CONVENIENT ALMOST OPTIMAL AND ROBUST TUNING OF PI AND PID CONTROLLERS

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Abstract: Tuning rules for PI and PID controllers for stable non oscillating plants are presented. These rules are based on an optimization procedure, where both low frequency performance, mid frequency stability margins and high frequency control activity are taken into account. The required plant knowledge is either a specific frequency point, like in Ziegler-Nichols frequency design method, or a step response for a first order plant with time delay. Almost optimal tuning rules are presented based on this knowledge, both for PI and PID controllers.

Keywords: PID control, PI control, autotuning, optimization, robustness

1. INTRODUCTION

A method for general and objective evaluation of controllers for all kinds of plants has during the last years been introduced, see (Kristiansson and Lennartson, 1990; Kristiansson and Lennartson, 2000; Kristiansson, 2000). It is based on three criteria, related to vital performance and robustness characters. This method can be used to compare controllers of different structures but also as a synthesis tool to find the best tuning of one single controller for a given plant.

Over the years a lot of different design methods for the most common of all regulators, the PID controller, have been presented. A survey up to 1993 is given in (Aström et al., 1993). Many of these methods are modifications of the frequency response method introduced by J.G. Ziegler and N.B. Nichols in (Ziegler and Nichols, 1942) almost sixty years ago.

Today it may be argued that modern data programs with optimization routines have made traditional tuning rules needless. However, these programs demand a good and reliable model for the plant to be controlled. In reality the plants are very often just marginally known. Then there is still a need to find good control design parameters for incompletely known plants. The tuning rules presented in this paper are characterized by their minimal demands on plant knowledge.

The focus in this contribution is set on stable and non oscillating plants, presumably the largest group of plants that may be of interest to control. Tuning rules for PI and PID controllers are introduced. These rules may be used to find almost optimal parameter settings for these controllers also when the plant in question is rather anonymous and no complete model is given. Furthermore, they may be used to find a starting point to get an optimization routine to converge. Comparisons between the tuning rules and the optimal parameter settings show very small differences in almost all cases investigated.

2. EVALUATION CRITERIA

It is a well known fact that improvement of a controller design in one respect will very often bring deterioration in another. Different system qualities are not independent of each other. Especially we note that changes of some character in one frequency region usually will have influences in other frequency ranges. Therefore a method for
Fig. 1. Closed loop SISO system with plant \( G(s) \) and controller \( K(s) \).

Comparison of two controllers must, if it claims to be fair, guarantee that all aspects that are not immediately compared are equally restricted during the comparison. The method proposed here will fulfill this demand. It is based on three criteria, each of them related to essential performance and robustness qualities of the actual system and also roughly related to different frequency ranges.

Consider the SISO system in Figure 1, where a plant \( G(s) \) is controlled by a controller \( K(s) \). Also introduce the loop transfer function \( L(s) = G(s)K(s) \) and the following four sensitivity functions, including corresponding closed loop transfer functions with related output and input signals as indices: the sensitivity function \( S(s) = 1/(1 + L(s)) = G_{er}(s) \), the complementary sensitivity function \( T(s) = L(s)/(1 + L(s)) = G_{yr}(s) = G_{yw}(s) \), the disturbance sensitivity function \( G_S(s) = G(s)/(1 + L(s)) = G_{wr}(s) \), and the control sensitivity function \( K_S(s) = K(s)/(1 + L(s)) = G_{uw}(s) \).

**Disturbance rejection** The first of the proposed evaluation criteria is related to the low frequency (LF) region and can be defined as

\[
J_b = \| G_{yr} \|_\infty = \| G_S(s) \|_\infty
\]

This is a measure of the systems ability to handle low frequency load disturbances. For \( J_b \) to be finite the controller must include integral action. This criterion also happens to be related to the closed loop bandwidth \( \omega_b \) for \( G_{yr}(s) \), cf. (Kristiansson, 2000), and hence it can be regarded as a general performance measure.

**Stability margins** Two classical measures are common to characterize the mid frequency (MF) robustness, the phase margin \( \phi_m \) and the gain margin \( G_m \). However, in recent years a restriction of the maximum sensitivity function

\[
\| S \|_\infty = \max_{\omega} | S(j\omega) | \leq M_S
\]

has been more and more accepted as an exclusive robustness measure, (Åström and Hägglund, 1995; Langer and Landau, 1999). The reason is that \( \| S \|_\infty \) is equal to the inverse of the minimal distance from the loop transfer function to the critical point \((-1, 0)\) in the Nyquist plot. In many situations it is also a fully sufficient MF robustness measure.

When further damping of the step response or increased phase margin is required, without slowing down the system response too much, a restriction on the maximum complementary sensitivity function

\[
\| T \|_\infty \leq M_T
\]

should be added, especially for plants with integral action, see (Kristiansson, 2000; Schei, 1994).

Hence, the proposed mid frequency robustness criterion, the Generalized Maximum Sensitivity \( GM_S \) is defined as

\[
GM_S = \max(\| S \|_\infty, \alpha \| T \|_\infty)
\]

where \( \alpha = M_S/M_T \). The default values throughout this article are \( M_S = 1.7 \) and \( M_T = 1.3 \).

**Control activity** When reasonable stability margins are fulfilled, design of a control system is typically a question of trade-off between performance and control activity. It is therefore suitable to introduce a cost criterion related to the mid to high frequency (MHF) region, around or slightly above the closed loop bandwidth, where the maximum of the control sensitivity is mostly to be found.

\[
J_c = \| G_{uw} \|_\infty \leq \| G_{uw} \|_\infty = \| K_S(s) \|_\infty
\]

**Evaluation procedure** To evaluate a controller we assume that there are one or more free parameters available for tuning. These parameters are represented by the vector \( \rho \). Based on the proposed criteria we then suggest the following evaluation method. Solve the optimization problem

\[
\min_\rho J_c(\rho) \quad GM_S \leq C_1 \quad J_c \leq C_2
\]

The default value of \( C_1 \) in this paper is 1.7, while \( C_2 \) may be given different values depending on the desired control activity.

3. PARAMETERS OF PID AND PI CONTROLLERS

Traditionally a PID controller with a low pass filter on the derivative part used to be formulated as

\[
K_{PID}(s) = K_p(1 + \frac{1}{sT_i} + \frac{sT_d}{1 + sT_f})
\]

Mostly the filter on the derivative part has not been regarded as part of the design, just added afterwards to prevent the HF gain of the controller to grow too much. Then the filter constant is given an ad hoc value, small enough not to influence on the mid frequency properties. Introduce the ratio \( b = T_d/T_f \), which is often chosen as \( b = 10 \).
Fig. 2. $J_e/J_u$-relations for $G(s) = e^{-0.3s}/(1 + s)^2$
controlled by an optimal PI controller, an optimal
PID controller and a traditional PID controller with
fixed relations $a = 4$ and $b = 10$ and the two
remaining parameters optimized.

corresponding to $T_f = 0.1T_d$, see e.g. (Åström and
Hägglund, 1995a). Moreover the ratio $a = T_i/T_d$
is often fixed to $a = 4$, see for example (Ziegler
and Nichols, 1942; Åström and Hägglund, 1984;
Mantz and Tacconi, 1989). That leaves only two
parameters for tuning.

In e.g. (Kristiansson and Lennartson, 2000; Kristi-
ánsson, 2000) it has been shown that a PID
controller, optimized with the filter included and
all parameters free, implies complex zeros in the
controller. This has been verified for a large
number of stable non-oscillating plants (plants with
poles strictly on the negative real axis). Hence, a
suitable alternative parameterization of the PID
controller is

$$K_{PID}(s) = k_1 \frac{1 + 2\zeta \tau s + (\tau s)^2}{s(1 + \frac{s}{\tau})}$$

where $k_1$ is the integral gain. The high frequency
gain is then $k_\infty = K_{PID}(\infty) = k_1\tau/\beta$. The
remaining parameters are the zero damping $\zeta$ and
the zero natural frequency $1/\tau$. Correspondingly,
The PI controller may be formulated as

$$K_{PI}(s) = \frac{k_1}{s} + k_\infty = k_i \frac{1 + \tau s}{s}$$

In this paper the parameters $k_i$, $\tau$, $k_\infty$, and $\zeta$ are
used as tuning parameters.

4. CONTROLLERS FOR STABLE NON-OSCILLATING PLANTS

In Fig. 2 the performance criterion $J_e$ is shown
for different control activities $J_u$, but a fixed
$GM_S = 1.7$. Three graphs show the $J_e/J_u$
relations for a plant with the transfer function
$G(s) = e^{-0.3s}/(1 + s)^2$ controlled by an optimal
PI controller, an optimal PID controller and a tra-
tditional PID controller with fixed relations $a = 4$
and $b = 10$ and the two remaining parameters
optimized. These graphs demonstrate well typical
behaviours for the three cases.

In the PI case there is a clear optimum, where
$J_e$ has its absolute minimum and the goal of the
ccontroller design procedure must be hit that
point. When we have a PID controller it is not
quite so obvious, but for the optimal controller it
can be noticed that the profit in terms of decrease
of $J_e$ is smaller for the same increase of $J_u$ the
higher $J_u$ gets. Then it can be argued that there
is some kind of "economic" limit, above which it
will not pay to go. For this plant it is somewhere
in the neighborhood of $J_u = 8 - 10$. The PID case
with fixed $a$ and $b$ will not show the same tendency
to get horizontal, but for higher values of $J_u$ the
demanded $GM_S$-value can not be reached.

This figure is also a good demonstration on the
indisputable fact that a well tuned PID controller
can always offer better performance than a PI
controller to the price of a little higher $J_u$, while
a traditionally tuned PID controller is even worse
than a well tuned PI controller.

It might also be interesting to look at some step
responses. Figure 3 shows reference and distur-
bance step responses for the same plant as above. For
the PI controller, parameters corresponding to the
minimum of $J_e$ is chosen. For the PID controllers
parameters close to the "economic" limit $J_u = 8$
are used.

4.1 Nyquist crossover point known

When the point, where the Nyquist plot of the
plant crosses the negative real axis, is known, this
knowledge together with the static gain can form
a sufficient base for formulation of tuning rules.
Based on this, the plant can be characterized by its
$k$-value,

$$k_{80} = \frac{|G(j\omega_{80})|}{|G(0)|}$$

(7)
a number proposed in (Hang et al., 1991) as a 
measure of the complexity of the plant. Here \( \omega_{180^\circ} \)

is the frequency where the plant has a phase lag 
of 180\(^\circ\), which corresponds to the crossover point.
This \( \kappa \)-number usually falls in the interval [0, 1].

Sets of tuning rules for this situation have earlier 
been published in (Kristiansson and Lennartson, 1999; Kristiansson and Lennartson, 2000; Kristiansson, 2000). However, over the time they have 
been reformulated and simplified. Now the most simple formulation of the rules for the PID controller so far, and also the one giving results closest to the optimum for plants with \( \kappa \geq 0.1 \) is

\[
k_\infty = \frac{1}{|G(0)|} \left( 4 + \frac{1}{\kappa} \right) \quad (8)
\]

\[
\zeta = 0.75 \quad (9)
\]

\[
\frac{1}{\tau} = \omega_{180^\circ} (0.4 + 0.75 \kappa) \quad (10)
\]

\[
k_i = \frac{\omega_{180^\circ}^2}{|G(0)|} (0.13 + \frac{0.16}{\kappa} - \frac{0.007}{\kappa^2}) \quad (11)
\]

With these rules \( J_u \) hits the ”economic” limit quite well; the stability margin falls in the interval [1.7, 1.9] and the \( J_u \)-values differ from the optimal ones with less than 5% for a large amount of different plants. It should also be noticed that for these plants \( J_u = k_\infty \), so choosing a value for \( J_u \) is the same as fixing one parameter. When better performance is needed or the control costs should be reduced, \( \zeta \) and \( \tau \) may well be computed according to (9) and (10) respectively, while another trade-off between \( J_u \) and \( J_e \), that is between \( k_\infty \) and \( k_i \), is made.

Among the considered models are those with and without delay, with multiple as well as spread poles, with and without positive zeros etc. Figure 4 shows how well the rules for the individual parameters (8)-(11) connect to optimal values for a set of different plants.

For PI controllers corresponding rules may be formulated as

\[
\frac{1}{\tau} = \omega_{180^\circ} (0.18 + \kappa) \quad (12)
\]

\[
k_i = \frac{\omega_{180^\circ}^2}{|G(0)|} (0.33 - 0.15 \kappa) \quad (13)
\]

Using these rules has resulted in systems with \( GM_\tau \)-values in [1.59, 1.76], \( \tau \)-values that differ from the optimal ones with less than 10% and \( J_e \)-values that differ from the optimal ones with less than 5%.

4.2 Nyquist crossover point unknown

However, there are common plants, whose Nyquist plots never crosses the negative real axis and also those, for which the crossover point occur at very high frequencies, causing \( \kappa_{180} \) to get very small (\( \kappa < 0.1 \)). For such plants \( \kappa_{180} \) can not be used to

express the plant knowledge. For these situations a modified number has been introduced by the authors in (Kristiansson and Lennartson, 2000; Kristiansson, 2000), which utilizes the frequency \( \omega_{135^\circ} \), where the plant phase is \(-135^\circ\). This number is

\[
\kappa_{135} = \frac{|G(j \omega_{135^\circ})|}{|G(0)|} \quad (14)
\]

Based on this number a set of tuning rules for 
PID controllers for these ”light” plants has been derived. These formulas are
\[
k_\infty = \frac{1}{|G(0)| \min(14 + \frac{0.5}{\kappa_{135}}, 25)} \tag{15}
\]
\[
\zeta = 0.75
\tag{16}
\]
\[
\frac{1}{\tau} = \frac{\omega_{135,0}(0.44 + 1.4\kappa_{135})}{|G(0)|} \tag{17}
\]
\[
k_i = \frac{\omega_{135,0}}{|G(0)|} \left(1.35 + \frac{0.35}{\kappa_{135}} - \frac{0.006}{\kappa_{135}^2}\right) \tag{18}
\]

For these plants there is no clear "economic" limit. \(J_o\) can always be decreased by increasing \(J_i\). For some of these plants \(J_o\) is also significantly higher than \(k_\infty\). However, the \(J_o\)-values reached by the proposed rules will mostly be reasonable. The regularity for these plants is less marked than for the plants with \(\kappa_{180} > 0.1\). It should be noticed that the \(J_o\)-values are so small that a marginal absolute difference may show up as a big difference in percentage. With the proposed rules, the stability margin \(GM_s\) falls in \([1.7, 1.9]\) and \(J_o\) differs from its optimal values with up to about 15%.

For PI controllers corresponding rules may be formulated as
\[
\frac{1}{\tau} = \omega_{135,0}(0.09 + 1.2\kappa_{135}) \tag{19}
\]
\[
k_i = \frac{\omega_{135,0}}{|G(0)|} \left(0.25 + \frac{0.09}{\kappa_{135}}\right) \tag{20}
\]

These rules make \(\tau\) differ from the optimal value in most cases with less than 10 %, while \(J_i\) differs up to 15 % and sometimes even a little more. \(GM_s\) falls in the interval \([1.6, 1.9]\).

### 4.3 General rules

When a relay-experiment including hysteresis is used to get the necessary plant knowledge, the controlled self-oscillating frequency will not be \(\omega_{180,0}\), but somewhat lower, typically around \(\omega_{150,0}\). To make things easy it could also be valuable to have a set of tuning rules that are usable for all stable non-oscillating plants, independent of \(\kappa_{180}\). With this motivation a set of tuning rules based on \(\kappa_{150}\) is also given,
\[
\kappa_{150} = \frac{|G(j\omega_{150,0})|}{|G(0)|} \tag{21}
\]

The rules are for PID controllers
\[
k_\infty = \frac{1}{|G(0)| \min(6 + \frac{1}{\kappa_{150}}, 25)} \tag{22}
\]
\[
\zeta = 0.75
\tag{23}
\]
\[
\frac{1}{\tau} = \omega_{150,0}(0.32 + 1.6\kappa_{150} - 0.8\kappa_{150}^2) \tag{24}
\]
\[
k_i = \frac{\omega_{150,0}}{|G(0)|} \left(0.4 + \frac{0.075}{\kappa_{150} + 0.05}\right) \tag{25}
\]

and for PI controllers
\[
\frac{1}{\tau} = \omega_{150,0}(0.06 + 1.6\kappa_{150} - 0.06\kappa_{150}^2) \tag{26}
\]
\[
k_i = \frac{\omega_{150,0}}{|G(0)|} \left(0.2 + \frac{0.075}{\kappa_{150} + 0.05}\right) \tag{27}
\]

With these rules the differences from the optimal values are for PID controllers in \(J_o\) mostly less than 10%, and \(GM_s\) falls in the interval \([1.6, 1.9]\). For PI controllers the differences are in \(\tau\) less than 5%, but in \(J_i\) it may in some cases go up to 40% for plants with very low \(\kappa\)-values. The values of \(k_i\) are here very crucial. For higher \(\kappa\)-values it is less than 5%. The stability margin \(GM_s\) is in \([1.65, 1.8]\), except for those cases with big differences in \(J_o\), for which it may rise over 2.0.

In Fig. 5 step responses are given both from the reference and disturbance inputs. They show that the simple tuning rules based on \(\kappa_{180}\) \((8)-(11)\) and \(\kappa_{150}\) \((22)-(25)\) give the same closed loop behavior as the optimal solution given by \((6)\).

### 4.4 First order plants with delay

A special group of models constitute those with one time constant and a time delay. Very often they are results of a rough step response experiment. Their transfer functions may be formulated
\[
G(s) = K \frac{e^{-sL_d}}{1 + sT} \tag{28}
\]

When the parameters \(K\), \(T\) and \(L_d\) in this model are approximately known and nothing else about the process, it might be of no interest to find out any \(\kappa\)-value. In those cases these parameters can be used to formulate the demanded rules.

Here is a proposal for tuning of PID controllers, useful as long as the relation \(L_d/T\) is in the interval \([0.2, 5]\). When this demand on \(L_d/T\) is not fulfilled more complex rules have to be used.
A couple of different tuning rules for PI and PID controllers are shown to give almost optimal responses for stable non-oscillating plants. The rules are based on either a frequency or a step response, and hence they can be used e.g. for auto-tuning. For manual tuning the most crucial parameter and the best one to use for final tuning is the integral gain $k_i$. This parameter is then adjusted to give a desired stability margin (damping of the closed loop system). Compared to existing tuning rules the main contribution of this paper is the systematic treatment of the filtering of the derivative action. This means that significant performance improvements can be achieved for a given control activity, compared to standard tuning rules.

5. SUMMARY

A couple of different tuning rules for PI and PID controllers are shown to give almost optimal responses for stable non-oscillating plants. The rules are based on either a frequency or a step response, and hence they can be used e.g. for auto-tuning. For manual tuning the most crucial parameter and the best one to use for final tuning is the integral gain $k_i$. This parameter is then adjusted to give a desired stability margin (damping of the closed loop system). Compared to existing tuning rules the main contribution of this paper is the systematic treatment of the filtering of the derivative action. This means that significant performance improvements can be achieved for a given control activity, compared to standard tuning rules.

6. REFERENCES


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Fig. 6. Reference ($r$) and disturbance ($d$) step responses for $G(s) = e^{-0.05 t}/(1 + s)$ with PID controllers, optimal, and tuned by rules (29) - (32).

$$k_\infty = \frac{1}{|G(0)|} \min (5 + \frac{2T}{L_d}, 25)$$  \hspace{1cm} (29)

$$\zeta = 0.7 + \frac{0.12}{L_d/T + 0.05}$$  \hspace{1cm} (30)

$$\tau = T (0.13 + 0.5 \frac{L_d}{T} - 0.04 (\frac{L_d}{T})^2)$$  \hspace{1cm} (31)

$$k_i = \frac{1}{|G(0)|} \frac{0.9}{L_d/T - 0.05}$$  \hspace{1cm} (32)

These rules will keep $GM_N$ in $[1.65, 1.85]$ and $J_c$ within $+15$ to $-5$ % from the optimum. In (Kristiansson, 2000) the simple rule $\tau = \frac{T}{60}/3$ was suggested for general stable non-oscillating plants, where $T_{63}$% is the time when the reference step response has reached 63% of its final value. For the model (28) this rule implies that $\tau = (T + L_d)/3$ which in fact is close to (31) for $0.7 < L_d/T < 4$.

For tuning of PI controllers our proposal is

$$\tau = 0.5 T_{63} = 0.5 (T + L_d)$$  \hspace{1cm} (33)

$$k_i = \frac{1}{T |G(0)|} \frac{0.57}{L_d/T - 0.055}$$  \hspace{1cm} (34)

as long as $L_d/T$ is in $[0.2, 1]$. Then $J_c$ differs less than 8% and $GM_N$ is in $[1.65, 1.80]$.

For $L_d/T$ outside this interval $\tau$ should for corresponding results be calculated by one of the formulas

$$\tau = 0.4 T + 0.7 L_d \quad \frac{L_d}{T} < 0.5$$  \hspace{1cm} (35)

$$\tau = 0.6 T + 0.3 L_d \quad \frac{L_d}{T} \geq 0.5$$  \hspace{1cm} (36)

In Fig. 6 step responses show that results from a design based on (29)-(32) almost coincide with the optimal solution given by (6).