A SINGULARLY PERTURBED MODEL FOR ROBUST CONTROL OF LINEAR SINGLE-LINK FLEXIBLE MANIPULATOR

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Abstract: This paper deals with the modelling of a single-link flexible manipulator utilizing the singular perturbation method. Authors’ attention is focused on the robust regulation of the tip-position based on a new modelling approach under the assumption of norm-boundedness of the fast dynamics (deflection modes). In this approach, the deflection modes may be treated as norm-bounded disturbance. Hence, the controller synthesis is performed only for the certain dynamics of the system.

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Keywords: Flexible link Manipulator; disturbance attenuation; singularly perturbed model.

1 INTRODUCTION
Flexible link manipulators are attractive because they avoid the large inertia forces associated with traditional, large-section, rigid-link manipulators. However, the introduction of flexibility and the consequent tendency of the links to oscillate during motion create a control problem for which a very accurate model of the flexible link system is required; see (Cannon and Schmitz, 1984; Hussain, et al., 1998). A new controller design for controlling a single-link flexible manipulator based on variable structure theory has been presented by (Qian and Ma, 1992). Also, a singular perturbation approach in (Siciliano and Book, 1988) has been developed for the control of lightweight flexible-link manipulators. Singularly perturbed systems often occur naturally because of the presence of small parasitic parameters multiplying the time derivatives of some of the system states. Singularly perturbed control systems have been intensively studied for the past three decades; see (Kokotovic, et al., 1986). A popular approach adopted to handle these systems is based on the so-called reduced technique. The composite design based on separate designs for slow and fast subsystems has been systematically reviewed in (Saksena, et al., 1984). Also, the robust stabilization of singularly perturbed systems based on a new modeling approach has been investigated in (Karimi and Yazdanpanah, 2000).

The system under consideration, with slow and fast dynamics is described in the standard singularly perturbed form by

\[ \dot{x}_s = a_{11} x_s + a_{12} x_f + b_1 u_s, \quad x_s(0) = \tilde{\eta} \]

\[ \dot{x}_f = a_{13} x_s + a_{22} x_f + b_2 u_s, \quad x_f(0) = \tilde{\xi} \]

\[ y = C x_f + F x_f \]

where \( a_{11}, a_{12}, a_{13}, a_{22} \in R^{n_s \times n_s}, a_{21}, a_{22} \in R^{n_f \times n_s}, b_1 \in R^{n_s \times k}, b_2 \in R^{n_f \times k}, C \in R^{n_r \times n_s}, F \in R^{n_r \times n_f} \) are the certain matrices and \( x_s = [x_{s_1}, x_{s_2}, \ldots, x_{s_n}]^\top \in R^{n_s} \), \( x_f = [x_{f_1}, x_{f_2}, \ldots, x_{f_m}]^\top \in R^{n_f} \), \( y(t) \in R^{n_r} \) and \( u(t) \in R^k \) represent the state vectors of the slow and fast dynamics and measured output and control input, respectively. Also, \( \tilde{\eta} \) and \( \tilde{\xi} \) are, respectively, the initial states of \( x_s(t) \) and \( x_f(t) \). The singularly perturbed parameter \( \varepsilon \) is nonnegative and always represents the response time of the fast dynamics.
According to (Karimi and Yazdanpanah, 2000), our objective is to view a portion of the fast dynamics as norm-bounded uncertainty. Therefore, we call it unmodeled dynamics. Although the term unmodeled refers to a subsystem whose dynamics are not known, it is used to emphasize that the complete characteristics of this subsystem will not be utilized in the controller synthesis. If this is feasible, then the synthesis has to satisfy the design specifications for only the known dynamics, hereafter referred to as the plant nominal dynamics. The unmodeled dynamics, on the other hand, may be considered as a subsystem that is connected to the plant nominal dynamics.

We showed in (Karimi and Yazdanpanah, 2000) that a portion of the fast dynamics may be treated as norm-bounded uncertainty and the remaining part can be augmented to the slow dynamics. In this view, (1-3) will read:

Nominal system:

\[ E \dot{X} = A_1 X + A_2 v + B_x u \]  
\[ y = C_1 X + D_1 v \]  

Uncertain system:

\[ \varepsilon \dot{v} = A_1 X + A_2 v + B_x u \Delta A \dot{v} + [A_{e_1}, A_{e_2}] Z \]  
\[ y_v = v \]  
\[ Z = C_1 X + D_1 u \Delta \]  

such that

\[ E \Delta \begin{bmatrix} \text{diag} \{ I_{i_1}, \varepsilon I_{i_1} \} & i_1 > 1 \\ \text{I} \end{bmatrix} = \begin{bmatrix} A_{e_1} & A_{e_2} \\ A_{e_2} & A_{e_2} \end{bmatrix}, \]

\[ B_x = \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix}, \]

\[ C_1 = C_{11}, \]

\[ A_{e_1} = \begin{bmatrix} A_{e_1} \\ A_{e_2} \end{bmatrix}, \]

\[ C_2 = C_{21}, \]

In which \( v = (v_1, v_{i_1}, ..., v_n)^T \) is the vector of fast dynamics, which is to be treated as a norm-bounded uncertainty and \( X = (x_1, x_2)^T \) (in which \( x_i = (v_1, v_{i+1}, ..., v_n)^T \) for \( i>1 \)) is the vector of dominant dynamics, where \( i \) is the index of the first state of the uncertainty dynamics. The value of this index will be determined using the algorithm mentioned in (Karimi and Yazdanpanah, 2000). The vectors \( \eta_1 \) and \( \xi \) are the initial states.

The stability problem (\( \varepsilon \)-bound problem) in singularly perturbed systems differs from conventional linear systems, which can be designed as: characterizing an upper bound \( \varepsilon_0 \) of the positive perturbing scalar \( \varepsilon \) such that the stability of a reduced-order system would guarantee the stability of the original full-order system for all \( \varepsilon \in (0, \varepsilon_0) \). Researchers have tried various ways to find either the stability bound \( \varepsilon_0 \) or a less conservative lower bound for \( \varepsilon_0 \), see (Chen and Lin, 1990; Karimi and Yazdanpanah, 2002b; Kokotovic, et al., 1986). Recently, the problem of robust stabilization and disturbance attenuation for a class of uncertain singularly perturbed systems with norm-bounded nonlinear uncertainties has been considered in (Karimi and Yazdanpanah, 2001). Also, the robust stability analysis and stability bound improvement of perturbed parameter (\( \varepsilon \)) in the singularly perturbed systems by using linear fractional transformations and structured singular values approach (\( \mu \)) has been investigated in (Karimi and Yazdanpanah, 2002b).

The references (Karimi and Yazdanpanah, 2002a; Karimi and Yazdanpanah, 2001) present the new results on control synthesis for robust stabilization and robust disturbance attenuation for linear state-delayed singularly perturbed systems with norm-bounded nonlinear uncertainties. The class of plants considered in this paper consists of systems in state-space form with linear nominal parts and norm-bounded nonlinear uncertainties only in the slow state variable.

The proposed methodology in (Karimi and Yazdanpanah, 2000) may be applied to many physical systems. The principle behind the proposed methodology is that the system under control should possess a two-time scale separation, namely, low and high frequency subsystems. The restriction imposed is that the high frequency subsystem should be stable to result in the norm-bounded property. One of the practical applications that fit into this framework is the single-link flexible manipulator, which is studied in this section. In fact the rigid dynamics that characterize the dominant motion of the joints correspond to the low frequency subsystem and the deflection dynamics due to flexibility of the links correspond to the high frequency subsystem. Assuming that all damping including the flexural damping, are positive and nonzero, the high frequency subsystem is stable and hence norm-bounded (Gawronski, 1993). The modelling approach is applicable to linear as well as nonlinear models of single-link flexible manipulators.

2 SINGLE-LINK FLEXIBLE MANIPULATOR DYNAMICS

Only the linearized model of a single-link flexible manipulator in the modal coordinates will be considered. The model is given by (Cannon and Schmitz, 1984)

\[ \dot{X} = AX + Bu \]  
\[ y = CX \]  

with \( X = (x_1^T, x_2^T, ..., x_m^T)^T \), \( x_i = (\theta, \dot{\theta})^T \), \( x_i = (q, \dot{q})^T \) for \( i = 1, 2, ..., m \) and \( A = \text{diag} \{ A_1, A_2, ..., A_m \} \).
Following (Siciliano and Book, 1988), a singularly perturbed model of (9-10) can be obtained as follows, where the singularly perturbed parameter $\varepsilon$ is defined as

$$\varepsilon \Delta = \frac{1}{\min\{|w_i^2|\}}, \quad i = 1, \ldots, \infty$$

Now with defining $q_i = \varepsilon^i \delta_i$, we can reformulate (9-10) to a singularly perturbed system in this form

$$E \dot{x} = \hat{A} x + B u$$

$$y = \hat{C} x$$

with

$$E = \left[ \begin{array}{cc} I_z & \varepsilon I_{r_m} \\ 0 & 0 \end{array} \right], \quad \hat{C} = (C_0, \varepsilon C_1, \ldots, \varepsilon^2 C_m),$$

$$\hat{A} = \text{diag}(\hat{A}_0, \hat{A}_1, \ldots, \hat{A}_m), \quad \hat{A}_i = \left[ \begin{array}{cc} 0 & 1 \\ -\tilde{w}_i^2 & -\xi \tilde{w}_i \end{array} \right],$$

and $\delta_i = \delta_i$.

To apply the modelling approach of (Karimi and Yazdanpanah, 2000) to (12-13), we arrange the state of fast dynamics (12), on the basis of decreasing order of their performance levels. Let

$$T_i = \text{diag}(T_{ii}, \ldots, T_{in})$$

denote the balancing transformation matrix for the fast dynamics, where $T_{ii}$ is the balancing transformation of $i$th fast subsystem (Shahruz and Behtash, 1988) and it is represented by

$$T_{ii} = \frac{1}{\tilde{w}_i^2 \sqrt{1-\xi^2_i}} \left[ \begin{array}{cc} \xi_i & \sqrt{1-\xi^2_i} \\ -\tilde{w}_i & 0 \end{array} \right].$$

Using $T_i$ in (12-13), we obtain the balanced system in this form

$$E \dot{x} = \bar{A} x + \bar{B} u$$

$$y = \bar{C} x$$

where $x = (x_0^T, \tilde{x}_1^T, \ldots, \tilde{x}_m^T)^T$, $\tilde{x}_i = (\delta_i, \tilde{\delta}_i)^T$, $\bar{A} = \text{diag}(\bar{A}_0, \bar{A}_1, \ldots, \bar{A}_m)$, $\bar{B} = [B_0^T, B_1^T, \ldots, B_m^T]^T$, and $\bar{C} = [C_0, \bar{C}_1, \ldots, \bar{C}_m]$. The matrices $\bar{A}_i$, $\bar{B}_i$, and $\bar{C}_i$ are defined as follows:

$$\bar{A}_i = \left[ \begin{array}{cc} -\tilde{\xi}_i & \tilde{w}_i \sqrt{1-\xi_i^2} \\ -\tilde{w}_i \sqrt{1-\xi_i^2} & -\xi_i \tilde{w}_i \end{array} \right],$$

and

$$\bar{C}_i = \frac{\varepsilon^i \phi(i) \tilde{w}_i}{\tilde{w}_i} \left[ \begin{array}{cc} \xi_i & \sqrt{1-\xi_i^2} \end{array} \right].$$

Consequently, for simplicity of design, the first state is taken as the output, i.e., the tip-position. Taking the output as the first state implies that the first row of $\bar{A}$ should be changed (Yazdanpanah, et al., 1997). The state space model in this case is

$$E \varepsilon = A x + B u$$

$$y = C x$$

with

$$\begin{array}{c}
\bar{A} = \bar{A}_0 \\
0 \\
0 \\
\end{array}, \quad \bar{A}_0 = \begin{bmatrix} 0 & L \\
0 & 0 - \alpha \end{bmatrix}, \quad \bar{A} = \text{diag}(\bar{A}_0, \bar{A}_1, \ldots, \bar{A}_m), \\
\bar{B} = [r_1 \ldots r_m]^T, \quad C = [1 \ 0 \ 0 \ldots, 0]^T. \\
\end{array}$$

Now, let $v = (\tilde{x}_1, \ldots, \tilde{x}_m)^T$, $(i \geq 1)$ denote the part of the state that is to be treated as uncertainty corresponding to unmodeled dynamics. The certain dynamics in this setting corresponds to the state vector $X = [y, \theta, \tilde{x}_0, \ldots, \tilde{x}_m]^T$. Therefore, the state-space representation of the system is in this form

$$E \dot{X} = A_x X + A_{x_{x}} v + B_x u$$

$$y_x = C_x X$$

with

$$A_x = \begin{bmatrix} \bar{A}_0 & \bar{A}_1 & \cdots & \bar{A}_m \end{bmatrix}, \quad A_{x_{x}} = \text{diag}(\bar{A}_0, \bar{A}_1, \ldots, \bar{A}_m),$$

$$A_{x_{x}} = \begin{bmatrix} r_1 & \cdots & r_m \\
0 & \cdots & 0 \\
0 & \cdots & 0 \end{bmatrix}, \quad A_x = \text{diag}(\bar{A}_0, \bar{A}_1, \ldots, \bar{A}_m),$$

$$B_x = [B_0^T \ B_1^T \ \cdots \ B_m^T]^T, \quad B_{x_{x}} = [B_0^T \ \cdots \ B_m^T]^T.$$
In the above representation form, the subsystem (21) may play the role of an uncertainty coupled to the certain subsystem (20) provided that the \( H_\infty \) norm of the uncertainty is bounded. The above representation is shown schematically in Figure 1. The plant has two inputs \((v, u)\) and two outputs \((Z, \eta)\). The first input \( y = v \) represents the disturbances to be rejected. The second input is the control input \( u \) that is used for feedback design. The controlled output \( Z \) represents a penalty variable, which may include a tracking error, as well as a cost of the control input, needed to achieve the prescribed goal. The second output is the measurement output that is made on the system. This is used to generate the control input, which in turn is the tool we have to minimize the effect of the exogenous input on the controlled output. A constraint that is imposed is that the mapping from the measurement to the control input should be such that the closed-loop system is internally stable. The effect of the exogenous input on the controlled output after closing the loop is measured in terms of their energies and the worst-case disturbance of the closed-loop system. Our measure is the closed-loop \( H_\infty \) norm, which is simply the \( L_2 \) induced norm. Suppose the objective is to only stabilize the system, i.e., the system has no exogenous input. By virtue of the small gain theorem, if the plant is stable, the overall system would remain stable if the product of the \( L_2 \) gains of the plant and unmodelled dynamics is less than one.

\[ y = v \]

Fig. 1. Partitioning the flexible-link dynamics into two subsystems: plant and unmodelled dynamics.

4 SIMULATION RESULTS

In this section the methodology proposed in this paper is applied to a single-link flexible manipulator of 14th–order with six modes of deflection as the fast dynamics. The objective is to design a regulator so that the tip-position \( x_t(t) \) is robustly regulated to zero. The control design proceeds by utilizing the five steps introduced in (Karimi and Yazdanpanah, 2000). The link parameters as well as the natural modes and the corresponding damping ratios used for design and simulation are given in (Yazdanpanah, et al., 1997). By utilizing the procedure of (Karimi and Yazdanpanah, 2000), the minimum value for \( i \) was found to be four. In other words, three deflection modes are eligible to be considered as uncertainty. By using the relation (11), we obtain \( \epsilon = 0.011 \), also, \( x_i(0) = 1 \) and the initial value of other dynamics are zero. In the mean time, we obtain \( \gamma_1 = 0.00372 \) and \( \gamma_2 = 67.5 \) and \( \gamma_3 = 1 \).

4.1 The state feedback control

Consider \( \gamma = 68 \) and according to Theorem A1, we apply the state feedback controller to nominal system (20). Figure 2, depicts the regulation of tip-position \((x_t(t))\) and other states of the nominal system, also Figure 3, depicts the regulation of uncertainty dynamics \((\Delta)\). The controller has been depicted in Figure 4 and Figure 5 depicts correctness of the attenuation level of uncertainty dynamics on the controlled output.

4.2 Two-Time Scale Sliding-Mode Control

To illustrate the methodology proposed in this paper and its performance we compare it with at least one of the robust control design approaches according to (Alvarez-Gallegos and Silva-Navarro, 1997). Therefore, we consider the singularly perturbed system

\[
\hat{x} = A_\delta x + A_\mu z + B_\mu u, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^r
\]

\[
\epsilon \dot{z} = A_\epsilon z + A_\mu z + B_\mu u, \quad z \in \mathbb{R}^m
\]

where \( A_\delta \) and \( B_\mu (i, j=1,2)\) are non-singular matrices with appropriate dimensions. The slow subsystem and the fast subsystem are given by

\[
\dot{x}_s = A_s x_s + B_\mu u
\]

\[
\frac{d\hat{\eta}}{d\tau} = A_{\hat{\eta}} \hat{\eta} + B_\mu u
\]

where

\[
A_s = A_{s1} - A_{s2} A_{s1}^{-1} A_{s2}, \quad B_\mu = B_1 - A_{s2} A_{s1}^{-1} B_2.
\]

Suppose that both slow and fast subsystems are stabilizable and slow switching surface \( \sigma_s(x_s) = S_s x_s \) and fast switching surface \( \sigma_f(\hat{\eta}) = S_f \hat{\eta} \) with constant matrices \( S_s \) and \( S_f \) are such that

\[
\Re\lambda\{I_s - B_{s1}(S_s B_{s1})^{-1} S_s A_{s1}\} \preceq -c_s < 0
\]

\[
\Re\lambda\{I_{f1} - B_{f1}(S_f B_{f1})^{-1} S_f A_{f1}\} \preceq -c_f < 0.
\]

Then the two-time scale sliding-mode control

\[
u = -G_s z - G_f (z - Hx)
\]

with
\[ G_s = (S, B_s)^{-1}(S, A_s + L_s S_s), \]
\[ G_f = (S, B_f)^{-1}(S, A_f + L_f S_f), \]
\[ H = -A_f^2 (A_s - B_s G_s) \]

stabilizes the singularly perturbed system asymptotically. Select \( L_s \) and \( L_f \) as positive definite matrices such that \( \Re \lambda (L_s) \leq -c_s < 0 \) and \( \Re \lambda (L_f) \leq -c_f < 0 \).

The design parameters for the sliding-mode control are selected as
\[ S_s = \begin{bmatrix} 10^7 & -10^{-4} \end{bmatrix}, \]
\[ S_f = \begin{bmatrix} 2 & -2 & 2 & -2 & 2 & -2 & 2 & -2 & 2 & -2 \end{bmatrix}, \]
\[ L_s = L_f = 1. \]

We apply the two-time scale sliding-mode control to single-link flexible manipulator. Figure 6, depicts the regulation of tip-position \( (x_\text{tip}(t)) \), also Figure 7, depicts the regulation of other states of the system. The controller has been depicted in Figure 8.

5 CONCLUSION
In this paper the modelling of a single-link flexible manipulator utilizing the singular perturbation method was presented. The robust regulation of the tip-position was considered based on a new modelling approach under the assumption of norm-boundedness of the fast dynamics. In this approach, norm-bounded disturbances and their effect on the tip-position are minimized. Hence, the controller synthesis was performed only for the certain dynamics of the singularly perturbed system. In the comparison between the results obtained with state feedback control and two-time scale sliding-mode control, it was observed that the methodology proposed in this paper achieved a desirable performance.
REFERENCES


Appendix

Theorem A1 (Green and Limebeer, 1996)

Under the following assumptions on the nominal system (4-8):

i. $(A_x, B_x, C_x)$ is stabilizable-detectable.

ii. $(A_v, A_w, C)$ is stabilizable-detectable.

iii. Rank of matrix $D_v$ is $k$ and rank of matrix $D_w$ is $r$.

Then, the algebraic Riccati equation

$$A'_x X + X A_x + X (\gamma^2 A_w A'_w - B_u B'_u) X + C'_w C_x = 0$$

has a positive semi-definite solution $X \geq 0$ such that $A_x - (B_x B'_x - \gamma^2 A_w A'_w) X_{ss}$ is asymptotically stable.

Then the control law $u = -B_x X_{ss} \Delta k X$ is stabilizing and satisfies $\|u\|_2 < \gamma$. 