)
\[ |V_n(x) - V_n(y)| \leq L\|x - y\|, \quad n \in \mathbb{Z}_+, \quad x \in B_H, \quad y \in B_H, \quad L > 0. \] (2.5)
Then the zero solution of system (1.1) is uniformly asymptotically stable.

In particular case, when system (1.1) is autonomous, i.e. $f_n(x) = f(x)$, the following theorem is valid (Halanay & Wexler, 1971, p.34):

Theorem 2.3. If there exists a continuous function $V(x)$ such that $a(\|x\|) \leq V(x) \leq b(\|x\|)$, $a \in \mathcal{K}$, $b \in \mathcal{K}$, and
\[ V(x_{n+1}) - V(x_n) \leq 0 \] (2.6)
for every nonzero solution $x_n$ of system (1.1), and equality sign in (2.6) holds in some set which does not contain entire semitrajectories, then solution (1.2) of system (1.1) is asymptotically stable.

The purpose of this paper is to obtain conditions of asymptotic stability of solution (1.2) of system (1.1) assuming that sequences $\{f_n(x)\}$ are almost periodic.

3. STABILITY IN ALMOST PERIODIC SYSTEMS

Definition 3.1. The sequence of numbers $\{u_k\}_{k=1}^{\infty}$ is called finally nonzero if for any natural number $M$ there exists $k > M$ such that $u_k \neq 0$.

Lemma 3.1. (Halanay & Wexler, 1971, p.125) Let sequences $\{u_n^1\}$, $\{u_n^2\}$, ..., $\{u_n^M\}$ be almost periodic. Then for every $\varepsilon > 0$ there exists $l = l(\varepsilon) \in \mathbb{N}$ such that each segment of the form $[sl, (s + 1)l]$, $s \in \mathbb{Z}$ contains at least one $\varepsilon$-almost period, common for all these sequences.

Lemma 3.2. If for every $x \in B_H$, a sequence $\{F_n(x)\}$ is almost periodic, and each function $F_n(x)$ satisfies Lipschitz condition uniformly in $n \in \mathbb{Z}$, $x \in B_H$, then this sequence is uniformly almost periodic.

Proof. Functions $F_n(x)$ satisfy Lipschitz condition, hence
\[ \|F_n(x) - F_n(y)\| \leq L_1\|x - y\|, \] (3.1)
where $L_1$ is Lipschitz constant. Let $\varepsilon$ be any positive number. $B_H$ is bounded and closed, therefore
it is a compact. It means that there exists a finite set of points \( z_1, \ldots, z_M \) such that \( z_j \in B_H \) \((j = 1, \ldots, M)\), and for any \( x \in B_H \) there exists a natural number \( i \) \((1 \leq i \leq M)\) such that

\[
\|x - z_i\| < \frac{\varepsilon}{3L_1}.
\]

(3.2)

From Lemma 3.1 it follows that there exists \( l = l(\varepsilon) \in \mathbb{N} \) such that every segment \([s\ell, (s+1)\ell], s \in \mathbb{Z}\) contains a number \( m \in \mathbb{Z} \) such that

\[
\|F_n(z_i) - F_{n+m}(z_i)\| < \frac{\varepsilon}{3}
\]

(3.3)

for all \( 1 \leq i \leq M \), \( n \in \mathbb{Z} \).

Now let show that for every \( x \in B_H \), any integer \( m \) satisfying inequality (3.3) is \( \varepsilon \)-almost period of the sequence \( \{F_n(x)\} \). Let \( z_k \) be the same element of the set \( z_1, \ldots, z_M \), for which \( \|x - z_k\| < \varepsilon/(3L_1) \).

Then (3.1)-(3.3) imply

\[
\|F_{n+m}(x) - F_n(x)\| \leq \|F_{n+m}(x) - F_{n+m}(z_k)\| + \|F_{n+m}(z_k) - F_n(z_k)\| + \|F_n(z_k) - F_n(x)\|
\]

\[
< \frac{\varepsilon}{3} + 2L_1 \cdot \frac{\varepsilon}{3L_1} = \varepsilon.
\]

(3.4)

Inequality (3.4) completes the proof of Lemma 3.2.

**Theorem 3.1.** Let a sequence of continuous functions \( \{V_n(x)\} \) satisfy conditions

\[
a(||x||) \leq V_n(x), \quad a \in \mathcal{K}, \quad x \in B_H, \quad V_n(0) = 0,
\]

(3.5)

and for every \( n_0 \in \mathbb{Z}_+ \) there exists \( \Delta(n_0) > 0 \) such that \( \|x_{n_0}\| < \Delta \) implies that the sequence \( \{V_n(x_{n_0})\} \) does not increase and tends to zero. Then the zero solution of system (1.1) is equiattractive.

**Proof.** Pick arbitrary \( \delta = \delta(n_0) \in (0, \Delta) \). According to conditions of the theorem, for any \( \varepsilon > 0 \), \( n_0 \in \mathbb{Z}_+ \), and \( x_{n_0} \in B_\delta \) there exists \( \sigma = \sigma(\varepsilon, n_0, x_{n_0}) \in \mathbb{N} \) such that

\[
V_{n_0+\sigma}(x_{n_0+\sigma}(n_0, x_{n_0})) < \frac{1}{2} \varepsilon.
\]

Because of the continuity of functions \( V_n(x) \) and continuous dependence of solutions on initial data, there exists a neighbourhood \( Q(x_{n_0}) \) of the point \( x_{n_0} \) in which the inequality

\[
V_{n_0+\tau}(x_{n_0+\sigma}(n_0, y)) < \varepsilon \quad \text{for} \quad y \in Q(x_{n_0})
\]

(3.6)

is valid. Since the sequence \( \{V_n\} \) monotonically does not increase along solutions of system (1.1), from (3.6) it follows \( V_n(x_{n_0}) < \varepsilon \) for \( n \geq n_0 + \sigma(\varepsilon, n_0, x_{n_0}) \), \( y \in Q(x_{n_0}) \). So the compact set \( B_\delta \) is covered by the system of neighbourhoods \( \{Q(x_{n_0})\} \) from which, by Heine - Borel’s Lemma, it is possible to select the finite subcovering \( Q_1, \ldots, Q_j \) with corresponding numbers \( \sigma_1, \ldots, \sigma_j \). Let \( \sigma(\varepsilon, n_0) = \max_{j} \sigma_j \) \( (\sigma \) depends only on \( \varepsilon \) and \( n_0) \). Then \( V_n(x_{n_0}(n_0, x_{n_0})) < \varepsilon \) for all \( n \geq n_0 + \sigma(\varepsilon, n_0) \) if \( \|x_{n_0}\| \leq \delta(n_0) \). This inequality proves that solution (1.2) of system (1.1) is equiattractive.

Later on throughout this section, we shall assume that the sequence \( \{f_n(x)\} \) in the right-hand side of system (1.1) is almost periodic for every fixed \( x \in B_H \), and functions \( f_n(x) \) satisfy Lipschitz condition uniformly in \( n \).

**Lemma 3.3.** Consider the solution \( x_n(n_0, x_{n_0}) \) of system (1.1). We suppose that \( x_n(n_0, x_{n_0}) \) belongs to \( B_r \) \((0 < r < H)\) for \( n \geq n_0 \). Let \( \varepsilon_k \) be a monotonically approaching zero sequence of positive numbers, and \( \{m_k\} \) a sequence of \( \varepsilon_k \)-almost periods of \( \{f_n(x)\} \) for every \( \varepsilon_k \). Then the limit relation

\[
\lim_{k \to \infty} \|x_{n_0}(n_0, x^{(k)}) - x_{n_0 + m_k}(n_0, x_{n_0})\| = 0,
\]

(3.7)

holds where \( x^{(k)} = x_{n_0 + m_k}(n_0, x_{n_0}) \), and \( n_k \) is a fixed natural number which is more than \( n_0 \) \((n_k > n_0)\).

**Proof.** Consider solutions

\[
x_n(n_0, x^{(k)})
\]

(3.8)

and

\[
x_n(n_0 + m_k, x^{(k)})
\]

(3.9)

of system (1.1). After \( \Delta n = n_k - n_0 \) steps the point \( x^{(k)} \) passes to \( x_{n_0 + m_k}(n_0, x_{n_0}) \) along solution (3.8), and \( x^{(k)} \) passes to the point \( x_{n_0 + m_k}(n_0 + m_k, x^{(k)}) = x_{n_0 + m_k}(n_0, x_{n_0}) \) along solution (3.9). Solution (3.9) of system (1.1) with initial condition \((n_0 + m_k, x^{(k)})\) can be interpreted as the solution of the system

\[
x_{n+1} = f_{n+m_k}(x_n)
\]

(3.10)

with initial data \((n_0, x^{(k)})\). The sequence \( \{f_n(x)\} \) is almost periodic, and every function \( f_n(x) \) satisfies Lipschitz condition, hence right-hand sides of (1.1) and (3.10) differ arbitrary small from each other for \( k \) large enough. This implies limit relation (3.7).

**Theorem 3.2.** Suppose that there exists a sequence of functions \( \{V_n(x)\} \) such that

a) for every fixed \( x \in B_H \), the sequence \( \{V_n(x)\} \) is almost periodic;

b) each member \( V_n(x) \) satisfies condition (3.5) and Lipschitz condition uniformly in \( n \);

c) \( V_n(x) \geq V_{n+1}(x_{n+1}) \) along any solution of (1.1);

d) the sequence \( \{V_n(x_n)\} \) is finally nonzero along any nonzero solution of (1.1).

Then the zero solution of system (1.1) is equiasymptotically stable.
Proof. First let us show that solution (1.2) of system (1.1) is stable. Pick arbitrary $\varepsilon \in (0, H)$ and $n_0 \in \mathbb{Z}_+$. Let $\delta = \delta(\varepsilon, n_0) > 0$ be such that $V_{n_0}(x) < \varepsilon(\delta)$ for $x \in B_\delta$. Then
\[
a(||x_n||) \leq V_n(x_n) \leq V_{n_0}(x_{n_0}) < \varepsilon(\delta)
\]
whence we have $\|x_n\| < \varepsilon$ for $n > n_0$.

Now let us show that solution (1.2) is equiattractive. Take arbitrary $x_n, x_0 \in B_\delta$. The sequence $\{V_n(x_n(n_0, x_{n_0}))\}$ monotonically does not increase, therefore there is the limit
\[
\lim_{n \to \infty} V_n(x_n(n_0, x_{n_0})) = \eta \geq 0,
\]
and $V_n(x_n(n_0, x_{n_0})) \geq \eta$ for $n \geq n_0$. Let us show that $\eta = 0$. Suppose the opposite: $\eta > 0$. Consider a monotonically approaching zero sequence of positive numbers $\{\varepsilon_k\}$ where $\varepsilon_1$ is sufficiently small. By lemmas 3.2 and 3.1, for every $\varepsilon_k$ there exists a sequence of $\varepsilon_k$-almost periods $m_{1,1}, m_{1,2}, \ldots, m_{i, k}, \ldots, m_{k,k}, \ldots, (m_i, k < m_{i+1, k}, \lim_{i \to \infty} m_{i, k} = +\infty)$ for sequences $\{f_n(x)\}$ and $\{V_n(x)\}$ such that inequalities
\[
|V_{n+m_i,k}(x) - V_n(x)| < \varepsilon_k,
\]
\[
\|f_{n+m_i,k}(x) - f_n(x)\| < \varepsilon_k
\]
hold for any $n \in \mathbb{Z}$, $x \in B_\delta$. Without loss of generality one can suppose $m_i, k < m_{i+1, k}$ for all $i \in \mathcal{N}, k \in \mathcal{N}$. Designate $m_k = m_{k,k}$.

Consider the sequence $\{x^{(k)}\}$ where $x^{(k)} = x_{n_0+m_k}(n_0, x_{n_0})$ ($k = 1, 2, \ldots$). This sequence is bounded, therefore there exists its subsequence which converges to some point $x^*$. Without loss of generality we suppose that the sequence $\{x^{(k)}\}$ itself converges to $x^*$. The sequence $\{V_n(x)\}$ is almost periodic for every fixed $x \in B_H$, and each function $V_n(x)$ satisfies Lipschitz condition uniformly in $n$ and next conditions:
\[
|V_n(x)| \leq b(||x||), b \in \mathcal{K}, n \in \mathbb{Z}_+ \quad \text{for} \quad x \in B_H;
\]
for any $n \in \mathbb{Z}_+$ and $\delta > 0$ there is $x \in B_\delta$ such that $V_n(x) > 0$;
\[
V_{n+1}(x_{n+1}) \geq V_n(x_n) \quad \text{along any solution} \ x_n;
\]
Then solution (1.2) of system (1.1) is unstable.

Proof. Let $\varepsilon \in (0, H)$ be an arbitrary number. Take any $n_0 \in \mathbb{Z}_+$ and sufficiently small $\delta > 0$. Choose $x_0 \in B_\delta$ by such way that $V_{n_0}(x_{n_0}) > 0$.

From conditions of the theorem, it follows that there exists $n_* > n_0$ ($n_0 \in \mathcal{N}$) such that the inequality
\[
V_{n_*}(x_{n_*}(n_0, x^*)) = \eta_1 \leq \eta\n\]
holds. Functions $f_n(x)$ satisfy Lipschitz condition, hence
\[
\lim_{k \to \infty} \|x_{n_*}(n_0, x^{(k)}) - x_{n_*}(n_0, x^*)\| = 0
\]
because
\[
\lim_{k \to \infty} \|x^{(k)} - x^*\| = 0.
\]
This implies
\[
\lim_{k \to \infty} V_{n_*}(x_{n_*}(n_0, x^{(k)})) = \eta_1. \quad (3.11)
\]
The almost periodicity of the sequence $\{f_n(x)\}$ and limit relation (3.7) imply
\[
\|x_{n_*}(n_0, x^{(k)}) - x_{n_*+m_k}(n_0, x_{n_0})\| \leq \gamma_k \quad (3.12)
\]
where $\gamma_k \to 0$ as $k \to \infty$. Since the sequence $\{V_n\}$ is almost periodic, we have
\[
|V_{n_*}(x) - V_{n_*+m_k}(x)| < \varepsilon_k \quad (3.13)
\]
for every $x \in B_H$, and conditions (3.11), (3.12) imply
\[
|V_{n_*}(x_{n_*+m_k}(n_0, x_{n_0})) - \eta_1| < \xi_k \quad (3.14)
\]
where $\xi_k \to 0$ as $k \to \infty$. From (3.13) it follows
\[
|V_{n_*}(x_{n_*+m_k}(n_0, x_{n_0})) - \varepsilon_k| < \eta_k \quad (3.15)
\]
Inequalities (3.14), (3.15) imply
\[
|V_{n_*+m_k}(x_{n_*+m_k}(n_0, x_{n_0})) - \xi_k| < \xi_k + \varepsilon_k \quad (3.16)
\]
where $\varepsilon_k + \xi_k \to 0$ as $k \to \infty$.

On the other hand
\[
\lim_{k \to \infty} V_{n_*+m_k}(x_{n_*+m_k}(n_0, x_{n_0})) = \eta_1. \quad (3.17)
\]
Inequality (3.16) and limit relation (3.17) are in contradiction to the inequality $\eta_1 \leq \eta$. This contradiction proves that $\eta = 0$, hence, according to Theorem 3.1, we derive that solution (1.2) of system (1.1) is equiasymptotically stable.

Theorem 3.3. Suppose that there exists a sequence of functions $\{V_n(x)\}$ such that for every $x \in B_H$, the sequence $\{V_n(x)\}$ is almost periodic, and each function $V_n(x)$ satisfies Lipschitz condition uniformly in $n$ and next conditions:
\[
|V_n(x)| \leq b(||x||), b \in \mathcal{K}, n \in \mathbb{Z}_+ \quad \text{for} \quad x \in B_H;
\]
for any $n \in \mathbb{Z}_+$ and $\delta > 0$ there is $x \in B_\delta$ such that $V_n(x) > 0$;
\[
V_{n+1}(x_{n+1}) \geq V_n(x_n) \quad \text{along any solution} \ x_n;
\]
Then solution (1.2) of system (1.1) is unstable.
4. EXAMPLES

Example 4.1. Consider the system
\[ x_{n+1} = -y_n \sin(3\sqrt{n}), \quad y_{n+1} = x_n \sin n \]  \hspace{1cm} (4.1)
and the function \( V_n(x_n, y_n) = x_n^2 + y_n^2 \).
\[ V_{n+1}(x_{n+1}, y_{n+1}) - V_n(x_n, y_n) \]
\[ = -\cos^2 n \, x_n^2 = (\cos^2 \sqrt{n}) \, y_n^2. \]  \hspace{1cm} (4.2)
According to Corduneanu (Corduneanu, 1989) for any sufficiently small \( \varepsilon > 0 \) there exists a sequence \( n_1, n_2, \ldots, n_k, \ldots \rightarrow \infty \) such that \( 0 < \cos^2 n_k < \varepsilon, \quad 0 < \cos^2(\sqrt{n_k}) < \varepsilon \) \((k = 1, 2, \ldots)\). This means that there is not a function \( c \in \mathcal{K} \) such that the left-hand side of (4.2) satisfies inequality (2.4), so Theorem 2.2 cannot be applied to this system. System (4.1) is not an autonomous one, therefore Theorem 2.3 cannot be applied to this system. System (4.1) is almost periodic, and right-hand side of (4.2) is negative for each nonzero solution of system (4.1). Hence, according to Theorem 3.2, the zero solution of system (4.1) is equiasymptotically stable.

Example 4.2. Consider the system
\[ x_{n+1} = X_n(x_n, y_n), \quad y_{n+1} = Y_n(x_n, y_n) \]  \hspace{1cm} (4.3)
where
\[ X_n = y_n - x_n^2 y_n (2 - \sin^2 n - \cos^2 \sqrt{2n}), \]
\[ Y_n = x_n + x_n y_n (2 - \sin^2 n - \cos^2 \sqrt{2n}). \]
If we choose \( V_n(x_n, y_n) = x_n^2 + y_n^2 \), then
\[ V_{n+1}(x_{n+1}, y_{n+1}) - V_n(x_n, y_n) \]
\[ = x_n^2 y_n^2 (x_n^2 + y_n^2) (2 - \sin^2 n - \cos^2 \sqrt{2n})^2. \]
By Theorem 3.3, we can state that the zero solution of system (4.3) is unstable.

5. CONCLUSION

In this paper, the discrete system which is described by difference equations
\[ x_{n+1} = f_n(x_n), \quad f_n(0) = 0, \quad n = 0, 1, 2, \ldots \]
is considered. This system has the trivial (zero) solution \( x_n = 0 \). Sufficient conditions of its asymptotic stability are obtained in the cases when functions \( f_n(x) \) are almost periodic in \( n \).

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