EXEMPLARY STABILISABILITY OF MULTIDIMENSIONAL LINEAR SYSTEMS WITH FINITE DATA RATES

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Abstract: A critical notion in the field of communication-limited control is the lowest data rate above which a given system is stabilisable. In this paper the objective is to derive this rate for a multidimensional LTI system when exponential stability with a given decay is desired. By assuming an initial state probability density, a new quantiser distortion lower bound and an asymptotic quantisation result are able to be applied to derive the infimum stabilising data rate, under very mild requirements on the initial density. Furthermore, a stabilising scheme is explicitly constructed.

Keywords: Stabilizability, Communication Channels, Vector Quantization

1. INTRODUCTION

In recent years, rapid progress has been made in the field of control with limited communication rates, particularly for deterministic linear systems. Beginning with (Delchamps, 1990) and continuing with (Wong and Brockett, 1999; Baillieul, 1999; Elia and Mitter, 1999; Brockett and Liberzon, 2000; Petersen and Savkin, 2001), a number of different coding and control schemes have been proposed and analysed. A common feature of all these articles were inequalities which gave sufficient data rates for stabilisability but which were tight, i.e. the least possible, only for dimension one. In (Nair and Evans, 2000), asymptotic quantisation theory (Gersho, 1979; Graf and Luschgy, 2000) was used to explicitly derive the smallest data rate above which a given auto-regressive moving average (ARMA) system of arbitrary dimension could be asymptotically stabilised. However, in a sense this work was not far from the one-dimensional case, as the initial condition was scalar.

In this paper, the methods of asymptotic quantisation are extended to the deal with a general, noiseless, discrete-time, LTI system in R^n. A precise formulation is given in the next section. Assuming that the initial state is governed by a non-singular probability density with finite (m + ε)th absolute moment, the objective is to find the least data rate above which a coding and control law exists that takes the mth absolute state moment to zero faster than a given exponential decay. The main result of this paper, Theorem 1, is stated here and gives an expression for it in terms of the ratios of the unstable eigenvalue magnitudes to the decay constant. A similar result for asymptotic stabilisability has also been presented in (Tatikonda and Mitter, 2000). The proof outlined there is quite different, employing a covering argument and relying on the existence of a known bound for the deterministic unknown initial state. Here in contrast, the initial state is random, possessing an unknown density that may have infinite support and ‘fat’ tails. The fact that essentially the same result is obtained indicates that it is in some sense fundamental.

The remainder of the paper basically constitutes the proof of Theorem 1. In section 3, the system dynamics are explicitly decoupled and the control objective is shown to be equivalent to recursively quantising the initial state so that a mean mth power error term...
approaches zero exponentially fast. In the subsequent section, the necessity of the data rate specified in Theorem 1 is established by means of a new quantiser distortion lower bound. In the penultimate section, its sufficiency is confirmed by explicitly constructing a coding and control scheme and using a result on scalar asymptotic quantisation (Linder, 1991; Graf and Luschgy, 2000) to characterise its performance.

2. FORMULATION

Certain conventions are followed in this paper. Sequences \( \{a_t\}_{t=0}^{\infty} \) are denoted \( a_k \) and \( \| \cdot \| \) represents either the Euclidean norm on a real vector space or the matrix norm induced by it. However when subscripted as in \( \|f\|_p \) it denotes the \( L_p \)-norm \( \left( \int |f(x)|^p \, dx \right)^{1/p} \) of a function \( f \) with respect to Lebesgue measure \( \lambda \). Vectors are written in bold-face type, matrices in bold-face upper-case, random variables in upper-case and their realisations in corresponding lower-case letters. All random variables are assumed to exist in a common probability space. The probability density of \( X \) is written \( p_X \), expectation \( E \) and probabilities, densities and expectations conditioned on an event \( A = a \) are given a subscript \( a \). The \( d \times d \) identity matrix is represented by \( I_d \), the \( m \times n \) matrix by \( \mathbf{0}_{m \times n} \), the set of real numbers by \( \mathbb{R} \), complex numbers by \( \mathbb{C} \). Sets of positive integers by \( \mathbb{N} \), non-negative integers by \( \mathbb{Z}_{\geq 0} \) and finally \( \mathbb{Z}_n = [0, 1, \ldots, n-1] \).

Consider the discrete-time, linear time-invariant system

\[
x_{k+1} = Ax_k + Bu_k, \quad y_k = Cx_k, \quad \forall k \in \mathbb{W},
\]

where \( x_k \in \mathbb{R}^d \) is the state, \( y_k \in \mathbb{R}^r \) the sensor measurement and \( u_k \in \mathbb{R}^m \) the control vector, \( \forall k \in \mathbb{W} \). It is assumed that \((A, B)\) is reachable, \((A, C)\) is observable and that \( x_0 \) is realisation of a random variable \( X_0 \) with a density \( p_{X_0} \) that is non-singular with respect to \( \lambda \) and satisfies \( E[|X_0|^{m+\epsilon}] < \infty \) for some \( m, \epsilon > 0 \).

The formulation of the coder, controller and digital link below are essentially the same as in (Nair and Evans, 2000). The sensor is connected to a distant controller by a digital channel that can carry only one symbol \( s_k \) from a coding alphabet \( Z_\mu \equiv [0, 1, \ldots, \mu - 1] \) during each sampling interval. Each symbol is generated via

\[
s_k = \gamma_k (y_k, s_{k-1}), \quad \forall k \in \mathbb{W},
\]

where \( \gamma_k : \mathbb{R}^r \times Z_\mu \mapsto Z_\mu \). Each is then transmitted over the digital link, with data rate \( R \overset{\Delta}{=} \log_2 \mu \) bits per interval, and arrives in entirety at the controller after one sampling interval. The controller then calculates

\[
u_k = \delta_k (s_{k-1}), \quad \forall k \in \mathbb{W},
\]

where \( \delta_k : Z_\mu \mapsto \mathbb{R}^m \). In the spirit of Shannon’s source coding theory, no structural or computational constraints have been placed on the coder and controller, apart from causality, and the digital channel is assumed to be error-free. In addition, as discussed in (Nair and Evans, 2000) \( R \) should be regarded as the overall data rate of the entire feedback loop, not just the path from sensor to controller.

Define the coder-controller as the pair of mapping sequences \( (\gamma, \delta) \overset{\Delta}{=} \{(\gamma_k), (\delta_k)\}_{k=0}^{\infty} \). For a given constant \( \rho > 0 \), the objective is to find a coder-controller that \( \rho \)-exponentially stabilises the plant (1) in the sense that

\[
\rho^{-md} E[|X_k|^m] \to 0 \text{ as } k \to \infty,
\]

while using as low a data rate as possible. As remarked in the previous section, for noisless ARMA systems with a scalar initial condition there is a critical data rate, given by the base-2 logarithm of the largest eigenvalue magnitude, which determines whether closed-loop asymptotic stability is possible. It may therefore be expected that a critical data rate will also exist for the case of a general multidimensional LTI system. The main result of this paper is now stated:

**Theorem 1.** Assume that the initial state of the system (1) is governed by a probability density that is non-singular with respect to Lebesgue measure on \( \mathbb{R}^d \) and has finite \( (m + \epsilon) \)th absolute moment for some \( m, \epsilon > 0 \). Then for a given data rate \( R \) (bits per interval), a coder-controller that \( \rho \)-exponentially stabilises the system in the sense (4) exists if and only if

\[
R > \sum_{\nu = [1, \ldots, d]} \log_2 \left[ \sum_{|s| \geq \rho} \frac{\eta_s}{\rho} \right],
\]

where \( \eta_1, \ldots, \eta_d \) are the possibly repeated and conjugate eigenvalues of the open-loop system.

This result assumes nothing about the coding and control laws but causality. In a very general sense, it therefore draws a fundamental line of demarcation between what is and is not achievable when communication bandwidth is limited, in addition to providing a conceptual measure of the difficulty of stabilising a given system as remarked in (Baillieul, 1999).

The remainder of this paper is devoted to outlining the proof of Theorem 1. In the next section, the problem is transformed in order to simplify its analysis and clarify its connection to asymptotic quantisation. In section 4, the necessity of (5) is then established via a new quantiser distortion lower bound. Finally, its sufficiency is established in Section 5 by explicitly constructing a coder-controller that achieves (4) for any data rate satisfying (5).

3. QUANTISATION OF INITIAL STATE

Instead of dealing with the state \( x_k \) directly, it is convenient to transform it so as to simplify the system
dynamics. In order to avoid the complex-valued transformation matrix required by the Jordan canonical form, the real Jordan canonical form (Horn and Johnson, 1985) will be used.

Let \( \lambda_1, \ldots, \lambda_n, n \leq d \), be the distinct eigenvalues of \( A \in \mathbb{R}^{d \times d} \), not counting conjugates and ordered by non-increasing magnitude, and let the algebraic multiplicity of each \( \lambda_i \) be \( m_i \). The real Jordan canonical form \( J \) has the block diagonal structure

\[
J = \text{diag}(J_1, \ldots, J_n) \in \mathbb{R}^{d \times d},
\]

where \( J_i \in \mathbb{R}^{d_i \times d_i} \) has either exactly one eigenvalue \( \lambda_i \) or a pair of complex conjugate eigenvalues \( \lambda_i, \lambda_i^* \), each with multiplicity \( m_i \), and

\[
d_i = \begin{cases} m_i & \text{if } \lambda_i \in \mathbb{R} \\ 2m_i & \text{otherwise} \end{cases}
\]

Furthermore there exists a real similarity matrix \( T \in \mathbb{R}^{d \times d} \) such that \( T^{-1} J T = \Lambda \). Defining the transformed state

\[
\tilde{x}_k = T x_k, \quad \forall k \in \mathbb{W},
\]

the system equations (1) can then be written

\[
\tilde{x}_{k+1} = J \tilde{x}_k + T B u_k, \quad \tilde{y}_k = C T^{-1} \tilde{x}_k.
\]

By partitioning the transformed state vector into the vectors \( \tilde{x}_k^{(1)}, \ldots, \tilde{x}_k^{(n)} \) corresponding to each subsystem, the dynamical equation above can be rewritten more explicitly \( \forall k \in \mathbb{W}, i \in [1, \ldots, n] \) as

\[
\tilde{x}_k^{(i)} = J_i \tilde{x}_k^{(i)} + (T B u)_{(i)}^{(i)} \in \mathbb{R}^{d_i},
\]

where \((T B u)_{(i)}^{(i)}\) denotes that portion of the \( T B u \) control vector that feeds into the \( i \)th subsystem. In the form (10), it can be seen that the original system has been decoupled into \( n \) real subsystems, with open-loop dynamics characterised by either a single eigenvalue or a pair of complex conjugate eigenvalues, possibly repeated. As \( T \) is invertible, the problems of exponentially stabilising (9) and (1) are equivalent.

The key idea in the proof of Theorem 1 is the equivalence of the control problem to recursive quantisation of the initial state. Expanding (9) out \( \forall k \in \mathbb{W} \) and using (3),

\[
\tilde{x}_k = J \tilde{x}_0 + \sum_{j=0}^{k-1} J^{j-1} - j T B \delta_j(\tilde{x}_{j-1}).
\]

Observe that the sum above is a function of the symbol sequence \( \tilde{s}_{k-1} \) which is in turn completely determined by \( \tilde{s}_0 \) for a given coder. Thus a function \( q_{k-1} : \mathbb{R}^d \to \mathbb{R}^d \), may be defined \( \forall k \in \mathbb{W} \) by

\[
q_{k-1}(\tilde{s}_0) = \sum_{j=0}^{k-1} J^{j-1} - j T B \delta_j(\tilde{s}_{j-1}), \quad \forall \tilde{s}_0 \in \mathbb{R}^d.
\]

As the symbol sequence \( \tilde{s}_{k-1} \in \mathbb{Z}_{\mu_i}^{d_i} \) can take \( \mu_i \)-1 distinct values, \( q_{k-1} \) can only assume up to \( \mu_i^{d_i-1} \) distinct values in \( \mathbb{R}^d \), i.e. it can be viewed as a \( \mu_i^{d_i-1} \)-point quantiser for the initial state. Hence finding a coder-controller that \( \rho \)-exponentially stabilises the system is equivalent to finding a sequence \( \{ q_k \}_{k \in \mathbb{W}} \) of vector quantisers which i) are expressible in the form (12) for some sequence of coder and controller mappings and ii) achieve

\[
\rho - \text{min} E \| J \tilde{x}_0 - q_{k-1}(\tilde{x}_0) \|^m \to 0 \text{ as } k \to \infty.
\]

Note that by (12) this sequence of quantisers is generally recursive, since \( q_{k-1}(\tilde{x}_0) \) depends on past quantiser outputs \( q_{k-2}(\tilde{x}_0), \ldots, q_0(\tilde{x}_0) \) via its dependence on \( \tilde{s}_{k-2} \).

Apart from the time-varying coefficients and matrices, the term on the LHS is the mean \( mth \) power error (MMPE) distortion criterion that is commonly used in asymptotic quantisation theory, e.g. (Graf and Luschgy, 2000). This equivalence to a quantisation problem is not surprising, since the initial state is the only unknown quantity in the system. However the presence of the time-varying matrix \( J_k \) makes achieving (13) somewhat more complicated than in standard asymptotic vector quantisation, primarily because it makes the quantiser errors of the subsystems grow at different speeds. It is intuitively obvious that the recursive quantiser for the initial state should somehow allocate more quantisation levels - and hence a larger proportion of the data rate - to unstable subsystems, so as to balance the MMPE’s of all. In section 5, it is shown how this can be done. First however, very general quantisation arguments are used to show that the data rate must satisfy (5) for exponential stability in the sense of (4), irrespective of the coder-controller used.

4. NECESSITY

The first step towards proving Theorem 1 is to establish the necessity of (5). The approach used is to find a lower bound for \( \rho - \text{min} E \| \tilde{x}_k \|^m \) which is independent of the coder-controller and is also easier to analyse in terms of its dynamics and the data rate. More precisely, this bound is sought for the \( mth \) absolute moment of the overall state vector corresponding to subsystems with eigenvalue \( | \lambda_i | \geq \rho \), since the remaining subsystems are \( \rho \)-exponentially stable without needing any control.

With this aim in mind, let

\[
L \triangleq \{ i \in [1, \ldots, n] : | \lambda_i | \geq \rho \}, \quad f \triangleq \sum_{i \in L} d_i,
\]

\[
\tilde{s}_k \triangleq [\tilde{s}_k^{(1)} \ldots \tilde{s}_k^{(L)}]^{T} \equiv R \tilde{x}_k \in \mathbb{R}^f, \quad \forall k \in \mathbb{W},
\]

\[
R \triangleq [I_f 0_{f \times (d-f)}] \in \mathbb{R}^{f \times d}.
\]

It is straightforward to show that \( \| R \| = 1 \) and

\[
R J_k = J^{k} R, \quad J \triangleq \text{diag}(J_1, \ldots, J_n) \in \mathbb{R}^{f \times f}.
\]

As \( \rho - \text{min} E \| R \tilde{x}_k \|^m \leq \rho - \text{min} E \| \tilde{x}_k \|^m \), which by hypothesis \( \to 0 \), it follows that as \( k \to \infty \).
\[
\rho^{-kn}E[\|RX_0\|^m] = \rho^{-kn}E[\|R_jX_0 - R_{t-1}(\hat{X}_0)\|^m],
\]
\[
= \rho^{-kn}E[\| R_j X_0 - \hat{R}_{t-1}(\hat{X}_0)\|^m],
\]
\[
= E[\| (\rho^{-1}j)^j \hat{X}_0 - \rho^{-1}R_{t-1}(\hat{X}_0)\|^m] \rightarrow 0. \quad (17)
\]

In order to lower-bound the RHS expectation over all coder-controllers, the following lemma is used:

**Lemma 1.** Let \( X \in \mathbb{R}^f \), \( Y \in \mathbb{R}^d \) be random variables, \( c_v : \mathbb{R}^f \times \mathbb{R}^d \rightarrow \mathbb{R}^f \) a quantiser with no greater than \( \nu \) distinct points and \( c_v' : \mathbb{R}^f \rightarrow \mathbb{R}^f \) the mean-mth-power-error-optimal, \( \nu \)-point quantiser for \( X \). Then \( \forall r \in (f/[f+m], 1), \nu \in \mathbb{N}, \)
\[
E[\|X - c_v(X, Y)\|^m] \geq \beta \nu^{-m/f} \|p_X\|_{m/f(1-r)}^m,
\]
where \( \beta > 0 \) depends only on \( m, f \) and \( r \).

**Proof:** Omitted for lack of space, but briefly it uses Holder’s and Jensen’s inequalities.

Applying this lemma to the R.H.S. of (17) at each time \( k \in \mathbb{N} \), with \( X = (\rho^{-1}j)^j X_0, Y = \hat{X}_0, \nu = \mu^{-1} \) and \( c_v(X, Y) = \rho^{-1}R_{t-1}(X_0) \), it follows that \( \forall k \in \mathbb{N}, \)
\[
E[\| (\rho^{-1}j)^j \hat{X}_0 - \rho^{-1}R_{t-1}(\hat{X}_0)\|^m] \geq \frac{\beta}{\mu^{(k+1)m/f}} \left( \int p_{(\rho^{-1}j)^j \hat{X}_0}(x) d\hat{\lambda}(x) \right)^m 
\]
As \( \text{det} L = \prod_{\ell \in L} |\lambda_\ell|^{p_\ell} > 0 \), change the integration variable in (19) to \( y = (\rho^{-1}j)^j x \) to obtain
\[
E[\| (\rho^{-1}j)^j \hat{X}_0 - \rho^{-1}R_{t-1}(\hat{X}_0)\|^m] \geq \frac{\beta \|p_X\|_{m/f(1-r)}^m}{\mu^{(k+1)m/f}} \left( \rho^{-f} \prod_{\ell \in L} |\lambda_\ell|^{p_\ell} \right)^m.
\]

From (17) the left-hand side (LHS), and hence RHS, \( \rightarrow 0 \) as \( k \rightarrow \infty \). As the non-singularity of \( p_X \) with respect to Lebesgue measure implies that \( \|p_X\| > 0 \), it is necessary that
\[
\mu > \prod_{\ell \in L} |\lambda_\ell|^{p_\ell} = \prod_{\ell \in L} |\lambda_\ell|^{p_\ell} = \prod_{\ell \leq \mu \leq \ell \geq 1} \prod_{\ell \leq \mu \leq \ell \geq 2} |\lambda_\ell|^{p_\ell}.
\]

where the last product includes conjugate and repeated eigenvalues. Taking logarithms yields (5).

5. **SUFFICIENCY**

The final step in proving Theorem 1 is to establish that any data rate satisfying the inequality (5) is sufficient to be able to somehow \( \rho \)-exponentially stabilise the system (1). In order to do so, a specific coder-controller will be constructed and its convergence properties analysed. This scheme is described below.

**Coder 1.** Divide times \( k \geq d \) into epochs \( [d + j \tau, \ldots, d + j \tau + \tau - 1], j \in \mathbb{N}, \) of uniform integer duration \( \tau \geq d \). With \( L \) defined by (14), subdivide each epoch into \( |L| \) subepochs of duration \( d \tau_i \), where
\[
\tau_i = \begin{cases} 
\tau R^{-1} \log_2 |\lambda_\rho| + 1 & \text{if } i \in L \\
\tau & \text{otherwise}
\end{cases} \quad (20)
\]
with \( |\cdot| \) denoting rounding down, and define
\[
c(x) = \begin{cases} 
1 - 0.5(1 + x)^{1/d} & \text{if } x > 0 \\
0.5(1 - x)^{1/d} & \text{if } x \leq 0
\end{cases} \quad (21)
\]

At times \( k \in \{0, \ldots, d - 1\} \), record the measurements of the system (1). At time \( k = d \), calculate the transformed initial state \( \hat{X}_0 \) of (9) by solving
\[
W_s T^{-1} \hat{X}_0 = [X_0^T \; Y_0^T \; \cdots \; Y_{d-1}^T]^T,
\]
where \( W_s = [C^T \; (CA)^T \; \cdots \; (CA^{d-1})^T]^T \).

This is possible since the observability matrix \( W_o \) has rank \( d \). Indicating each scalar component of \( \hat{X}_0 \) with an additional superscript \( h \in \{1, \ldots, d\} \), apply the compressor \( c \) component-wise and expand in base-\( \mu \),
\[
c \left( \hat{x}_0^{(i,h)} \right) \equiv \sum_{i=1}^{d} \gamma_i^{(i)} \mu^{|\gamma_i^{(i)}|} \forall h \in \{1, \ldots, d\}, i \in \mathbb{N} \quad (22)
\]
where \( \gamma_i^{(i)} \in Z_\mu \). During the \( i \)-th sub epoch of the \( (j + 1) \)-th epoch, transmit the \( (j + 1) \)-th block of \( \tau_i \) successive digits in this expansion, \( \forall h \in \{1, \ldots, d\} \).

**Controller 1.** Set \( u_0, \ldots, u_{d-|\tau|-1} = 0 \). At time \( k = d + j \tau \), \( j \in \mathbb{N} \), the symbol sequence \( \hat{s}_{d+j\tau-1} \) has been received, comprising the first \( j \tau \) digits \( \hat{s}_1^{(j)} \), \ldots, \( \hat{s}_{j\tau}^{(j)} \) in the base-\( \mu \) expansion of each compressed scalar component \( c \left( \hat{x}_0^{(i,h)} \right) \), \( \forall h \in \{1, \ldots, d\}, i \in \mathbb{L} \). Estimate each component of the transformed initial state \( \hat{X}_0 \) by
\[
c^{(i,h)}_{j} = \begin{cases} 
-1 \left( \frac{0.5}{|\mu^{\gamma_i^{(i)}}|} \sum_{i=1}^{d} \gamma_i^{(i)} \mu^{|\gamma_i^{(i)}|} \right) & \text{if } i \in \mathbb{L} \\
0 & \text{if } i \notin \mathbb{L}
\end{cases} \quad (23)
\]

Then calculate control signals \( u_{d+j\tau}, \ldots, u_{d+j\tau-1} \) by using the reachability of \((A, B)\) to solve
\[
\sum_{k=d+j\tau}^{2d+j\tau-1} J^{2d+j\tau-k} TBu_k = J^{d+j\tau} (e_{j-1} - e_j), \quad (24)
\]
where \( e_0 = \Delta \). Set the remaining control signals \( u_{2d+j\tau}, \ldots, u_{d+(j+1)\tau-1} \) in the epoch to 0.

Coder-Controller 1 can be cast into a more practical form in which the current transformed state is recursively quantised, not just its initial value. However, in the form above it is easier to apply results from asymptotic quantisation, which typically deals only with static random variables.

Before proceeding, it needs to be verified that the sum of sub-epoch durations \( d \tau_i \) is not greater than
the epoch duration $\tau$, for otherwise the time-sharing protocol above is infeasible. Observe that
\[
\sum_{j=1}^{n} d_{j}\tau \leq \sum_{i \in I} d_{i} \left( \frac{\log_{2} |\lambda_{i}/\tau|}{R} + 1 \right),
\]
\[
= \tau \sum_{|\lambda_{i}| > \nu} \log_{2} |\lambda_{i}/\nu| \frac{R}{|\lambda_{i}|} + f, \forall \tau \in \mathbb{N},
\]
where the final sum includes repeated and conjugated eigenvalues. As the coefficient of $\tau$ is $< 1$ by the hypothesis (5), it follows that the RHS and hence LHS \leq any sufficiently large epoch length $\tau$.

Upper bounds on the $n$th state absolute moments generated by this coder-controller will now be derived under the assumption that (5) is satisfied. This will be done at times $k \equiv j\tau + d$ corresponding to the start of an epoch. The extension to times within an epoch follows readily. Observe that by (11) and (24), $\forall j \in \mathbb{N},$
\[
\|x_{d+j}\|_{m}^{m} = \|J_{d+j}^{m} e_{0} - e_{j-1}\|_{m}^{m} \leq n^{m/2} \sum_{j=1}^{n} \|J_{d+j}^{m} x_{0} - c_{j}\|_{m}^{m},
\]
using the diagonal block structure (6) of the real Jordan form $J$. As $J_{i}$ is similar to the block diagonal matrix consisting of all Jordan blocks associated with $\lambda_{i}$, a trivial adaptation of a result in (Horn and Johnson, 1985) (pg. 138) states that $\exists \Phi > 0$ such that $\forall i \in \{1, \ldots, n\}, k \in \mathbb{N}$,
\[
\|J_{i}\|_{m} \leq \Phi k^{\gamma-1} |\lambda_{i}|^{k}.
\]
Substituting into (25), taking expectations and using $c_{j} \equiv 0$ where $i \notin I$,
\[
E\|\hat{x}_{d+j}\|_{m} \leq n^{m/2} \Phi^{m} \sum_{d+j} \|J_{d+j}^{m} e_{0} - e_{j-1}\|_{m}^{m}
\times |\lambda_{i}|^{(d+j)/m} d_{i}^{m/2} \sum_{1}^{d_{i}} E\|\hat{x}_{i} - e_{j-1}\|_{m}^{m}
+ \sum_{i \in I} |(d+j+1)|^{\gamma-1} |\lambda_{i}|^{d+j} m E\|\hat{x}_{j}^{i}\|_{m}^{m}.
\]

Now, by (22) and (23), the sequence of base-\mu digits $x_{d+(j-1)\tau}, e_{j-1}$ received by the controller at time $d + (j-1)\tau$ identify the unique interval $[x_{d+(j-1)\tau}, (z+1)\mu^{-(j-1)\tau}), z \in \mathbb{Z}_{\mu^{-(j-1)\tau}}, \tau \geq \log_{2} \mu^{-(j-1)\tau}$, which contains each component $c_{i}$ of the transformed and compressed initial state, with $z \in \sum_{j=1}^{d+j} c_{i} \mu^{-(j-1)\tau}$. From the initial state estimate equation (23), the transformed initial state component $c_{i}$ is then simply $c^{(-1)}$ applied to the midpoint of this interval and can be represented as
\[
c_{i}^{(j)} \equiv c^{(-1)} Q_{d+j}\tau c_{i}^{(0)}, \forall j \in \mathbb{N},
\]
where $Q_{d+j}\tau [0, 1] \rightarrow \left\{ \frac{1}{2\tau}, \frac{1}{\tau}, \frac{3}{2\tau}, \cdots, \frac{\tau-1}{2\tau} \right\}$ is the the uniform, \nu-level, mid-point quantiser on the unit interval.

The function on the RHS above describes a so-called companding quantiser for a scalar random variable. A result in (Linder, 1991; Graf and Luschgy, 2000) states that if $X \in \mathbb{R}$ is a random variable with probability density $p_{X}$ which is absolutely continuous with respect to Lebesgue measure $\lambda$ and the derivative $\alpha$ of a function $c : \mathbb{R} \rightarrow (0, 1)$ is continuous, positive, decreases monotonically for sufficiently large argument magnitudes and satisfies certain mild technical conditions then as $\nu \rightarrow \infty$,
\[
v^{m}E|X - c^{-1} Q_{\nu c}|(X)^{m} \rightarrow E(\alpha(X)^{-m}) < \infty.
\]
The prerequisites for this limit can be shown to hold here, with $X = X_{i}^{(j,h)}$ and $\nu = \mu^{(j-1)\nu}$. Consequently, $\forall \nu > 0, h \in \{1, \ldots, d_{j}\}$ and $i \in I$, there is a $j^{(j,h)} \in \mathbb{N}$ such that $\forall j \geq j^{(j,h)}$,
\[
E|X_{i}^{(j,h)} - c^{-1} Q_{\nu c}|(X)^{m} \leq (1+w)E(\alpha(X)^{-m}) < \infty.
\]
Applying this to each term in the first sum in (27) and dividing by $\rho^{d+j}m$, $\forall j \geq \max_{i,j^{(j,h)}},$
\[
\frac{E|\hat{x}_{d+j}\|_{m}}{\rho^{d+j}m} \leq n^{m/2} \Phi^{m} \sum_{d+j} \|J_{d+j}^{m} e_{0} - e_{j-1}\|_{m}^{m}
\times \left[ |(d+j+1)|^{\gamma-1} |\lambda_{i}|^{d+j} m \right] \sum_{i \in I} E(\alpha(X_{i}^{(j,h)})^{-m})
+ \sum_{i \in I} \left[ |(d+j+1)|^{\gamma-1} |\lambda_{i}|^{d+j} m \right] E|\hat{x}_{j}^{i}\|_{m}^{m}.
\]

Now, each term in the last sum above $\rightarrow 0$ as $j \rightarrow \infty$ since, by definition of $\mu, |\lambda_{i}| < \rho$ and so $|(d+j+1)|^{\gamma-1} |\lambda_{i}|^{d+j} m \rightarrow 0, \forall i \notin I$. Looking next at the first sum, observe that by (20), $\tau_{\gamma} > \tau \rightarrow \gamma^{1} \log_{2} |\lambda_{i}/\nu|$, with strict inequality, $\forall i \in I$. This is equivalent to $\rho^{\gamma} |\lambda_{i}|^{1} > |\lambda_{i}|^{1}$, which implies that $\forall i \in I$, each summand $\rightarrow 0$ as $j \rightarrow \infty$. As the number of summands is finite, the RHS and hence LHS of (29) also $\rightarrow 0$ as the epoch number $j \rightarrow \infty$. It is then trivial to show that $\rho^{d+j}m E|\hat{x}_{j}^{i}\|_{m} \rightarrow 0$ as the integer time $k \rightarrow \infty$. Thus Coder-Controller 1 achieves exponential stability in the sense of (4) for any data rate $R$ satisfying (5). This completes the proof of Theorem 1.

6. CONCLUSION

In this paper the problem of exponentially stabilising a multidimensional, noiseless, linear time-invariant system when the feedback loop can only carry a finite data rate was posed. Particular attention was given to the question of determining the smallest data rate above which stability with a specified exponential decay can be achieved, when no restrictions apart from causality are placed on the coder and controller. By casting the problem as one of moment stabilisation, the problem was shown to be equivalent.
to finding a sequence of recursive quantisers for the initial state that yielded an exponentially diminishing mean $m$th power error. Asymptotic quantisation theory and a new lower bound were used to derive the smallest data rate in terms of the eigenvalues of the system and the desired decay constant and a stabilising coder-controller was explicitly constructed. Similar techniques are currently being investigated for jump Markov linear systems (Nair et al., 2002) and digital networks consisting of two or more coupled linear systems.

7. REFERENCES


