ON FDI FILTERS AND SYSTEM INVERTIBILITY

Addison RIOS-BOLIVAR *,1 Ferenc SZIGETI *
Germain GARCIA **

* Universidad de los Andes - Facultad de Ingeniería
Departamento de Sistemas de Control,
Mérida 5101, Venezuela
email: ilich@ing.ula.ve, szigeti@ing.ula.ve
** LAAS-CNRS
7, Avenue du Colonel Roche
31077, Cedex 4, Toulouse, France
email: garcia@laas.fr

Abstract: In this paper, an approach for fault detection and isolation filter design based on the reconstruction of the fault modes is presented. Input reconstruction and a very closely related concept, the system inversion is addressed to the fault detection and isolation problem. Moreover, the basic concept of that field, the fault detectability, output separability, and the fault detection and isolation, that is, fault diagnosis are also defined in terms of the system inversion. System invertibility can also be characterized as maximal rank condition which allows to prove the existence of FDI filters under minimal possible conditions.

Keywords: Fault Detection and Isolation, System Inversion, Input Observability, Input Reconstruction, Linear Time Invariant System

1. INTRODUCTION

In the context of the analytic redundancy approaches to fault detection and isolation (FDI) problem, the faults are represented as additive inputs, (Chen and Patton, 1999; Douglas and Speyer, 1996; Gertler, 1998; Massoumnia, 1986). The diagnostic models are characterized by failure directions, which are supposed to be well-known, and failure modes, which are unknown functions.

Fault detection and isolation begin with residual generating (detection), and are completed with a logical decision procedure, in order to achieve the diagnosis of the diagnostic problem (Edelmayer, 1996; Iserman, 1984; Jones, 1973; Rios-Bolivar et al., 1999). (Beard, 1971),(Jones, 1973), and principally (Massoumnia, 1986), have developed geometric methods to generate residuals based on observers, in the case of LTI of the diagnostic systems. These approaches are based on the knowledge of the failure directions.

In some cases, it is important to determine the information provided by the failure modes, in order to isolate the causes that give origin to the anomalous operation of the process. For example, if a certain fault appears in an actuator, the origin of that malfunction can have different causes: zero deviation, error of the range of measurement, deviations of the dead area, problems of linearity and hysteresis, etc. Each of these problems can be represented by fault patterns. The reconstruction and evaluation of the failure modes are required in relation of that pattern. Moreover the failure mode reconstruction allows the automatic reconfiguration of the fault patterns and FDI filters, because in the case of more than one
simultaneous fault a hypothesis can be verified: instead of simultaneous faults a new fault occurs, and causes the reconstructed simultaneous faults. That hypothesis can be verified by identification of a system with unknown failure mode as single output and the known reconstructed failure modes as outputs. Therefore, the FDI problem, the basic concepts and the existence of FDI filters are closely related to system invertibility, which is the backbone of this paper. As a starting point, input observability showed in (Hou and Patton, 1998) will be related to system invertibility. The basic concepts as the detectability of one fault, fault separability, the detectability and the invertibility with respect to distinguishing whether the changes of the input of system are reflected as changes at the output. In the classical framework these concepts are treated indirectly, supposing the existence an observer satisfying the mentioned conditions. In our framework obvious necessary conditions to solve the FDI problem are the detectability and the invertibility with respect to failure modes. The sufficiency of these conditions are obtained by construction of the system inverse. The inversion based FDI filters can be obtained using only the first order derivative in the same case when methods based on unknown input observers are working, (Chen and Patton, 1999). Hence, our method is more general.

2. INPUT OBSERVABILITY

In the problem of input reconstruction, the first task consists in evaluating the input observability, distinguishing whether the changes of the input of a dynamic system are reflected as changes at the output.

If a system is input observable, the input reconstruction problem consists in the synthesis of a device or a mechanism which has as input the measured outputs, and it should take place as output a signal that should converge to the observable input. One can easily notice that the input reconstruction problem is closely linked to the problem of system inversion.

Consider a LTI dynamic system with output \( \dot{y}(t) \) and input \( u(t) \):

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t), & x(0) = x_0, \\
y(t) &= Cx(t) + Du(t);
\end{align*}
\]

(1)

with \( x \in \mathbb{R}^n \) is the state, \( y \in \mathbb{R}^p \) is the output, and \( u \in \mathbb{R}^m \) is the unknown input. The matrices \( A, B, C, D \) are of appropriate dimensions.

**Definition 2.1.** Consider the system (1). The input \( u(t) \) is said observable if that input can be distinguished from zero by the output \( y(t) \), i.e., if \( y(t) = 0 \) for \( t \geq 0 \), implies \( u(t) = 0 \) for \( t > 0 \).

Definition 2.1 implies the following simple facts:

1. If the initial state \( x(0) = x_0 = \zeta \) is unobservable, the linear transformation \( \Sigma_{x_0} \) which transfers the input \( u(t) \) to the output \( y(t) \)

   \[
u(t) \xrightarrow{\Sigma_{x_0}} y(t),
\]

   is left invertible transformation.

2. If \( x(0) = x_0 = \zeta \) is not unobservable, then \( y(t) = \Sigma_{x_0}u(t) \neq 0 \).

3. Fixing a known initial condition \( x(0) = x_0 \), the transfer mapping \( \Sigma_{x_0} \) is affine and linear if and only if \( x_0 \) is unobservable. However, input observability implies the left invertibility of \( \Sigma_{x_0} \), when left inverse satisfies the same initial condition.

4. If system (1) is input observable and \( y = 0 \), then \( x_0 \) is unobservable. If \( x_0 \) is unobservable then the input observability implies the left invertibility for the system (1).

In this case \( u \longrightarrow y \) is an application with null kernel, which admits the left inverse for the system (1). This means that for all \( x_0 \in \mathbb{R}^n \), the affine transformation \( u(t) \xrightarrow{\Sigma_{x_0}} y(t) \) of system (1) is left invertible.

However, input observability is a stronger property than invertibility. Indeed, consider the input observable system (1) with states \( x, \bar{x} \), inputs \( u, \bar{u} \), and initial values \( x_0, \bar{x}_0 \), respectively; with the same outputs \( y = \bar{y} \). Then, \( x_0 - \bar{x}_0 \) is unobservable, and \( u = \bar{u} \).

By subtraction of the system equations

\[
\frac{d}{dt}(x - \bar{x}) = A(x - \bar{x}) + B(u - \bar{u}),
\]

\[
0 = C(x - \bar{x}) + D(u - \bar{u});
\]

consequently, \( (u - \bar{u}) = 0 \) according to the Definition 2.1, and

\[
0 = C(x - \bar{x}) = Ce^{At}(x_0 - \bar{x}_0).
\]

are obtained. Therefore, \( (x_0 - \bar{x}_0) \) is unobservable.

However, if the system inversion is restricted to a particular input class, i.e.

\[
U_0 = \left\{ u(t) : \quad u(0) = 0, \bar{u}(0) = 0, \ldots, u^{(n-1)} = 0 \right\}
\]

(2)

then, the left invertibility for the system (1) implies the input observability, i.e., those concepts are equivalent. Indeed, suppose that \( u \in U_0 \); the corresponding output \( y \) is zero. Thus,

\[
0 = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)
\]

(3)

By calculating the successive derivatives for the equation (3), the following equations are obtained:
Consider the orthogonal projection obtained:

\[ 0 = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t) \]

\[ 0 = CA^{(n-1)}e^{At}x_0 + CAb^{(n-2)}Bu(t) + \cdots + CBu^{(n-2)} + \int_0^t CA^{(n-1)}e^{A(t-\tau)}Bu(\tau)d\tau + Du^{(n-1)}(t) \]

Substituting for \( t = 0 \), \( 0 = Cx_0, 0 = CAx_0, \ldots, 0 = CA^{(n-1)}x_0 \); which means that \( x_0 \) is unobservable, and all solutions for the system (1) satisfy the following equation:

\[ y(t) = \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau = 0. \]

Therefore, the invertibility implies that \( u = 0 \), which corresponds to the input observability.

Once identified the observability property for one input, the arising problem is the construction of an observer, which is a dynamic system that has as input the output of the original system and produces as output an estimate of the unknown input of the original system.

The existence of an observer in order to estimate an unknown input is based on a necessary and sufficient condition for the invertibility of systems with known initial condition initial. We can notice that the observer can depend on the derivatives of the output. These derivatives are also required to systems invertibility, (Rios-Bolivar, 2001). Indeed, applying the successive differentiation to the output of the system (1), the following relations are obtained:

\[ y(t) = Cx(t) + D_{00}u(t) \]
\[ \dot{y}(t) = CAx(t) + D_{10}u(t) + D_{11}\dot{u}(t) \]
\[ \vdots \]
\[ y^{(k)}(t) = CA^kx(t) + \sum_{i=0}^k D_{ki}u^{(i)}(t); \]

where the coefficients \( D_{ki} \), depend on \( A, B, C, D \): \( D_{ii} = D \), and \( D_{ki} = CA^{k-i-1}B \).

Consider the orthogonal projection \( P_k \) from \( R^n \oplus \cdots \oplus R^n = R^{(k+1)n} \) onto \( D_k \) given by

\[ D_k = \text{Im}(\begin{pmatrix} \phantom{D_{00}} D_{11} & \cdots & \phantom{D_{10}} 0 \\ \phantom{D_{21}} D_{22} & \cdots & \phantom{D_{10}} 0 \\ \vdots & \cdots & \phantom{D_{10}} 0 \\ D_{kk} & \cdots & \phantom{D_{10}} 0 \end{pmatrix}). \]

Let \( D_k = (D_{00}^T D_{11}^T \cdots D_{kk}^T)^T \). Then, \( D_k \) and the subspaces \( \text{Im}(I - P_k)D_k \) and \( (D_k + \text{Im}D_k)D_k \) are the same.

Consequently, the following theorem can be established, (Rios-Bolivar and Szigeti, 2001):

**Theorem 2.1.** Consider system (1) and a \( k_0 \) such that \( \text{rank}(I - P_{k_0})D_{k_0} \) is constant for all \( k \geq k_0 \).

\[ \text{rank}(I - P_k)D_k = m, \] the number of inputs, if and only if the system (1) is invertible. Consequently, the input \( u \) can be reconstructed.

**Remark 2.1.** Matrix \( (I - P_k)D_k \) is an analytic function of \( A, B, C \), which is a consequence that the orthogonal projection \( P_k \) can be constructed by the Gram-Schmidt orthogonization of column vectors of the matrices

\[ (0, \ldots, 0, D_{11}^T, \ldots, D_{kk}^T)^T. \]

Hence, considering system

\[ \dot{x} = (A - GC)x + (B - GD)u \]
\[ y = Cx + Du, \]

instead of (1), the corresponding matrix function \( G \rightarrow (I - P_k(G))D_k(G) \) will also be analytic matrix valued function. Then, using the maximal rank property of the analytic matrix functions, i.e., the set \( G \),

\[ G = \{ G \in \mathbb{R}^{n \times p} : \text{rank}(I - P_k(G))D_k(G) \text{ is maximal} \}, \]

is an open and dense subset of \( \mathbb{R}^{n \times p} \). Thus, the following theorem is obtained:

**Theorem 2.2.** Suppose that system (1) is invertible, then there exists an open and dense subset \( G \subset \mathbb{R}^{n \times p} \) such that system (4) is also invertible for all \( G \in G \).

Thus, invertibility is a generic property for the parametrized class of systems (4).

### 3. FDI FILTERS AND INPUT RECONSTRUCTION

Consider the diagnostic model

\[ \dot{x}(t) = Ax(t) + Bu(t) + \sum_{i=1}^k L_i \nu_i(t), \quad x(0) = x_0 \]

\[ y(t) = Cx(t) + \sum_{i=1}^k M_i \nu_i(t). \]

- It is supposed that there exists a \( t_0 > 0 \), such that \( \nu_i(t) = 0 \) for all \( i = 1, \ldots, k, t \in [0, t_0] \).
- \( k \) is the number of statistically and functionally significant faults.

The first phase for FDI is the generation of residues. In particular, we can consider a filter based on classical state observers, (Beard, 1971;
Massoumnia, 1986). In this case, the dynamics for the estimation error corresponds to:

\[
\begin{align*}
\dot{e}(t) &= (A - GC)e(t) + \sum_{i=1}^{k} (L_i - GM_i) \nu_i(t) \\
\eta(t) &= Ce(t) + \sum_{i=1}^{k} M_i \nu_i(t); \\
\end{align*}
\]

with \( e(0) = x_0 - \hat{x}_0 \), where \( G \in \mathbb{R}^{n \times p} \) is the observer gain. The gain should be such that the observer is asymptotically stable.

**Definition 3.1.** Consider diagnostic system (5). The i-th fault is detectable if there exists an asymptotically stable observer with gain matrix \( G_i \), such that the i-th error system

\[
\begin{align*}
\dot{e}_i(t) &= (A - G_i C)e(t) + (L_i - GM_i) \nu_i(t) \\
\eta_i(t) &= Ce(t) + M_i \nu_i(t); \\
\end{align*}
\]

is input observable.

This means that faults are inputs observable in the error equation. However, input observability of the error equation, with the failure modes \( \nu_1, \ldots, \nu_k \) as inputs, is equivalent to invertibility, by the generally required hypothesis, that there exists a \( t_0 > 0 \), such that all \( \nu_i(t) = 0 \) if \( 0 < t < t_0 \). Hence the failure modes \( \nu_i \) satisfy the conditions \( \nu_i(0) = 0, \dot{\nu}_i(0) = 0, \ldots, \nu_i^{(n-1)}(0) = 0 \), required in (2).

Combining Theorem 2.2 and the fact that all gains \( G \) with asymptotically stable error equation form an open subset of \( \mathbb{R}^{n \times p} \), the following theorem is obtained, (Ríos-Bolívar, 2001):

**Theorem 3.1.** Suppose the detectability of diagnostic system (5) and the invertibility of the partial system

\[
\dot{x} = Ax + L_i \nu_i, \quad y = Cx + M_i \nu_i. 
\]

Then the i-th fault of system (5) is detectable.

**Proof**

By Theorem 2.2, and the invertibility of (8), there exists a dense and open subset \( \mathbb{G}_i \subset \mathbb{R}^{n \times p} \), such that system defined by (7) is invertible, for all \( G \in \mathbb{G}_i \). Detectability means that there exists a gain matrix \( G \in \mathbb{R}^{n \times p} \), such that system (7) is asymptotically stable, then, the subset of all gain matrices \( G \in \mathbb{R}^{n \times p} \), with asymptotically stable error equation, is a non empty (by detectability) open subset \( \mathbb{G}_st \subset \mathbb{R}^{n \times p} \). Hence, the intersection \( \mathbb{G}_i \cap \mathbb{G}_st \) is non empty. Therefore, there exists a gain matrix, satisfying the statement of Theorem 3.1.

**Definition 3.2.** Consider the diagnostic system (5). We say that the faults are output separable if there exists an observer with gain \( G_i \) and the subspaces \( \mathbb{Y}_1, \mathbb{Y}_2, \ldots, \mathbb{Y}_k \subset \mathbb{R}^{p} \), in the output space, such that, (Massoumnia, 1986):

1. The outputs for the estimation error:

\[
\begin{align*}
\dot{e}_i(t) &= (A - G_i C)e(t) + (L_i - GM_i) \nu_i \\
\eta_i(t) &= Ce(t) + M_i \nu_i(t),
\end{align*}
\]

satisfy \( \eta_i(t) \in \mathbb{Y}_i \) for all \( \nu_i : [0, \infty) \rightarrow \mathbb{R} \), \( t > 0 \).

2. and

\[
\mathbb{Y}_i \bigcap \left( \bigcup_{j \neq i} \mathbb{Y}_j \right) = 0.
\]

Consider now the matrices

\[
L = (L_1 \quad L_2 \quad \ldots \quad L_k) \quad M = (M_1 \quad M_2 \quad \ldots \quad M_k).
\]

Let us consider the faults \( \nu_i \) in a single form, with their corresponding residues \( \eta_i \). The error equation for each one of the faults is described by (7).

Now, detectability of the failure modes \( \nu_i \) are not supposed. Failure mode \( \nu_i \) is unobservable with the observer corresponding to the gain matrix \( G_i \), if error equation satisfies that \( \eta_i = 0 \). The unobservable failure modes \( \nu_i \) form a subspace \( V_{io} \) of the vector space \( V_i \) of the possible failure modes. The factor space \( V_i/V_{io} \) can be identified with a subspace \( V_{io} \subset V_i \) for good spaces. For example, if \( V_i \subset L_2[0, \infty) \), i.e., all \( \nu_i \) are square integrable, then the \( V_{io} = V_i \bigcap V_{io} \perp \) where \( \perp \) means the orthogonal complement of the subspace in question. Then, the following theorem can be proven easily:

**Theorem 3.2.** Diagnostic system (5) is output separable if and only if, there exists a gain matrix \( G_i \) such that the error equation is asymptotically stable, and the error system, restricted to the input set \( V_{io} \times \cdots \times V_{kio} \), is left invertible.

**Remark 3.1.** Output separability is also equivalent to the following statement:

\[
\eta_1 + \eta_2 + \cdots + \eta_k = 0,
\]

which implies that \( \eta_1 = 0, \ldots, \eta_k = 0 \). However, this latter characterization can not be generalized to the nonlinear case.

**Definition 3.3.** Diagnostic system (5) is diagnostics if and only if, there exists a gain matrix \( G_i \),
such that: a) the i-th failure mode $\nu_i$ is detectable, for all $i = 1, \ldots, k$. b) Diagnostic system is output separable.

Combining the previous theorems, the following theorem is easily obtained.

**Theorem 3.3.** Diagnostic system (5) is diagnosable if and only if, there exists a gain matrix $G \in \mathbb{R}^{n \times p}$ such that the error equation (6) is asymptotically stable and invertible.

Using the genericity of the matrix valued analytic functions, similarly to Theorem 3.1, the following theorem can be proven:

**Theorem 3.4.** Suppose that diagnostic system (5) is detectable and

$$\dot{x} = Ax + \sum_{i=1}^{k} L_i \nu_i, \quad y = Cx + \sum_{i=1}^{k} M_i \nu_i \quad (9)$$

is invertible. Then system (5) is diagnosable.

**Remark 3.2.** Instead of the invertibility of (9) the existence of a gain matrix $G_0$, such that the error equation (6), with $G = G_0$, is invertible, without the hypothesis on the error equation stability, can also be supposed, in the Theorem 3.4, which is also a necessary condition under the detectability of (5).

**Remark 3.3.** All of theorems are addressed to the first method of Massoumnia (Massoumnia, 1986). The second method, showed in (Massoumnia and Vander, 1988) can also be treated in the framework of invertibility.

If each of the faults $\nu_i$ generates a residual $\eta_i$, then these faults are detectable.

Establishing a residual $\eta_i$ for each fault $\nu_i$, we can notice that some different faults can activate the same residual. Therefore, if the equivalence class of the outputs is defined $(\eta_1(t), \ldots, \eta_k(t))$ to $\eta(t)$, i.e., the system $(\eta_1(t), \ldots, \eta_k(t)) \mapsto \eta(t)$, which corresponds to:

$$\dot{\epsilon}(t) = (A - GC)\epsilon(t) + (L - GM)\nu(t), \quad \eta(t) = Ce(t) + M\nu(t), \quad (10)$$

where $\nu(t) = (\nu_1^T, \nu_2^T, \ldots, \nu_k^T)^T$. In this case, $\eta(t)$ does not depend on the selection of $\nu_1(t), \ldots, \nu_k(t)$. It depends on its respective outputs $\eta_1(t), \ldots, \eta_k(t)$, then the residuals equals $\eta_i$’s have the same image on $\eta$.

Thus, considering the error equation for the individual faults given in (7), and the output system $(\eta_1(t), \ldots, \eta_k(t)) \mapsto \eta(t)$ described by (10),

Then, the error system is output separable if and only if the system

$$(\eta_1(t), \ldots, \eta_k(t)) \mapsto \eta(t)$$

is left invertible, (Ríos-Bolívar, 2001).

### 4. NUMERICAL EXAMPLE

Consider system

$$\dot{x}(t) = \begin{pmatrix} 0 & -1 & -2 \\ -1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u(t) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \nu_1(t) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \nu_2(t);$$

$$y(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} x(t).$$

In this case, the detectability and separability conditions are satisfied.

The conditions required in order to design an FDI filter based on $(A, C)$-invariance, (Massoumnia, 1986), does not hold, hence the fault separability is impossible, only the fault detectability can be achieved.

However, the fault separability can be achieved by reconstructing the failure modes. Indeed, if $y_1(t) = x_1(t)$ and $y_2(t) = x_3(t)$, are the system outputs, then:

$$\dot{y}_1(t) = -2x_2(t) - 2y_2(t)$$

$$\dot{y}_2(t) = 2y_1(t) + x_2(t) + \nu_2(t).$$

Therefore

$$x_2(t) = -2y_2(t) - \dot{y}_1(t).$$

Thus

$$\nu_2(t) = -2y_1(t) + 2y_2(t) + \dot{y}_1(t) + \dot{y}_2(t).$$

Since

$$\dot{x}_2(t) = -2\dot{y}_2(t) - \dot{y}_1(t) = -y_1(t) + y_2(t) + u(t) + \nu_1(t),$$

then

$$\nu_1(t) = y_1(t) - y_2(t) - u(t) - 2y_2(t) - \dot{y}_1(t).$$

Thus, the faults can be reconstructed from the measured signals.

For illustration purposes, figures 1, 2, and 3 show the simulation results.

The fault $\nu_1(t)$ is activated at time $t = 25s$. The fault $\nu_2(t)$ becomes present from $t = 65s$. For both situations, we can notice that the residuals, that is, the fault modes reconstructed remain very closely to zero until the faults are activated in a single way. So, the faults are detected and isolated.
Fig. 1. The system outputs

Fig. 2. The fault $\nu_1(t)$ (---), and the reconstructed signal $\nu_{1e}(t)$ (- - -).

Fig. 3. The fault $\nu_2(t)$ (---), and the reconstructed signal $\nu_{2e}(t)$ (- - -).

5. CONCLUSIONS

A technique for fault detection and isolation filter design based on reconstruction of the fault modes has been presented.

The conditions for the existence of FDI filter are derived from the invertibility of the system from the faults to outputs, which is a maximal rank condition for analytic matrix valued function.

The fault reconstruction, by means of system inversion, is a less restrictive methodology than the fault detection and isolation filter synthesis based on the geometric approach (sub-space $(A,C)$-invariant).

Noise sensitivity was not considered in this paper, however if is closely related to the asymptotic stability of the inverse system to the error equation.

Neither disturbances was considered. The treatment of the elimination of the effects of disturbances can be achieved by the well-known decoupling methods. We notice that output separability of the disturbance and the failure modes, was characterized by systems inversion in Theorem 3.2, hence the extension of our results seems straightforward.

6. REFERENCES


