STABILIZATION OF STATE-DELAYED NONLINEAR SYSTEMS

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Abstract: Dead-time or delay is a common problem in the processing industry. Sensors, actuators and transport phenomena often contribute to delay the process response or to prevent immediate action. Current control technology offers various alternatives to analyze and control delay systems. Among the most widely applied techniques are internal model control (IMC) and, to a lesser extent, model predictive control. Although these methods are useful in the control of most delay systems, they can be greatly affected by a poor knowledge of the dead-time. In the study of nonlinear systems, the problem of delay dependent designs remains unresolved. In the current paper, a sufficient condition is obtained for the existence of a state-feedback controller that guarantees stability for a class of nonlinear state-delay control systems.

Keywords: Time delay, nonlinear systems, stabilization

1. INTRODUCTION

The study of linear time-delay systems has been of great interest over the last few years and a number of approaches currently exist (see (Jeung et al. 1996), (Jong et al. 1996), (Mahmoud and Al-Muthairi 1994), (Phoojaruenchananachai and Furuta 1992) and (Verriest 1994)). In the study of nonlinear systems, the problem of delay dependent memoryless feedback designs remains poorly understood. Most studies of nonlinear delay systems have dealt with delay-independent techniques where the effect and magnitude of the delay is assumed to be known. In this setting, controller can be designed to handle very large, even infinite, but known delays (see (Antoniades and Christofides 1999), (Haddad et al. 1997), (Y. Wang and de Souza 1992) and (Kwon and Pearson year) and references therein).

Over the last few years, a number of authors have considered the problem of delay-dependent stabilization of nonlinear systems. In (Teel 1998), a relationship between the Ramizukhin-type theorems (see (Hale and Lunel 1993)) and the ISS small gain theorem is analyzed. It is shown that the stabilization of a state-delayed nonlinear system can be ensured for arbitrary time delay less than a computable upper bound. Exponential stability of nonlinear systems with delayed states has been considered in (Mao 1996) who considered delay-dependent stability conditions for nonlinear systems. A delay-dependent design of nonlinear state-delayed control systems, where delayed states appear linearly, are given in (Yanushkevsky 1999).

In a recent paper ((Jankovic 2000)), the problem of robust stabilization of state-delayed nonlinear systems is discussed. The approach taken is based on the concept of control Lyapunov Razumikhin functions (CLRFS) whose existence guarantees the existence of a stabilizing feedback. Use of the Razumikhin condition avoids the need the construction of Lyapunov functionals and facilitates the development of a feedback law through domination re-design.

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In the current paper, we treat a similar problem by adopting the approach described in (Li and de Souza 1997b), (Li and de Souza 1997a) and (Li and de Souza n.d.). In particular, we develop a sufficient condition for the stability of a class of nonlinear state-delayed systems. As in (Li and de Souza n.d.), the problem is solved by constructing a Lyapunov functional that reduces to a delay dependent condition. In doing this, we do not strictly impose the Razumikhin condition as in (Jankovic 2000). However, we show that, through an appropriate choice of Lyapunov functional, the conditions for stability can be related to the existence of a Lyapunov function for the undelayed system. In this context, the stability of nonlinear state-delayed systems reduces to the construction of a Lyapunov function for the undelayed system that fulfills a certain growth condition. Given a particular Lyapunov function, the conditions can be checked easily to verify the stability of a system for various values of the delay. Using this Lyapunov function, the stabilization problem can be handled in a standard fashion by introducing the concept of a delay-dependent control Lyapunov function (DDCLF). The existence of a DDCLF may prove to be useful for the development of appropriate design strategies for state-delayed nonlinear systems.

The paper is as follows. In Section 2, we provide some background on the approach. The stabilization of state-delayed nonlinear control-affine systems is presented in Section 3. Some conclusions are presented in Section 4.

2. PRELIMINARY

In this paper, we consider nonlinear delay control systems of the form

\[
\begin{align*}
\dot{x}(t) &= f(x(t)) + f_d(x(t - \tau)) + g(x(t))u(t) \\
\end{align*}
\]

where \( x \in M \subset \mathbb{R}^n \) are the process states, \( u \in \mathbb{R}^p \) are the process inputs, \( y \in \mathbb{R}^q \) are the process outputs. The vector fields \( f(x) \), \( f_d(x) \), \( g(x) \) and \( k(x) \) are smooth along with the function \( h(x) \). The delay \( \tau \) appears in the vector field \( f_d(x) \). The initial condition of this system is assumed to be a continuous function \( x(t) = \varphi(t) \) for \( t \in [-\tau, 0) \) such that \( \varphi(0) = x_0 \) and \( \varphi(-\tau) = x_0 \). Without loss of generality, we assume that \( f(0) = 0 \), \( f_d(0) = 0 \) and \( h(0) = 0 \).

It is to show that the nonlinear delay system eq.1 can be rewritten as follows

\[
\begin{align*}
\dot{x}(t) &= f(x(t)) + f_d(x(t)) - \\
&\quad \int_{-\tau}^{0} \left. \frac{\partial f_d(x(t + \theta))}{\partial x} \right|_{\theta=0} x(t+\theta) d\theta + [f(x(t + \theta)) + f_d(x(t - \tau + \theta))]
\end{align*}
\]

\[
\times \left[ g(x(t + \theta))u(t + \theta) + g(x(t))u(t) \right] d\theta + g(x(t))u(t).
\]

Eq. 2 is called the functional differential equation associated with the state-delayed nonlinear system eq.1. System eq.2 is a more general representation of the delay system eq.1. Furthermore, it is more amenable to the application of Lyapunov functionals, the main ingredient in the developments given in Sections 3 and 4. In the sequel, we will require the following assumption.

**Remark 2.1.** Any solutions of eq.1 for a given \( u(t) : \mathbb{R} \rightarrow \mathbb{L}_2 \) are also solutions of eq. 2.

As a result, stability of eq.2 will also be interpreted as stability of the original state-delayed system eq.1. We first consider the following definitions.

**Definition 2.1.** Delay-dependent Stability: The time-delay system eq.1 is said to be closed-loop stable if the trivial solution \( x(t)=0 \) of the functional differential equation associated to eq.1 with \( u(t) = K(x(t)) \) is globally asymptotically stable (GAS) for any constant delay satisfying \( 0 \leq \tau \leq \bar{\tau} \).

**Definition 2.2.** Delay-dependent Stabilization Problem: Find a nonlinear memoryless state feedback control law \( u(t) = K(x(t)) \) for the nonlinear system eq.2 such that the resulting closed-loop system is stable for any constant time-delay \( \tau \) such that \( 0 \leq \tau \leq \bar{\tau} \).

3. STABILIZATION

We first consider the stabilization problem. Delayed differential equations (often referred to as differential-difference equations) make up a special class of so-called functional differential equations.

3.1 Stability considerations

The stability of this class of systems was initially addressed in (Hale and Lunel 1993) where sufficient conditions for stability were derived. The conditions generalize the Lyapunov stability results of finite dimensional systems to function differential equation systems, which includes the class of systems under study. As in the classical approach, stability is shown to be related to the existence of positive definite functionals whose time derivatives can be made sufficiently negative in a region of the origin. The standard result, taken from (Hale and Lunel 1993), is given in the following theorem. The reader is referred to (Hale and Lunel 1993) for the proof.

**Theorem 3.1.** Consider the functional differential equation system
\[ \dot{x}(t) = f(x(t)) + f_d(x(t)) - \int_{-\tau}^{0} \frac{\partial f_d(x(t + \theta))}{\partial x} \, d\theta \times [f(x(t + \theta)) + f_d(x(t - \tau + \theta))] \frac{d\theta}{\theta} \] (3)

Let the initial conditions be \( \phi : [-2\tau, 0] \to C \) where \( C \) denotes the set of continuous functions from the interval \([-2\tau, 0]\) mapping to \(R^N\). Suppose \( V : C \to R \) is continuous on the set \( U_\tau = \{ \phi \in C \mid V(\phi) \leq \ell \} \) and there is a \( K = K(\ell) \) such that \( \phi \in U_\tau \Rightarrow |\phi(0)| < K \). Furthermore, let \( \alpha(\cdot) \) and \( \sigma(\cdot) \) be nonnegative functions such that \( \alpha(r) \to \infty \) as \( r \to \infty \)

\[ \alpha(\|\phi(0)\|) \leq V(\phi) \]

\[ V(\phi) \leq -\sigma(\|\phi(0)\|) \]

then every solution of eq.3 is bounded. If, in addition, \( \sigma \) is positive definite, then the origin is globally asymptotically stable.

The main ingredient of the last theorem is the construction of a Lyapunov functional for the system eq.3. Since such functionals may be difficult to obtain, most approaches used to prove the stability of time-delay problems seek Lyapunov functions whose existence and properties can be related to the existence of a Lyapunov functional that fulfill the conditions of Theorem 3.1. A standard approach, due to Razumikhin, consists in finding a Lyapunov function whose value at \( x(t) \) provides an upper bound for all possible solutions \( x(t-\tau) \) for a given delay \( \tau \). Although the use of this condition alleviates the need for Lyapunov functionals, it may be difficult to obtain a Lyapunov functions that meets the Razumikhin condition. In the following, it is shown how an appropriate Lyapunov functional can be constructed to solve the problem. By making a judicious choice of functional, the stability conditions can be expressed in terms of a Lyapunov function for the undelayed process, thus avoiding the Razumikhin condition.

Before stating and proving the main result, we recall a simple standard identity, stated without proof, that will be used in the sequel.

**Lemma 3.1.** Let \( \alpha : R^n \to R \) be a smooth scalar function such that \( \alpha(0) \neq 0 \). This function can always be written in the form

\[ \alpha(x) = \alpha(0) + \psi(x)x \] (4)

The following notation will be used. Given a vector \( \omega \) and square symmetric matrix \( A \), we write

\[ \|\omega\|^2 = \omega^T \Gamma \omega \] (5)

where the superscript \( T \) represents the transpose operation.

We state the main result of this study.

**Theorem 3.2.** Consider the nonlinear functional differential system eq.3 and assume that the undelayed process is globally asymptotically stable with radially bounded positive definite function \( V(x(t)) \) such that for some positive definite matrices \( P_1 \) and \( P_2 \),

\[ 0 \geq \frac{dV}{dt} = \frac{\partial V}{\partial x} (f(x) + f_d(x)) \]

\[ - \left[ \int_{-\tau}^{0} \frac{\partial V}{\partial x}(x(t + \theta)) \frac{\partial f_d(x(t + \theta))}{\partial x} f(x(t + \theta)) d\theta \right] \]

\[ + W(\phi) \]

for the smooth functions \( \psi_{i,j}, 1 \leq i, j \leq n, \) defined in eq.(15) and for all \( \tau \) satisfying \( 0 \leq \tau \leq \tau_\bar{c} \). Then the origin of the functional differential system eq.3 is globally asymptotically stable for any \( \tau \) satisfying \( 0 \leq \tau \leq \tau_\bar{c} \).

**Proof:** We prove this result by first constructing a Lyapunov functional of the form

\[ V(\phi) = V(\phi(0)) + W(\phi) \] (7)

where \( \phi : [\tau, 0] \to C \) are the initial conditions of the functional differential equation eq.3 and \( V(x) \) is a positive definite radially unbounded function. The positive definite functional \( W \) is defined as

\[ W(\phi) = \int_{-\tau+\theta}^{\tau} \frac{1}{2} \left[ \frac{\partial f_d(x(s))}{\partial x} f(x(s)) \right]^2 ds \]

\[ + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ (\psi_{ij}(x(s))x(s))^2 \right] ds \]

\[ + \int_{-\tau+\theta}^{\tau} \frac{1}{2} \left[ \frac{\partial f_d(x(s))}{\partial x} f(x(s)) \right]^2 ds \]

\[ + \int_{-\tau+\theta}^{\tau} \frac{1}{2} \left[ \frac{\partial f_d(x(s))}{\partial x} f(x(s)) \right]^2 ds \]

Next we show that this particular choice of functional yields the required conditions.

First differentiate the Lyapunov functional with respect to \( t \). This gives

\[ \frac{dV}{dt} = \frac{\partial V}{\partial x}(f(x) + f_d(x)) \]

\[ - \left[ \int_{-\tau}^{0} \frac{\partial V}{\partial x}(x(t + \theta)) \frac{\partial f_d(x(t + \theta))}{\partial x} f(x(t + \theta)) d\theta \right] \]

\[ + W(\phi) \] (9)
By completion of squares
\[ -2d^T v \leq u^T Qu + v^T Q^{-1} v \] (10)

and using the fact that
\[ \int_{-\tau}^{0} \|\lambda(x(t + \theta))\|^2 d\theta \geq \left( \int_{-\tau}^{0} \lambda(x(t + \theta)) d\theta \right)^2 \] (11)

for some continuous function \( \lambda(x) \), we obtain the inequality
\[
\frac{d\tilde{V}}{dt} \leq \frac{\partial V}{\partial x} (f(x) + f_d(x)) + \frac{\tau}{2} \left\| \frac{\partial V}{\partial x} \right\|^2_{P_1 + P_2} + \frac{1}{2} \int_{-\tau}^{0} \left( \frac{\partial f_d(x(t + \theta))}{\partial x} f(x(t + \theta)) \right)^2 d\theta \\
+ \frac{1}{2} \int_{-\tau}^{0} \left( \frac{\partial f_d(x(t + \theta))}{\partial x} f_d(x(t + \theta - \tau)) \right)^2 d\theta \\
+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (\psi_{ij}(x(t + \theta))) (x(t + \theta))^2.
\]

Next, observe that
\[
\frac{\partial}{\partial x} f_d(x(t + \theta - \tau)) = \sum_{i=1}^{n} \sum_{j=1}^{n} f_{d_i}(x(t + \theta - \tau)) f_{d_j}(x(t + \theta - \tau)) \times \alpha_{ij}(x(t + \theta))
\]

where
\[
\alpha_{ij}(x) = \left. \frac{\partial f_d(x)}{\partial x} \right|_{\sigma = x(t + \theta)}
\]

By Lemma 3.1, smoothness of \( f_d(x) \) implies that \( \alpha_{ij}(x) \), \( 1 \leq i, j \leq n \) are smooth functions of the delayed states \( x(t + \tau) \). We can then write
\[
\alpha_{ij}(x(t + \theta)) = \alpha_{ij}(0) + \psi_{ij}(x(t + \theta)) x(t + \theta)
\]

where
\[
\psi_{ij} = \int_{0}^{1} \frac{\partial \alpha_{ij}(x(t + \theta))}{\partial v} dv \bigg|_{v = x(t + \theta)}
\]

Then eq.13 becomes
\[
\left\| \frac{\partial f_d(x(t + \theta))}{\partial x} f_d(x(t + \theta - \tau)) \right\|^2_{P_2^{-1}} \\
= \left\| \frac{\partial f_d(x(t + \theta))}{\partial x} f_d(x(t + \theta - \tau)) \right\|^2_{P_2^{-1}} \\
+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} f_{d_i}(x(t + \theta - \tau)) f_{d_j}(x(t + \theta - \tau))
\]

The second term in eq.17 can be broken down further if we consider the following
\[
\sum_{i=1}^{n} \sum_{j=1}^{n} f_{d_i}(x(t + \theta - \tau)) f_{d_j}(x(t + \theta - \tau)) \\
\times \psi_{ij}(x(t + \theta)) x(t + \theta) \leq \\
\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (f_{d_i}(x(t + \theta - \tau)) f_{d_j}(x(t + \theta - \tau)))^2 \\
+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (\psi_{ij}(x(t + \theta))) (x(t + \theta))^2.
\]

Using these identities it is straightforward to show that the definition of \( W(\phi) \) is such that 12 simplifies to
\[
\frac{d\tilde{V}}{dt} \leq \frac{\partial V}{\partial x} (f(x) + f_d(x)) + \frac{\tau}{2} \left\| \frac{\partial V}{\partial x} \right\|^2_{P_1 + P_2} + \frac{\tau}{2} f_d(x)^T P_2^{-1} \frac{\partial f_d(0)}{\partial x} f_d(x).
\]

The condition eq.6 ensures that
\[
\frac{d\tilde{V}}{dt} \leq 0
\]

for some \( \tau \). By global asymptotic stability of the undelayed system, we have that
\[
\frac{\partial V}{\partial x} (f(x) + f_d(x)) < 0
\]

for all \( x \neq 0 \). Since the RHS of condition eq.6 is a monotonically increasing function of \( \tau \) it follows that there exists a well-defined maximum value \( \tilde{\tau} \) for which
\[
0 = \frac{\partial V}{\partial x} (f(x) + f_d(x)) + \frac{\tau}{2} \left\| \frac{\partial V}{\partial x} \right\|^2_{P_1 + P_2} + \frac{\tau}{2} f_d(x)^T P_2^{-1} \frac{\partial f_d(0)}{\partial x} f_d(x).
\]

This ensures that
\[
\frac{d\tilde{V}}{dt} < -w(\phi(0)) = -w(x)
\]

for \( x \neq 0 \) where \( w(x) \) is positive definite for all \( \tau \) satisfying \( 0 \leq \tau < \tilde{\tau} \). Finally we can easily check that
the functional $W(\phi)$ is positive definite and such that $W(\phi) \to \infty$ as $||\phi(0)|| \to \infty$.

As a result, the functional $V(\phi)$ fulfills the conditions of Theorem 3.1. Global asymptotic stability follows. This completes the proof. \textit{Q.E.D.}

One of the main advantage of Theorem 3.2 is that it provides a condition for the closed-loop stability of the nonlinear state-delayed system under delay-dependent memoryless feedback $u(t) = \alpha(x(t))$. To see this, consider the nonlinear functional differential control system with input $u(t)$,

$$x(t) = f(x(t)) + f_d(x(t)) + g(x(t))u(t) - \int_{-\tau}^{0} \frac{\partial f_d(x(t + \theta))}{\partial x} x(t + \theta) d\theta. \quad (22)$$

Imposing the feedback $u(t) = \alpha(x(t))$ leads to a closed-loop system with a structure similar to the nonlinear functional differential systems treated in Theorem 3.2. Hence, the existence of a positive definite Lyapunov function that satisfies an equality similar to eq.6 guarantees the stability of the closed-loop for a range on unknown input delays. We state the corresponding inequality for the closed-loop system in the following.

\textit{Lemma 3.2.} Suppose that there exists a positive definite radially unbounded function $V$, $V(0) = 0$, that fulfills the inequality

$$0 \geq \frac{\partial V}{\partial x}(f(x) + f_d(x)) + \frac{\tau}{2} \left\| \frac{\partial V}{\partial x} f(x) \right\|_{P_3}^2 + \frac{\tau}{2} \left\| \frac{\partial f_d(0)}{\partial x} f(x) \right\|_{P_3}^2 + \frac{\tau}{2} \left\| \frac{\partial f_d(x)}{\partial x} g(x)u \right\|_{P_2}^2 + \frac{\tau}{4} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ (\psi_i(x) x)^2 + (f_d(x) f_d(x)) \right]$$

where $u$ is given by a delay-dependent state-feedback $\alpha(x)$. Then the closed-loop function differential system eq.22 is globally asymptotically stable for all $\tau$, $0 \leq \tau \leq \bar{\tau}$.

\textit{Proof:} This is a direct consequence of the proof of Theorem 3.2. \textit{Q.E.D.}

Lemma 3.2 provides a sufficient condition for the stability of a closed-loop nonlinear system subject to a range of unknown constant delays. We can consider this inequality directly to define a more general stabilization criteria. First, observe that eq.23 can be written as

$$\{ a_1(x) + \tau a_2(x) + b(x)^T u + \tau u^T R(x) u \} < 0 \quad (24)$$

where the functions $a_1(x)$, $a_2(x)$ and $b(x)$, and the positive semi-definite matrix-valued function $R(x)$ can be obtained readily from inspection of eq.23.

In fact, we can interpret the conditions of Lemma 3.2 as stabilization conditions. Thus, we say that the delayed system is stabilizable for unknown delays if there exists a smooth radially unbounded positive definite function $V(x)$ that fulfills the strict inequality

$$\inf_{u} \{ a_1(x) + \tau a_2(x) + b(x)^T u + \tau u^T R(x) u \} < 0 \quad (25)$$

is fulfilled for some $\tau$, $0 \leq \tau \leq \bar{\tau}$ and $\forall x \in R^N$. We call such a function a delay-dependent control Lyapunov function (DDCLF). In order to design a control law to ensure that eq.25 is fulfilled at all $x \neq 0$, we proceed as follows. Consider the corresponding Hamilton-Jacobi-Bellman type equation

$$a_1(x) + \tau a_2(x) + b(x)u + u^T (\Gamma(x) + \tau R(x)) u + l(x) = 0 \quad (26)$$

where $\Gamma(x)$ is a positive definite matrix function that is nonsingular for all $x$ and $l(x)$ is some positive definite function. It is clear that if the function $V(x)$ solves this equation then it is automatically a DDCLF. The corresponding optimal feedback

$$u = -\frac{1}{2} (\Gamma(x) + \tau R(x))^{-1} b(x)^T \quad (27)$$

yields the inequality

$$a_1(x) + \tau a_2(x) - \frac{1}{4} b(x)^T (\Gamma(x) + \tau R(x))^{-1} b(x) \leq 0. \quad (28)$$

This observation indicates that, under the feedback eq.27, a DDCLF must be chosen to satisfy eq.28. Furthermore, it must be such that

$$b(x) = 0 \Rightarrow a_1(x) + \tau a_2(x) < 0 \quad (29)$$

where eq.29 is the standard requirement for a robust control Lyapunov function as defined in (Krstic and Deng 1998). As a result, if a robust CLF exists for the undelayed plant then this CLF can be transformed to a DDCLF by imposing, in addition, that eq.28 is fulfilled for the fixed matrix $R(x)$. If the matrix $R(x)$ is nonsingular for all $x$ then the addition of $\Gamma(x)$ is not required and therefore, we consider the inequality

$$a_1(x) + \tau a_2(x) - \frac{1}{4} b(x)^T (\tau R(x))^{-1} b(x) \leq 0. \quad (30)$$

In this case, fulfillment of Eq.30 may be achieved by simply considering a simple Lyapunov redesign of the robust CLF-based controller design for the undelayed plant.
It is important to note that the application of Lemma 3.2 implies that the upper bound on the delay depends on the choice of Lyapunov function for the original system. This assumption may be exceedingly conservative unless a suitable choice of the DDCLF is available.

4. CONCLUSIONS

A sufficient condition was derived for the delay-dependent stability and stabilization of a class of state-delayed nonlinear systems. Through the construction of a suitable Lyapunov functional, the stability of a state-delayed systems is related to the existence of a Lyapunov function for the undelayed system that meets certain growth conditions dependent on the magnitude of the delay. Like for the Razumikhin condition, the explicit need for a Lyapunov functional is avoided. However, the explicit fulfillment of the Razumikhin condition is avoided.

5. REFERENCES


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