PARTIAL STABILIZATION AND GEOMETRIC PROBLEMS OF NONLINEAR CONTROL

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Abstract: The paper addresses problems of analysis and control of spatial behavior of dynamical systems associated with properties of invariance and attractivity of smooth geometric objects (goal sets). A comparative study of the main concepts of geometric theory, coordinating and synergetic approaches is given and the problems of attractivity of invariant sets are considered in their connection with the partial stability analysis. Design procedures, static and dynamic control laws ensuring the desired properties of spatial dynamics are proposed, simplified local conditions of attractivity of the goal sets and partial stabilization of the system are represented.

Keywords: Nonlinear control, attractors, partial stabilization

I. INTRODUCTION

New theoretical and applied problems of nonlinear dynamics concerning a sophisticated spatial behavior of complex systems and anisotropy of the appropriate state and output spaces are connected with invariance and attractivity of nontrivial sets (submanifolds). These problems are a subject of differential geometry, as well as the so-called geometric (Isidori, 1995; Elkin, 1998; Fradkov et al., 1999; Miroshnik et al., 2000) and synergetic (Kolesnikov, 2000) approaches of the modern control theory. Simultaneously, they are closely connected with partial stability when only a part of system variables, their combination or a certain function of state coordinates tends to a desired equilibrium (Vorotnikov, 1998, Fradkov et al., 1999, Miroshnik, 2001).

The paper focuses on the spatial behavior of nonlinear dynamical systems when the required performance is associated with achieving a desired mode of spatial motion prescribed by equations of curves, surfaces or other nontrivial geometric objects of the system’s space. We illustrate how the geometric objects and the peculiarities of spatial motion naturally arise in many problems of nonlinear control. The concepts of invariance and attractivity are considered taking account of their connection with partial stability theory and techniques of partial stabilization. This enable one to establish the relevant local properties and to provide unification of the analysis and control methodologies. Procedures of the design of static and dynamic control laws ensuring the desired properties of spatial dynamics are proposed and simplified local conditions of the controlled attractivity of the goal sets and partial stabilization of the system are represented.

2. COORDINATION and PROBLEMS of SPATIAL DYNAMICS

We consider properties of the composite (MIMO)
Figure 1: Coordinated motion and attractor of $R^2$

\[ \dot{x} = f(x) + g(x)u, \quad y = h(x), \]

\[ y \in R^m \text{ is the state vector, } y = \{y_j\} \in R^m, \text{ is the output vector, and } u = \{u_j\} \in R^n \text{ is the input (control) vector which is produced by a feedback controller.} \]

- An ordinary problem of output stabilization implies finding control actions $u_j$ which provide stability of the system output $y(t)$ with respect to the required value $y = y^*$. It is connected with a simplest attractivity property of the point $y^*$.

- More complex multivariable problems arise when the conventional task is complemented by rules of subsystem (or output) interaction in order to provide identical behavior of the subsystems, synchronization of several (oscillating, chaotic or nonperiodic) processes, orbital stabilization or trajectory (curve-following) motion of dynamical (mechanical) systems (Blekhman et al., 1997, Fradkov et al., 1999; Miroshnik et al., 2000). These rules are usually represented in the form of the coordination (or synchronization) conditions

\[ \varphi_{y_j}(y) = 0, \]

and predetermine the need for coordinating the controlling actions $u_j$ (Fradkov et al., 1999). The conditions (3) are associated with the implicit form of the curve (one-dimensional hypersurface)

\[ Z^*_y = \{y \in Y : \varphi_{y_j}(y) = 0\}, \]

being a one-dimensional invariant set of the system (Fig. 1). In the general case, when $y(0)$ fails to belong to $Z^*_y$, the condition (3) can be satisfied only with time, which is often associated with attractivity of the invariant set $Z^*_y$.

- The general problems of spatial motion (or curve/surface-following) control, concerning a variety of mechanical/robotic systems and their restricted spatial motion in physical (Cartesian) space, are directly defined through the description (3) of multidimensional geometric objects $Z^*_y$ of the space $R^m$ (see Fig. 2) and therefore intrinsically involve the problems of invariance and attractivity (Fradkov et al., 1999).

- At last, synergetic approaches to analysis and control of natural or artificial processes (Haken, 1988; Kolesnikov, 2000) are closely connected with non-trivial spatial dynamics and the concepts of coordination. Sinergeia (Gr.) implies a purposeful collaboration of the elements (subsystems) of a composite system aimed at their coordinated behavior, which is often associated with the system’s self-organization (Haken, 1988). The synergetic control enables one to use intrinsic system properties for reaching the required cooperative performances of the dynamical processes and gives solutions for many nontrivial problems without a considerable external compulsion by using the so-called weak control, i.e. an input of the bounded energy disappearing after the system’s preliminary adjustment. The main concepts of synergistics such as conservation laws, anisotropy of the system space (functional structures) and slaving principal, dissipation and field contraction principle, order parameters, can be formulated by using the terminology of geometric theory of control.

Usually the problems initially stated and considered in the space $R^m$ of the system’s output variables $y_j$ fail to be solved within the framework of input-output representations. In the general case, in order to analyze dynamics of the system with respect to some sets in the output space or find a correct solution of control problem, it is necessary to go over to the system’s state space $R^n$.

Let us consider the nonlinear equation

\[ \varphi(x) = 0, \]

where $\varphi = \{\varphi_i\} (i = 1, 2, \ldots, n - \nu)$ is the smooth $(n - \nu)$-dimensional vector function, $\nu < n$, and analyze the behavior of the system with respect to a goal set $Z^*$ ($\nu$-dimensional hypersurface) defined as

\[ Z^* = \{x \in X : \varphi(x) = 0\}. \]
The class of problems which can be directly formulated by using the descriptions of geometric objects $Z^*$ in the system’s state space includes the problems of disturbance attenuation, robust, tracking, qualitative and optimal control. Here well-known proper subspaces and submanifolds, boundary surfaces, invariant sets and optimal surfaces of $R^n$ are formed by the trajectories $x(t)$ corresponding to satisfactory, "best" or optimal evolution of the system.

In output stabilization problems a zero (or constant) output vector $y$ is often produced by a collection of the system trajectories $x(t)$ in the state space $R^n$ which belong to an invariant hypersurface (4) (zero dynamics submanifold), where the components of the vector-function $\varphi$ are found as

$$\varphi_1(x) = h(x), \varphi_i(x) = L^{i-1}_f h(x).$$

In coordination and more general curve/surface-following problems equation (4) is induced by conditions (3):

$$\varphi_1(x) = \varphi \circ h(x), \varphi_i(x) = L^{i-1}_f \varphi_1(x),$$

and the given set $Z^*_g \subset R^n$ is a projection of a hypersurface $Z^* \subset R^n$.

The phenomena mentioned lead to the necessity for achieving, by using appropriate control actions, special properties of the system to be designed such as invariance and attractivity, or asymptotic stability with respect to the state space hypersurfaces $Z^*$.

### 3. ATTRACTIVITY AND PARTIAL STABILITY

Let us analyze a smooth autonomous nonlinear system of the form

$$\dot{x} = f_\varepsilon(x), \quad (5)$$

where $f_\varepsilon$ is the smooth vector field supposed to be complete in $\mathcal{X}$. Behavior of the system is considered with respect to a connected geometric object $Z^*$ defined by equation (4), where the function $\varphi$ satisfies the local regularity condition: rank $\varphi(x) = n - \nu$, and therefore the $\nu$-dimensional set $Z^*$ is a regular hypersurface, or an embedded submanifold of $\mathcal{X}$ (Isidori, 1995; Fradkov et al., 1999). For the sake of simplicity, we also suppose that (4) is one-sheeted and define the vector of local coordinates $z = \{z_i\} \in Z \subset R^n$ as

$$z = \psi(x), \quad (6)$$

where $\psi$ is a smooth mapping from $Z^*$ to the open simply connected set $\mathcal{Z}$. In order to analyze the motion in the vicinity of $Z^*$, we define a neighborhood of $Z^*$ as the open simply connected set

$$\mathcal{E}(Z^*) = \{x \in \mathcal{X} : \psi(x) \in Z\} \supset Z^*$$

and, for $x \in \mathcal{E}(Z^*)$, introduce the vector $\xi \in \Xi \subset R^{n-\nu}$ as

$$\xi = \varphi(x). \quad (7)$$

**Definition 1.** The set $Z^*$ is called an invariant submanifold of the system (5) when, for all $x_0 \in Z^*$, the solutions $x = x(t, x_0), t \geq 0$, belong to $Z^*$. The invariant set $Z^*$ is called an attracting submanifold of the system (5) when

$$\lim_{t \to \infty} \text{dist}(x(t, x_0), Z^*) = 0 \quad (8)$$

uniformly with respect to $x_0 \in \mathcal{E}(Z^*)$.

**Definition 2.** The point $\xi = 0$ is called a partial equilibrium of the system (5), (7) when, for all $x_0 \in Z^*$ the solutions $\xi(t) = \xi(t, x_0) = \varphi(x(t, x_0)), t \geq 0$, satisfy the identity $\xi(t, x_0) = 0$. The system (5),(7) at the equilibrium point $\xi = 0$ is called partially (uniformly) asymptotically stable when

$$\lim_{t \to \infty} \xi(t, x_0) = 0 \quad (9)$$

uniformly with respect to $x_0 \in \mathcal{E}(Z^*)$.

The concept of partial equilibrium is associated with system invariance: $\xi = 0$ is a partial equilibrium iff for $x_0 \in Z^*$ it holds $x(t) \in Z^*$. However, in the general case, the invariant submanifold $Z^*$ of a partially stable system fails to be an attracting set.

Let us introduce the Jacobian matrix of the mapping (6),(7), i.e.

$$\begin{bmatrix} \Phi(x) \\ \Psi(x) \end{bmatrix} = \begin{bmatrix} \partial \varphi/\partial x \\ \partial \psi/\partial x \end{bmatrix}$$

and find the metric matrix

$$Q(x) = \begin{bmatrix} Q_{\xi} & Q_{\xi z} \\ Q_{\xi z}^T & Q_z \end{bmatrix} = \left( \begin{bmatrix} \Phi & \Psi \end{bmatrix} \begin{bmatrix} \Phi^T & \Psi^T \end{bmatrix} \right)^{-1}, \quad (10)$$

Then (see Fradkov et al., 1999) for small enough $\xi$ one can derive the expression

$$\text{dist}(x, Z^*) = |\xi|_{Q_\xi}^{-1}, \quad (11)$$

where $|\xi|_{Q_\xi}^2 = \xi^T Q_{\xi} \xi$. The latter shows that coincidence of the expressions (8) and (9), as well as of the corresponding attractivity properties, depends on the matrix $Q(x)$. Let us introduce an additional condition of metric regularity.

**Assumption 1.** For all $x \in \mathcal{E}(Z^*)$ it holds

$$q_1^2 I \leq Q_{\xi}^2(x) \leq q_2^2 I, \quad q_2 \geq q_1 > 0. \quad (12)$$

Assumption 1 ensures that the coordinate change (6),(7) does not arise unlimited distortion of the space metrics at least in the vicinity of $Z^*$. Then the attractivity of $Z^*$ can be associated with the local partial stability of the system (5),(7).

Sufficient conditions for attractivity and partial stability of an autonomous system are given by the
Theorem 1. Suppose Assumption 1 holds. If in the set \( \mathcal{E}(Z^*) \subset X \) the system (5) is complete and there exist a smooth function \( V(x) \) and positive definite functions \( v_1(\xi), v_2(\xi), w(\xi) \) such that

\[
v_1(\xi) \leq V(x) \leq v_2(\xi), \quad \dot{V}(x) \leq -w(\xi),
\]

then: (i) the system (5), (7) is partially asymptotically stable; (ii) the set \( Z^* \) is an invariant set and an attractor of the system (5).

Theorem 1 does not give a constructive way for the design of Lyapunov functions, and the problem of simplification of the sufficient conditions for the both cases arises. Therefore we transform the system and reduce the problem to that of stability with respect to part (18) which parameters are generated by the design of Lyapunov functions, and the problem of task-oriented model

\[
\dot{\xi} = f_\xi(\xi, z),
\]

\[
\dot{z} = f_z(\xi, z),
\]

where \( f_\xi = (\Phi f) \circ r(s, e), f_z = (\Psi f) \circ r(s, e) \), and in view of property (14)

\[
f_\xi(z, 0) = 0.
\]

Let us perform partial linearization of the system and, in view of (17), rewrite (15) in the form

\[
\dot{\xi} = A_c(z)\xi,
\]

where \( A_c = \partial f_\xi / \partial \xi \big|_{\xi=0} \), supposing that the higher order terms of appropriate Taylor series obey the standard conditions (Sastry, 1999). The linearized system is represented by the linear nonstationary part (18) which parameters are generated by the nonlinear model (16).

Suppose that for all \( z \in \mathcal{Z} \) the matrix \( A_c(z) \) is bounded and define a function \( \lambda(z) \) such that

\[
\lambda(z) = \max_i \Re \lambda_i \{A_c(z)\} + \Delta,
\]

where \( \Delta > 0 \) is a small number. Then we can choose a Lyapunov-like function \( V(x) \) as

\[
V(x) = \xi^T P(z) \xi,
\]

where the matrix \( P = P^T \) is found as a solution of Lyapunov-like equation

\[
A_c(z)^T P(z) + P(z) A_c(z) = -Q + 2\lambda(z) P(z)
\]

and \( Q = Q^T > 0 \). By using Lyapunov Lemma and Theorem 1, taking into account the equivalence of the system (5) and the model (15)-(16), one can prove the following result.

Theorem 2. Suppose that Assumption 1 holds and in the neighborhood \( \mathcal{E}(Z^*) \) the system (5) is complete and satisfies the property (14). If, additionally, there exists \( \lambda_0 > 0 \) obeying, for all \( z \in \mathcal{Z} \), the inequality

\[
\dot{P}(z) + 2(\lambda(z) + \lambda_0) P(z) \leq 0,
\]

then: (i) the system (5), (7) at the point \( \xi = 0 \) is partially asymptotically stable; (ii) the set \( Z^* \) is an invariant set and an attractor of the system (5).

It should be noted that Assumption 1, playing the principal role for establishing attractivity of the submanifold \( Z^* \), can be omitted when partial stability of the system is solely investigated.

4. STABILIZATION AND SPATIAL MOTION CONTROL

Let us consider the smooth controlled system (1), where \( f \) and \( g \) are the smooth vector fields defined in \( X \), and suppose, for the sake of simplicity, \( u \in R^3 \). The general problems of control is stated as follows.

Control problem. Find the control law \( u = U(x) \) which provides

(i) partial stabilization of the system (1) with respect to the equilibrium point \( \xi = 0 \) (Definition 2),

(ii) achieving attractivity of the submanifold \( Z^* \) (Definition 1).

The necessary condition for the problem solution is \((f,g)\)-invariance of the system (1) with respect to the submanifold \( Z^* \), which implies the existence of a solution \( u = U(x) \) (invariant control) of the algebraic equation

\[
\frac{\partial \phi}{\partial x} f(x) + \frac{\partial \phi}{\partial x} g(x) U(x) = 0
\]

Let us introduce additional hypotheses.

Assumption 2. For all \( x \in \mathcal{Z}^* \) it holds

\[
\frac{\partial \phi}{\partial x} f(x) \in \text{span} \left\{ \frac{\partial \phi}{\partial x} g(x) \right\},
\]

\[
\text{rank} \left( \frac{\partial \phi}{\partial x} g(x) \right) = 1,
\]

\[
\frac{\partial \psi}{\partial x} g(x) = 0.
\]
In a special case, for \( x \in \mathcal{Z}^* \) it holds
\[
\frac{\partial \varphi}{\partial x} f(x) = 0
\]
i.e. \( \mathcal{Z}^* \) is an invariant set of the open loop system. Then equation (22) has the trivial solution \( U(x) = 0 \). Thus, the expression (26) represents a sufficient condition for the weak control (see Section 2).

Considering behavior of the system in the neighborhood \( \mathcal{E}(\mathcal{Z}^*) \), we carry out its transformation into the coordinates \((\xi, z)\). Differentiating equation (6),(7) with respect to time under the assumption (25), we obtain the task-oriented model of the controlled system
\[
\dot{\xi} = f_\xi(\xi, z) + g_\xi(\xi, z)u, \quad \dot{z} = f_z(\xi, z),
\]
where \( f_\xi = (\Phi f)\circ r(z, \xi), g_\xi = (\Phi g)\circ r(z, \xi) \). In view of Assumption 2, for \( x \in \mathcal{Z}^* \) there exists a solution of equation (22), or the function \( U_\xi(z) = U(r(0, z)) \) such that
\[
f_\xi(0, z) + g_\xi(0, z)U_\xi(z) = 0.
\]

A static solution of the control problem is given by the control law
\[
u = U_\xi(z) + \tilde{u},
\]
where \( U_\xi \) is the invariant control and \( \tilde{u} \) is a stabilizing control, satisfying the condition
\[
\lim_{r \to 0} \tilde{u} = 0.
\]

The model (27) under the control (30) takes the form
\[
\dot{\xi} = f_\xi(\xi, z) + g_\xi(\xi, z)\tilde{u},
\]
where \( f_\xi = f_\xi + g_\xi U_\xi \), and in view of the property (29)
\[
f_\xi(0, z) = 0.
\]
Performing partial linearization of the transformed model (32) for small enough \( \xi \) in view of the property (33), we obtain
\[
\dot{\xi} = A(z)\xi + b(z)\tilde{u},
\]
where \( A = \frac{\partial f_\xi}{\partial \xi} |_{\xi=0}, b = g(0) \). The main control problem is reduced to finding a stabilizing control \( \tilde{u} \).

Suppose that for \( z \in \mathcal{Z} \) the matrices \( A(z) \) and \( b(z) \) are bounded and the pair \((A(z), b(z))\) is controllable uniformly in \( z \in \mathcal{Z} \) (the latter follows from the property (24)). Choose \( \tilde{u} \) in the form
\[
\tilde{u} = k(z)\xi,
\]
where \( k(z) \) is the row matrix of varying coefficients defined as
\[
k(z) = -b^T(z)P(z),
\]
\( P = P^T \) is a solution of the algebraic Riccati equation
\[
A^T(z)P(z) + P(z)A(z) - P(z)b(z)b^T(z)P(z) = 2\lambda(z)P(z)
\]
and the function \( \lambda(z) \) satisfies the inequality
\[
\lambda(z) \leq \min_i \text{Re } \lambda_i \{A(z)\} - \Delta,
\]
\( \Delta > 0 \) is a small number. The model (34) takes the form
\[
\dot{\xi} = A_c(z)\xi,
\]
where \( A_c = A + bk = A - b b^T P \). In order to find conditions for partial stability of the closed loop system (39),(28), we can choose a Lyapunov-like function \( V(x) \) of the form (20). Then, by using properties of algebraic Riccati equations and Theorem 1-2, one can prove the following result.

**Theorem 3.** Suppose that Assumption 1-2 hold and the system (28) for all \( z \in \mathcal{Z} \) is complete. If, additionally, there exists \( \lambda_0 > 0 \) obeying, for all \( z \in \mathcal{Z} \), inequality (21) then the nonlinear control law
\[
u = U_\xi \circ \psi(x) + \left(k \circ \psi(x)\right) \varphi(x),
\]
provides (i) partial asymptotic stabilization of the system (1), (7) at the point \( \xi = 0 \); (ii) invariance and attractivity of the set \( \mathcal{Z}^* \) of the system (1).

Consider a dynamic control of the form
\[
u = \tilde{U}_\xi + \tilde{u},
\]
where invariance of the set \( \mathcal{Z}^* \) is achieved by the invariant control \( \tilde{U}_\xi = \tilde{U}_\xi(t) \), the stabilizing control \( \tilde{u} \) is chosen in the form (35) and the matrix of feedback gains \( k(z) \) is computed as above. Let us introduce the error of invariant control:
\[
\tilde{U}_\xi = \tilde{U}_\xi - U_\xi(z),
\]
and perform partial linearization of the system for small enough \( \xi \) and \( \tilde{U}_\xi \). The model (34) takes the form
\[
\dot{\xi} = A_c(z)\xi + b(z)\tilde{U}_\xi,
\]
and the control problem is reduced to finding a law of invariant control which provides asymptotic stability of (43).

In the case considered a standard choice of the Lyapunov function \( V(x, \tilde{U}_\xi) \) leads to the following equation of the invariant control error:
\[
\tilde{U}_\xi = -\gamma \tilde{u} = -\gamma k(z) \xi,
\]
where \( \gamma > 0 \). After simple manipulations one can obtain that asymptotical stability of the transformed
The transformed system (49) is also achieved if for all $z \in \mathcal{Z}$ there exists a number $\lambda_0 > 0$ satisfying (21).

The resulting invariant controls can be derived by using the condition (44). The combined law

$$\hat{U}_\xi = U_\xi(z) - \gamma \int_0^t \hat{u}(\tau)d\tau \tag{45}$$

evidently satisfies the relation (44) and can be used for improving the system accuracy in presence of weak uncertainties of the model and external disturbances. The integral control

$$\hat{U}_\xi = \hat{U}_\xi(0) - \gamma \int_0^t \hat{u}(\tau)d\tau \tag{46}$$

satisfies (44) if $\hat{U}_\xi(z) \equiv 0$.

- Finally, consider a case when the function $U_\xi(z)$ can be parameterized as

$$U_\xi(z) = \theta^T(z)z \tag{47}$$

where $\theta(z)$ is a smooth vector function, and the adaptive control law

$$u = \widehat{\theta}^T z + \hat{u}, \tag{48}$$

where $\widehat{\theta}(z)$ is the estimate of the function $\theta(z)$. After partial linearization of the system for small enough $\xi$ and $\hat{\theta}$ the model (27) takes the form

$$\dot{\xi} = A_s(z)\xi + b(z) \widehat{\theta}^T z, \tag{49}$$

where $\widehat{\theta}$ is the error vector:

$$\hat{\theta} = \widehat{\theta} - \theta(z) \tag{50}$$

The control problem is solved if the error obeys the equation

$$\dot{\hat{\theta}} = -\Gamma z \hat{u} = -\Gamma z k(z) \xi, \tag{51}$$

where $\Gamma > 0$. The required asymptotic stability of the transformed system (49) is also achieved if for all $z \in \mathcal{Z}$ there exists the number $\lambda_0 > 0$ satisfying (21).

Equation (51) corresponds to the combined law of tuning

$$\dot{\hat{\theta}} = \theta(z) - \Gamma \int_0^t \hat{u}(\tau)d\tau \tag{52}$$

which can be used for improving the system accuracy in presence of weak uncertainties, and the integral control

$$\dot{\hat{\theta}} = \hat{\theta}(0) - \Gamma \int_0^t \hat{u}(\tau)d\tau \tag{53}$$

in the case $\hat{\theta}(z) \equiv 0$.

Note that the algorithms (46), (53) provide asymptotic stability with respect to $\xi$ for substationary disturbing factors. Nevertheless it is possible to involve additional hypothesis providing boundedness of the solutions $\xi(t)$ (Fradkov et al., 1999), while accuracy of the system can be improved by increasing the coefficients of $k(z)$, $\gamma$ and $\Gamma$.

7. CONCLUSION

We showed that many problems of invariance and attractivity of smooth submanifolds, as well as a variety of other problems of nonlinear dynamics, are relative to those of partial stability and can be solved by using the unified methodologies. Simplified local conditions of attractivity and partial stability based on the Lyapunov methods and techniques of partial linearization were represented. A similar approach was used in the design of nonlinear static and dynamic control laws. It worth noting that the required properties are achieved due to partial passivity of the closed loop system with specially chosen nonlinear feedbacks (Fradkov et al., 1999).

REFERENCES


