SAFETY DRIVING OF THE DUBINS’ CAR

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Abstract: In this paper we consider the problem of safe driving of the Dubins’ car on a straight road where the steering control is in the form \((1 - h)s + hv\) where \(v\) represents a free control and \(s\) is an automatic control preventing the system from undesired behaviors that may result from the action of \(v\). We design the safety control \(s\) steering the car to the center of the road in minimum time which fact implies to consider discontinuous feedbacks, hence to face the problem of defining a solution to a system with a discontinuous right hand side, and to deal with the definition of regular synthesis used in optimal control. We thus consider generalized solutions in Krasowskii and Filippov sense and guarantee that these solutions have a good behavior, i.e., when only \(s\) is acting \((h = 0)\) the generalized solutions coincide with the optimal synthesis trajectories. Finally, we illustrate our strategy by simulations.

Keywords: Cooperative Controls, Differential Inclusions, Optimal Feedbacks.

1. INTRODUCTION

Consider the system:

\[
\dot{x} = F(x) + (1 - h)G_1(x)s + hG_2(x)v \tag{1}
\]

where \(x \in \mathbb{R}^n\), \(F : \mathbb{R}^n \rightarrow \mathbb{R}^n\), \(G_i(x) \in \mathbb{R}^{n \times m_i}\), \(i = 1, 2\), \(h \in [0, 1]\), \(s \in \mathbb{R}^{m_1}\) and \(v \in \mathbb{R}^{m_2}\). Here \(v\) represents a free control chosen by some controller and \(s\) is a safety control that is an automatic control preventing the system from undesired behaviors that may result from the action of \(v\). For example, equation (1) may model a car with a non expert driver choosing the control \(v\), a teacher or an automatic control choosing \(s\), a safety control, and \(h\) to be equal to 1 when the driver is driving properly and equal to 0 in case of danger. We choose the model of Dubins’ car with controlled steering (see section 2.2) and tackle the problem of safe driving of this car on a straight road. Moreover we want to design the safety control \(s\) to steer the car to the center of the road in minimum time. Thus we have to consider discontinuous feedbacks, hence to face the problem of defining a solution to (1) with a discontinuous right hand side, and to deal with the definition of regular synthesis used in optimal control. The difficulty of the problem is not bounded to treat the discontinuities of \(s\). Indeed, even if solutions to the equation \(\dot{x} = s(x)\) are defined (in Caratheodory sense), the presence of the free control \(v\) may provoke non existence, see Example 1. We then consider generalized solutions in Krasowskii and Filippov sense. As a drawback, we should guarantee that these solutions have a good behavior, i.e., when only \(s\) is acting \((h = 0)\) the generalized solutions coincide with the optimal synthesis trajectories. This is not always the case if the feedback \(s\) corresponds to a synthesis in Boltyanskii-Brunovskii sense (see Boltyanskii 1966, Brunovsky 1980)).

We propose a general strategy to deal with the above problems. This consists of three steps. First we give conditions to ensure that the set of generalized solutions for \(s\) have a good behavior. More precisely, when only \(s\) is acting \((h = 0)\) the generalized solutions should coincide with the optimal synthesis trajectories. This is not always the case if the feedback \(s\) corresponds to a synthesis in Boltyanskii-Brunovskii sense (see Boltyanskii 1966, Brunovsky 1980)).

We propose a general strategy to deal with the above problems. This consists of three steps. First we give conditions to ensure that the set of generalized solutions for \(s\) have a good behavior. These conditions are based on a definition of regular stratified feedback, modeled on those given by Boltyanski and Brunovsky and reported in the Appendix.

The second step consists in studying the set of generalized solutions to the system obtained putting the
regular stratified feedback $s$ in equation (1). Finally, the third step amounts to ensuring that the generalized solutions to this complete system do not enter the set of bad configurations. Moreover, checking the strategy by simulations is possible if approximate solutions converge to generalized ones, which is the case because we consider Krasowskii and Filippov solutions. Based on this approach we then propose, in section 4, two regular stratified feedbacks that provide safety driving for the Dubin’s car. The analysis of the two dimensional case, developed in (Mariog and Piccoli 2002), permits to check that generalized solutions remain in the safe zone. The results are tested also by simulations described in the last section 5.

Our problem is similar to that of air traffic management, considered in (Tomlin et al. 1998, Bicchi et al. 1998). Another approach would be to consider $v$ and $s$ in competition, but we want rather to design $s$ and $h$ as feedbacks that should reject only some dangerous configuration towards which $v$ could lead the system. Indeed, the idea is that in some “safe” zone $v$ should act freely either meanwhile learning or because some risky choice can be convenient, e.g. in problems from economics. Even more, in case of danger we would like $s$ to steer the system towards the safe zone minimizing the time or some more complicate cost functional.

The paper is organized as follows. In section 2.1 we state the problem of safety control and highlight the drawbacks in the concept of solution that one has to face when treating this kind of control problem. A description of the safety control problem for the Dubins’ car is given in section 2.2. In section 2.3 we give the definitions of Krasowskii and Filippov admissible solutions for our safety control problem. In section 3.1 we give condition on a feedback, associated to a synthesis, such that the set of Krasowskii (Filippov) solutions coincide with the trajectory of the synthesis. Next in section 3.2 we show that the safety control system admits solutions in the sense given previously and give the definition of safety and stabilizing feedback. Finally in section 3.3 we give further conditions on $h$ ((H1), (H2)) under which a safety (stabilizing) feedback actually is safe (stabilizing) for the system. The problem of convergence for simulations and implementations is also treated. The solution and simulation results for the safety control problem of Dubins’ car are given in section 4 and 5 respectively. The Appendix contains the definition of regular stratified feedback.

2. BASIC DEFINITIONS

2.1 Basic Model for a Safety Control System

In the system (1) we assume that: $F, G_1, G_2$ smooth, $\exists C > 0$ s.t. $||F||, ||G_1||, ||G_2|| \leq C(1 + ||x||)$ and $s, v$ take values in compact sets.

We want $s$ to bring back the system from a bad region $B$ to a safe region $S \subset \mathbb{R}^n$ in the most efficient way that is minimizing the time or a more complicate cost. We assume that $S$ and $B$ are closed, $S \cap B = \emptyset$ and $L = \mathbb{R}^n \setminus (S \cup B) \neq \emptyset$. In order to obtain this behavior automatically we shall design $h$ to be a smooth feedback which is equal to 0 on the bad region $B$, 1 on the safe region $S$ and taking values in $[0, 1)$ in the zone $L$, which we call “learning” zone because in this region the not expert controller $v$ is helped by the “teacher” $s$. Moreover $s$ is an optimal feedback that in general may be not continuous. We thus obtain a system:

$$\dot{x} = F + (1 - h)G_1s(x) + hG_2v(t)$$

(2)

which is comprised of an ODE with an autonomous discontinuous term plus a time dependent smooth (in $x$) one. Hence when $h$ is not constantly equal to 0 or 1 we immediately face the problem of defining a solution as illustrated by the following example.

Example 1. Let $\dot{x} = (1 - h)s + hv$ where $x = (x_1, x_2) \in \mathbb{R}^2$, $h(x) = 1/2$,

$$s(x_1, x_2) = \begin{cases} (1, 0) & \text{if } x_2 \leq 0 \\ (1, -1) & \text{if } x_2 > 0 \end{cases}$$

(3)

and $v(t) = (0, 1/2)$. It is easy to check that there is no Caratheodory solution if we take the initial data on the line $x_2 = 0$ (see figure 1). \hfill $\Box$

2.2 The Dubins’ Car in the Safety Control Model

We introduce in this section the Dubins’ model of a car moving on the plane and the safe driving problem. We consider a car moving only forward on a plane at constant velocity, that we may assume is equal to $u_1$, $|u| \leq 1$, where $u$ represents the control. We assume to have a street described, up to change of coordinates, by the strip $X = \{(x, y) : |y| \leq L\} \subset \mathbb{R}^2$, where $L > 1$. The aim is to stay inside the street without hitting the boundaries. The not expert driver will be modeled by a random control $v$ and the safety control is a feedback again indicated by $s$, so the complete system becomes:
Deﬁnition 2. A Krasowskii (resp. Filippov) solution is a solution to the differential inclusion
\[ \dot{x} \in F + (1 - h)G_1 S_K(x) + h G_2 v(t) \] (4)

We want to use the concept of Krasowskii (Filippov) solutions to deﬁne a solution to our problem (2) with a discontinuous bounded \( v \) and meausrable bounded \( v \).

Deﬁnition 3. Given a bounded measurable control \( v \), a Krasowskii admissible solution to equation (2) is a solution to the differential inclusion

\[ \dot{x} \in F + (1 - h)G_1 S_K(x) + h G_2 v(t) \] (4)

We deﬁne Filippov admissible solutions in the same way by replacing \( S_K \) with \( S_F \) in equations (4) and (5).

3. GENERAL STRATEGY

To construct a safety feedback we use the deﬁnition of admissible Krasowskii (Filippov) solutions. Our strategy is explained by the following scheme.

Step 1. Introduce a deﬁnition of feedback such that the corresponding Krasowskii (Filippov) solutions steer the system to a prescribed set possibly with minimal cost.

Step 2. Construct a safety control \( s \) satisfying the assumptions of Step 1 and steering the system to the safe zone \( S \) possibly with minimal cost. Prove that for every bounded control \( v \) there is a solution to (4).

Step 3. Choose \( h \) and use Step 2, to prove that solutions do not enter the dangerous zone and tend to \( S \). Moreover prove that approximate solutions converge to generalized solutions.

3.1 Step 1: Regular Stratification

Now we give sufﬁcient conditions for good behavior of solutions, based on concepts of regular stratified feedback and stratified solutions which are given in the Appendix and are modeled on the deﬁnitions given by Boltianskii and Brunovsky, see (Boltianskii 1966, Brunovsky 1980).

Deﬁnition 4. The regular stratified feedback \( \Xi = (\mathcal{P}, \mathcal{P}_1, \mathcal{P}_2, \Pi, \Sigma, s) \) is Krasowskii admissible if the following holds:

1. If \( P \) is a cell of type I then
2. If $P$ is of type II then

2.I) For each $x \in P$ let $N(x)$ be the space normal to $T_x P$ then there exists $v \in N(x)$ such that for every sequence $(y_n)$ in $M$ with $y_n \to x$, if $\omega = \lim_n \frac{x - y_n}{x - y_n}$ then $X_M(x) \cdot v > 0$ and $\omega \cdot v \geq 0$.

2.II) For each cell $M$ of type I such that $\partial M \supset P$ and $M \not\in \Sigma(P)$ the assumption 2.III) holds.

The same definition applies to Filippov admissible regular stratified feedback.

Theorem 1. (Marigo and Piccoli 2002) Let $\Xi = (P, P_1, P_2, \Pi, \Sigma, s)$ be a Krasowskii admissible regular stratified feedback then Krasowskii solutions to $s$ (not passing through the origin) coincide with stratified solutions. The same conclusion holds for Filippov solutions.

3.2 Step 2: Krasowskii and Filippov Solutions

Using standard tools of theory of differential inclusions, it is easy to prove that for every fixed bounded measurable control $v$, we have a solution to (4). More precisely we have following:

Proposition 2. If $s$ is bounded and $v$ is measurable and bounded then the set of Krasowskii (Filippov) solutions to equation (4) is not empty.

Using Step 1 we give also the definition of safety and stabilizing feedback for equation (2).

Definition 5. We say that $\Xi$ is a safety feedback, if $\Xi$ is a Krasowskii (Filippov) stratified feedback, all trajectories of $\dot{x} \in F(x) + G_1(x)S_k(x)$ (for Filippov $\dot{x} \in F(x) + G_1(x)S_F(x)$) starting outside $B$, do not enter $B$. If, moreover, the solutions reach $S$ (possibly with minimal cost) we say that $\Xi$ is a stabilizing feedback to $S$.

3.3 Step 3: Safe Evolution

Given a safety feedback, we can choose $h$ in the following way.

(H1) There exists $U(B)$, open neighborhood of $B$, such that $h(x) = 0$ for every $x \in U(B)$.

We are now ready to prove that the set of Krasowskii (Filippov) admissible solutions do not enter the bad zone $B$.

Proposition 3. Assume that $\Xi$ is a safety feedback and $h$ is chosen as in (H1). Then every solution to (5) with initial state $x \not\in B$ does not enter the set $B$.

Proof. Assume by contradiction that $x(\cdot)$ is a solution to (5) with $x(0) \not\in B$ and there exists $t > 0$ such that $x(t) \in B$. Let $\ell$ be the first time such that $x(\ell) \in B$. From (H1) there exists $\delta > 0$ such that and $h(x(t)) = 0$ for every $t \in [\ell - \delta, \ell]$. But then, on the set $[\ell - \delta, \ell]$, $x$ is a solution to $\dot{x} \in F(x) + G_1(x)S_k(x)$, hence it does not reach $B$ at time $\ell$.

Given a stabilizing feedback to $S$, we can choose $h$ in the following way

(H2) There exists $U$, open neighborhood of $cl(\mathbb{R}^n \setminus S)$, such that $h(x) = 0$ for every $x \in U$.

With arguments similar to those used in the proof of the previous proposition we immediately get:

Proposition 4. Assume that $\Xi$ is a stabilizing feedback to $S$ and $h$ is chosen as in (H2). Then every solution to (5) with initial state $x \not\in B$ does not enter the set $B$ and reach $S$ (possibly with minimal cost).

When implementing or simulating the feedback $s$, we have to consider approximations of $s$. Indeed in the first case we have to use some sampling strategy, while in the second we should consider some numerical methods. In both cases we have convergence to Krasowskii solutions as guaranteed by the following theorem (see Theorem 1 of Chapter 1.4 of (Aubin and Cellina 1984)).

Theorem 2. Let $S : \mathbb{R}^n \to \mathbb{R}^n$ be an upper semicontinuous multifunction with compact convex values. Let $x_k, y_k : [a, b] \to \mathbb{R}^n$ be measurable bounded such that for a.e. $t \in [a, b]$ and every $\delta > 0$, there exists $k \equiv k(t, \delta)$ such that for every $k \equiv k$, there exists $\dot{x}, \dot{y}$ satisfying $\dot{y} \in S(\dot{x}), |x_k(t) - \dot{x}| < \delta$ and $|y_k(t) - \dot{y}| < \delta$.

If $x_k$ converges a.e. to $x$ and $y_k$ converges weakly in $L^1([a, b], \mathbb{R}^n)$ to $y$, then for a.e. $t, y(t) \in S(x(t))$.

To simulate the whole system (2) we have to guarantee that when $h$ is small the solutions are close to Krasowskii solutions to $\dot{x} = F(x) + G_1(x)s(x)$. From Theorem 2 we get:

Proposition 5. If $s$ is bounded, as the $L^1$–norm of $v$ tends to zero, Krasowskii solutions to (4) tend to solutions of $\dot{x} = F(x) + (1 - h(x))G_1(x)S_k(x)$. The same conclusion holds for Filippov solutions.
4. SAFETY CONTROL FOR THE DUBINS’ CAR

We take \( h : X \times [-\pi/2, \pi/2] \rightarrow [0, 1] \) smooth such that

\[
h(Z) = \begin{cases} 
1 & \text{if } Z \in S \\
0 & \text{if } Z \in U(B) \\
\delta(Z) & \text{if } Z \in L \setminus U(B)
\end{cases}
\]

where \( \delta(Z) \in (0, 1), U(B) \) is a neighborhood of \( B \) and \( \text{cl}(U(B)) \cap S = \emptyset \).

We propose two choices for \( s \). The first one comes from an optimal synthesis, while the second one is designed from the first one.

Consider the space \((\theta, y)\). Since the safe zone \( S \) surely contains the origin, we can consider the time optimal feedback to the origin. This feedback was described in (Bicchi et al. 2000) and the corresponding trajectories are represented in Figure 2. The optimal trajectories reach the origin with bang-bang controls, that is \( \pm 1 \), and only one switching unless they reach the value \( |\theta| = \pi/2 \). In the latter case we have to use control 0, following a singular trajectory called turnpike, and then reach the origin with control \( \pm 1 \). Two trajectories separate the zone where the feedback is +1 from the zone where it is \(-1\), these are precisely the curves \( y = 1 - \cos(\theta), \theta \in [-\pi/2, 0] \) and \( y = 1 + \cos(\theta), \theta \in [0, \pi/2] \). Analyzing the one dimensional and zero dimensional singularities, according to the classification of (Marigo and Piccoli 2002), we have the following:

**Proposition 6.** The time optimal control to the origin is a stabilizing to \( S \) feedback.

Notice that if we are in the zone where \(|\theta| \geq \pi/4\) and \( \text{sign}(\theta)y < -1 + \cos(\theta) \), then we could let \( s = 0 \). This results in a minor cost of the safe control. Thus we propose a second strategy \( s_A \) whose trajectories are depicted in Figure 3. This strategy has the advantage of not leading the car to the line \( |\theta| = \pi/2 \) and thus being more robust for disturbances. Also in this case we have:

**Proposition 7.** The control \( s_A \) is a stabilizing to \( S \) feedback.

5. SIMULATION RESULTS

In this section we present some simulation results obtained for the suboptimal safety control \( s_A \).

A first simulation tests the good behavior of the stabilizing feedback both in absence and in presence of disturbance (the random control \( \nu \)). In the first picture (Figure 4) we depict two trajectories on the \((\theta, y)\) plane: one, labeled “Teacher”, for the case of only the safe control acting \((h \equiv 0)\) and one, labeled “Driver”, when both the safe control and the free control are acting \((h \equiv 1/2)\). Then we check that the control \( s_A \) is a safe control or stabilizing to \( S \) control, depending on the feedback \( h \). For simplicity we choose the safe zone \( S \) (where \( h \equiv 1 \)) to be \( B(0, 0.3) \) the ball centered at the origin and radius 0.3 and the learning zone \( L \) to be \( B(0, 1) \setminus B(0, 0.3) \), so \( h \equiv 0 \) outside \( B(0, 1) \). In case of stabilizing control \( h \equiv 0 \) outside \( B(0, 0.3) \).

In pictures 5 we show the behaviour, on the \((y, \theta)\)-plane of the whole system under safe control (with \( h \) satisfying condition (H1)) starting from point \((0, 1)\), which is on the boundary of the bad region \( B \). Other simulations, not reported for space and file length constraints, confirm the theoretical results.
To define a regular stratified feedback, for a system \( \dot{x} = f(x, u) \), we first need to recall the definition of Whitney stratified set.

**Definition 6.** Let \( M \) be a subset of \( \mathbb{R}^n \) and assume \( M = \bigcup_{j \in J} M_j \), where \( J \subset \mathbb{N} \) and \( M_j \) are disjoint nonempty connected embedded \( C^1 \) submanifolds of \( \mathbb{R}^n \). Then \( M \) is a Whitney stratified set if the collection \( \mathcal{P} := \{M_j\}_{j \in J} \), called the stratification of \( M \), is locally finite and the following holds.

- If \( M_j \cap \text{cl}(M_k) \neq \emptyset (j \neq k) \) then \( M_k \subset \partial M_j \) and \( \dim(M_k) < \dim(M_j) \).
- Let \( x_n, y_n \in M_j, n \in \mathbb{N}, x_n, y_n \to \hat{x} \in M_k \subset \text{cl}(M_j) \) and denote by \( t_{\hat{x}} \) the straight line in \( \mathbb{R}^n \) containing the segment joining \( x_n \) with \( y_n \). If \( T_{x_n} M_j \to T \) (in the Grassmannian) and \( t_{\hat{x}} \to t \), then \( t \in T \) and \( T_{x_n} M_k \to T \).

The dimension of \( M \) is \( \text{dim}(M) = \max_j \dim(M_j) \).

We restrict to the case of point target assumed to be the origin. For our system we can choose any \( x \in S \) as target.

**Definition 7.** Let \( \Omega \) be an open set containing the origin. A regular stratified feedback on \( \Omega \) is a 6-tuple \( \Xi = (\{\mathcal{P}, P_1, P_2, \Pi, \Sigma, s\}) \) such that

- \( \mathcal{P} \) is a Whitney stratified set with stratification \( \mathcal{P} \setminus \{\emptyset\} \subset \mathcal{P} \). The elements of \( \mathcal{P} \) are called “cells.”
- \( \mathcal{P} \setminus \{\emptyset\} \) is the disjoint union of \( \mathcal{P}_1 \) (the set of “type I cells”) and \( \mathcal{P}_2 \) (the set of “type II cells”),
- \( \Pi : \{P_1\} \to U \) and \( \Pi : \{P_1\} \to \mathcal{P}_1 \) are maps, \( \Sigma : \mathcal{P}_2 \to \mathcal{P}_1 \) is a multifunction, with non empty values, such that the following properties are satisfied:
- \( \mathcal{P}_3.A \) The function \( s \) is of class \( C^1 \) on each cell.
- \( \mathcal{P}_3.B \) If \( P_1 \in \mathcal{P}_1 \) then \( f(x, s(x)) \in T_x P_1 \) (the tangent space to \( P_1 \) at \( x \)) for every \( x \in P_1 \).

In addition, for each \( x \in P_1 \), if we let \( \xi_x \) be the maximally defined solution of the initial value problem

\[
\dot{\xi} = f(\xi, s(\xi)) \quad \xi(0) = x, \quad \xi \in P_1,
\]

and define \( t_x = \sup \text{Dom}(\xi_x) \), then the limit \( \xi_x(t_x) \) exists and belongs to \( \Pi(P_1) \).

\( \mathcal{P}_3.C \) If \( P_2 \in \mathcal{P}_2 \), then for each \( P \in \Sigma(P_2) \) and \( x \in P_2 \) there exists a unique curve \( \xi_x^P : [0, t_x^P] \to \Omega \) such that the restriction of \( \xi_x^P \) to \( [0, t_x^P] \) is a maximally defined integral curve of the vector field \( f(\cdot, s(\cdot)) \) on \( P \), and \( \xi_x^P(0) = x \).

\( \mathcal{P}_3.D \) On every cell \( P \in \mathcal{P}_1 \), \( x \to t_x \) is a continuously differentiable function, and \( (t_x, x) \to \xi_x(t), (t, x) \to u_x(t) \) \( s(\xi_x(t)) \) are continuously differentiable maps on the set

\[ E(P) \quad \{ (t, x) : x \in P, t \in [0, t_x] \} \]

in the sense that they can be prolonged to maps of class \( C^1 \) on some open subset of \( \mathbb{R} \times P \) containing \( E(P) \). If \( P_2 \in \mathcal{P}_2 \) the same holds for every \( t_x^P, \xi_x^P, u_x^P \), with \( P \in \Sigma(P_2) \).

\( \mathcal{P}_3.E \) For every \( x \in \Omega \setminus \{0\} \), if we let \( \tilde{\xi}_x \) denote a curve, starting at \( x \), obtained by piecing together the trajectories on every single cell, then \( \tilde{\xi}_x \) ends at the origin in finite time.

The trajectories \( \tilde{\xi}_x \) of \( \mathcal{P}_3.E \) are called stratified solutions to \( \Xi \).

**REFERENCES**


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Fig. 5. \((\theta, y)\)-trajectory for safety control and initial conditions \((0, 1)\)

**APPENDIX**