GLOBAL OUTPUT REGULATION FOR LOWER TRIANGULAR NONLINEAR SYSTEMS

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Abstract: Recently a general framework for studying robust output regulation problem for nonlinear systems is established. This framework is able to convert the robust output regulation problem for nonlinear systems into a robust stabilization problem, thus offering greater flexibility to incorporate recent new stabilization techniques for tackling the global output regulation problem. This paper successfully applies this new framework to solve the global output problem for a class of lower triangular uncertain nonlinear systems.

Keywords: Global output regulation, Stabilization, Nonlinear control.

1. INTRODUCTION

Research on the nonlinear output regulation has made steady progress since the publication of the paper by (Isidori and Byrnes, 1990). The robust version of the same problem was studied in (Huang and Lin, 1991, 1993; Huang, 1995; Delli Priscoli, 1993; Khalil, 1994; Byrnes et al, 1997). These researches have led to various methods for synthesizing controllers for achieving asymptotic tracking and disturbance rejection in an uncertain nonlinear system with local stability. Recently, efforts have also been devoted to nonlinear output regulation with nonlocal stability (Khalil, 1994; Isidori, 1997; Serrari and Isidori, 2000; Serrari et al, 2001; Ye and Huang, 2001). Nevertheless, there are only very limited results on global nonlinear output regulation problem. The main reason is that the current framework has a fundamental limitation in that it inherently utilizes the Lyapunov’s linearization method to achieve stabilization, thus is handicapped in dealing with output regulation with global stability. Very recently, a general framework is proposed for tackling the output regulation problem that is able to convert the robust output regulation problem for nonlinear systems into a robust stabilization problem, thus offering greater flexibility to incorporate recent new stabilization techniques (Huang and Chen, 2002).

The objective of this paper is to apply the framework established in (Huang and Chen, 2002) to solve the global output problem for a class of lower triangular uncertain nonlinear systems. Our approach consists of three major tasks. First, find an appropriate internal model that can reproduce part of the solution of the regulator equation. Second, convert the output regulation problem for the given plant into a robust stabilization problem for a well defined augmented system. Finally, solve the robust stabilization problem for this augmented system. While the first two tasks can be systematically handled following our general framework, the third task involves nontrivial challenges in that the augmented system does not appear lower triangular, and may not automatically satisfy some conditions needed for executing the recursive procedure for stabilizing the triangular systems (Jiang et al, 1997; Isidori, 1999). Nevertheless, these difficulties have been overcome and the problem has been successfully solved.

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Consider the plant described by
\[
\dot{x}(t) = f(x(t), u(t), v(t), w), x(0) = x_0 \tag{2.1}
\]
\[
e(t) = h(x(t), u(t), v(t), w), t \geq 0
\]
\[
\dot{v}(t) = A_1 v(t), \quad v(0) = v_0 \tag{2.2}
\]
where \( x(t) \) is the n-dimensional plant state, \( u(t) \) the m-dimensional plant input, \( e(t) \) the p-dimensional plant output representing the tracking error, \( v(t) \) the q-dimensional exogenous signal representing the disturbance and/or the reference input, and \( w \) the N-dimensional plant uncertain parameters. And all the eigenvalues of \( A_1 \) are simple with zero real parts.

The class of control laws considered here are described by
\[
u(t) = k(x(t), z(t), e(t)), \quad \dot{z}(t) = f_2(x(t), z(t), e(t)), \quad z(0) = z_0 \tag{2.3}
\]
where \( z(t) \) is the compensator state vector of dimension \( n_c \) to be specified later. With \( x_c = (x, z) \), the closed-loop system can be written as
\[
\begin{align*}
\dot{x}_c(t) &= f_2(x_c(t), v(t), w), x_c(0) = x_0 \\
y(t) &= h_c(x_c(t), v(t), w)
\end{align*} \tag{2.4}
\]
For simplicity, all the functions involved in this setup are assumed to be sufficiently smooth and defined globally on the appropriate Euclidean spaces with the value zero at the respective origins. Also it is assumed that 0 is the nominal value of the uncertain parameter \( w \), and \( f(0, 0, 0, w) = 0 \) and \( h(0, 0, 0, w) = 0 \) for all \( w \in \mathbb{R}^N \).

### 2.1 Global robust output regulation problem:

Find a control law of the form (2.3) such that

(i) The equilibrium of \( \frac{\partial}{\partial x_c} (0, 0, 0) \) is globally asymptotically stable.

(ii) For all \( v(t) \in V \) and \( w \in W \) where \( V, W \) are prescribed compact sets of \( \mathbb{R}^q \) and \( \mathbb{R}^N \) containing the origins of \( \mathbb{R}^q \) and \( \mathbb{R}^N \), respectively, the trajectories of the closed-loop system (2.4) starting from any initial conditions \((x_c(0), v_0)\) exist and are bounded for all \( t > 0 \), and satisfy \( \lim_{t \to \infty} y(t) = 0 \).

In (Huang and Chen, 2001), a framework is established that can convert a robust output regulation problem into a robust stabilization problem. This framework is briefly described below.

### 2.2 Definition:

Assume the nonlinear system (2.1) and (2.2) satisfies:

A1: There exist sufficiently smooth functions \( x(v, w) \) and \( u(v, w) \) defined for all \( v \in \mathbb{R}^q \) and \( w \in \mathbb{W} \) with \( x(0, 0) = 0 \) and \( u(0, 0) = 0 \) satisfying the following
\[
\begin{align*}
\frac{dx(v(t), w)}{dt} &= f(x(v(t), w), u(v(t), w), v(t), w) \\
0 &= h(x(v(t), w), u(v(t), w), v(t), w), t \geq 0
\end{align*}
\]
Let \( g : \mathbb{R}^{n+m} \to \mathbb{R}^l \) be a mapping for some positive integer \( 1 \leq l \leq n + m \), \( \alpha : \mathbb{R}^l \to \mathbb{R}^r \), \( \beta : \mathbb{R}^r \to \mathbb{R}^l \) for some integer \( s \). Then the following system
\[
t = \alpha(t), \quad y = \beta(t) \tag{2.5}
\]
is said to be a steady state generator with output \( g \) if there exists a mapping \( \theta : \mathbb{R}^{n+m} \to \mathbb{R}^l \) such that
\[
\begin{align*}
\frac{d\theta(v(t), w)}{dt} &= \alpha(\theta(v(t), w)) \\
g(x(v(t), w), u(v(t), w)) &= \beta(\theta(v(t), w))
\end{align*}
\]

### 2.3 Lemma:

Let \( u(v(t), w) \) be a trigonometric polynomial of the following form,
\[
u(v(t), w) = \sum_{j=-n_k}^{n_k} C_j(w, v_0) e^{ij\omega_t} \tag{2.6}
\]
where \( n_k \) is some finite integer, and, for \( l = 0, \pm 1, \cdots, \pm n_k \), \( C_l \) are \( m \)-dimensional column vectors, and, for \( l \neq 0 \), \( \omega_l = -\omega_{-l} \), and \( C_l = C_{-l} \) where \( C_l \) is the conjugate complex of \( C_l \). Then there exist some set of \( r \) real numbers \( a_0, a_1, \cdots, a_{r-1} \) such that
\[
\begin{align*}
\frac{d^r u(v(t), w)}{dt^r} - a_0 u(v(t), w) - a_1 \frac{d u(v(t), w)}{dt} - \cdots - a_{r-1} \frac{d^{r-1} u(v(t), w)}{dt^{r-1}} &= 0 \tag{2.7}
\end{align*}
\]

### 2.4 Proposition:

Under assumption A1, and suppose \( x_i(v(t), w), \ i = 1, \cdots, d \) and \( u(v, w) \) are trigonometric polynomial of the form (2.6). Then there exists a linear steady state generator with output \( g(x, u) = [x_1, x_2, \cdots, x_d, u]^T \) as follows
\[
\begin{align*}
\frac{d\theta(v(t), w)}{dt} &= \Phi \theta(v(t), w) \\
g(x(v(t), w), u(v(t), w)) &= \Psi \theta(v(t), w)
\end{align*}
\]
where \((\Phi, \Psi)\) is observable. Moreover, let \( T \) be any nonsingular matrix of proper dimension, and let
\[
\begin{align*}
\theta(v(t), w) &= T \theta(v(t), w), \alpha(\theta) = T \Phi T^{-1} \theta, \\
\beta(\theta) = \Psi T^{-1} \theta
\end{align*} \tag{2.9}
\]
It is ready to verify that \((\theta(v(t), w), \alpha(\theta), \beta(\theta))\) is also a steady state generator with output \( g(x, u) \).

### 2.5 Remark:

It was shown in (Huang, 2001) that if the solution of the regulator equation is a polynomial in \( v \) or satisfying a partial differential equation as given in (Byrnes et al, 1997). Then the trigonometric polynomial condition of \( x(v(t), w) \) and \( u(v(t), w) \) is satisfied.

### 2.6 Definition:

Assume for some \( 0 \leq d \leq n \), the nonlinear system (2.1) and (2.2) has a steady state generator with output \( g(x, u) \). Then one calls the following system
\[
\dot{\eta} = \gamma(\eta, x, u, e) \tag{2.10}
\]
an internal model with output \( g(x, u) \) if
\[
\begin{align*}
\gamma(\theta(v(t), w), x(v(t), w), u(v(t), w), 0)) &= \alpha(\theta(v(t), w))
\end{align*}
\]

### 2.7 Remark:

Given any controllable pairs \((M, N)\) such that \( \Phi \) and \( M \) have disjoint spectrum, there exists a
unique, nonsingular matrix $T$ satisfying the Sylvester equation

$$T \Phi - M T = N \Psi$$

Let

$$\gamma(\eta, x, u, e) = M \eta + N g(x, u) \quad (2.11)$$

Then (2.11) is an internal model with output $g(x, u)$ corresponding to the steady state generator defined by (2.9). (2.11) can be viewed as a generalization of the canonical internal model used in (Nikiforov, 1998; Serrari and Isidori, 2000).

2.8 Theorem: Suppose $A_1$, and assume system (2.1) and (2.2) has a steady state generator described by (2.5) with output $g(x, u)$ and an internal model described by (2.10). Let the following coordinates and input transformation be given

$$\tilde{\eta} = \eta - \theta(v, w), \quad \tilde{x}_i = x_i - \beta_i(\eta), \quad \tilde{\xi}_i = x_i - \beta_i(\eta), \quad \tilde{u} = u - [\beta_{d+1}(\eta), \ldots, \beta_{d+n}(\eta)]^T - \beta_n(\eta)$$

Then the augmented system composed of (2.1), (2.2) and (2.10) in the new coordinates and input denoted by

$$\tilde{\eta} = \tilde{f}(\tilde{\eta}, \tilde{x}, \tilde{u}, v, w)$$

$$\tilde{x} = \tilde{g}(\tilde{\eta}, \tilde{x}, \tilde{u}, v, w)$$

$$\tilde{e} = \tilde{h}(\tilde{\eta}, \tilde{x}, \tilde{u}, v, w) \quad (2.12)$$

has the property that, for all trajectories $v(t)$ of the exosystem, and all $w \in W$,

$$\tilde{f}(0, 0, 0, v(t), w) = 0$$

$$\tilde{g}(0, 0, 0, v(t), w) = 0$$

$$\tilde{h}(0, 0, 0, v(t), w) = 0$$

2.9 Remark: Consider a controller of the form

$$\tilde{u} = k(\tilde{x}_1, \ldots, \tilde{x}_d, \tilde{\xi}, e),$$

$$\tilde{\xi} = k(\tilde{x}_1, \ldots, \tilde{x}_d, \tilde{\xi}, e)$$

where $\tilde{\xi} \in R^c$, and $k(0, \ldots, 0) = 0$, and $\tilde{\xi}(0, \ldots, 0) = 0$. If the equilibrium of the closed-loop system composed of (2.12) and (2.13) is globally asymptotically stable for all $v(t) \in V$ and $w \in W$, then the same controller also solves the global robust regulation problem of the augmented system since $\tilde{h}(0, 0, 0, v(t), w) = 0$. As a result, the following controller

$$u = \beta_0(\eta) + k(x_1 - \beta_1(\eta), \ldots, x_d - \beta_d(\eta), \tilde{\xi}, e)$$

$$\tilde{\eta} = \gamma(\eta, x, u, e)$$

$$\tilde{\xi} = \zeta(x_1 - \beta_1(\eta), \ldots, x_d - \beta_d(\eta), \tilde{\xi}, e)$$

solves the output regulation problem globally for the original system (2.1) and (2.2).

3. MAJOR RESULT

Consider the class of nonlinear system in lower-triangular form,

$$\dot{z} = f(z, x_1, v, w)$$

$$\dot{x}_1 = f_1(z, x_1, v, w) + g_1(z, v, w) x_2$$

$$\vdots$$

$$\dot{x}_r = f_r(z, x_1, \ldots, x_{r-1}, v, w) + g_r(z, \ldots, x_{r-1}, v, w) u$$

$$e = x_1 - q(v, w) \quad (3.1)$$

For the special case where $g_i$’s are constant, (Isidori, 1997) has presented the solution of the semiglobal output regulation. Here the global robust output regulation problem for this class of systems with state feedback will be further considered. At the outset, the following assumptions are made.

A2: There exists a sufficiently smooth function $z(x, w)$ with $z(0, 0) = 0$ satisfying the following equation for all $v(t) \in V, w \in W$,

$$\frac{dz(v(t), w)}{dt} = f(z(v(t), w), q(v(t), w), v(t), w)$$

Further $z(v, w)$ is a trigonometric polynomial of the form (2.6).

A3: $f_i(\cdot)$ and $g_i^{-1}(\cdot), i = 1, \ldots, r$, are polynomials in $z, x_1, \ldots, x_r, u, v$.

A4: For $i = 1, \ldots, r$, there exist $b_i > 0$ such that

$$g_i(z, x_1, \ldots, x_{r-1}, v, w) \geq b_i$$

for all $z, x_1, \ldots, x_{r-1}, v$ and all $w \in W$.

Under these assumptions, the solution of the regulator equation for this system exists globally, and will be denoted by $z(v, w), x_i(v, w), u_i(v, w)$ with $x_i(v, w) = (x_1(v, w), \ldots, x_i(v, w))$. Moreover, there exist $(\theta_i, \Phi_i, \Psi_i), i = 2, \ldots, r + 1$, such that

$$\dot{\theta}_i(v(t), w) = \Phi_i(\theta_i(v(t), w), v(t), w)$$

$$x_i(v(t), w) = \Psi_i(\theta_i(v(t), w), v(t), w)$$

$$u_i(v(t), w) = \Psi_{r+1}(\theta_{r+1}(v(t), w))$$

Also there exist controllable pairs $(M_i, N_i)$ with $M_i$ Hurwitz that define the following systems:

$$\dot{\eta}_i = \gamma_i(\eta_i, x, u) = M_i \eta_i + N_i x_i, i = 2, \ldots, r$$

$$\eta_{r+1} = \gamma_{r+1}(\eta_{r+1}, x, u) = M_{r+1} \eta_{r+1} + N_{r+1} u$$

Thus let $\theta_i = (\theta_2, \ldots, \theta_i, \theta_{i+1}), g(x, u) = (x_2, \ldots, x_r, u)$, and $\eta_i = (\eta_2, \ldots, \eta_i, \eta_{i+1})$, one can obtain the steady state generator of the form (2.8) and the internal model of the form (2.11). As a result, one can define the transformation

$$\dot{\eta}_i = \eta_i - \theta_i, i = 2, \ldots, r + 1$$

$$\dot{x}_1 = x_1 - x_1(v, w) = e$$

$$\dot{x}_i = x_i - \Psi_i(\eta_i), i = 2, \ldots, r$$

$$\dot{e} = u - \Psi_{r+1}(\eta_{r+1}, v) \quad (3.2)$$

that leads to the augmented system (2.12) into the following form

$$\dot{\eta}_i = (M_i + N_i \Psi_i) \dot{\eta}_i + N_i \dot{x}_i, i = 2, \ldots, r + 1$$

$$\dot{\tilde{\xi}} = \tilde{f}(\tilde{\eta}, \tilde{x}, v, w)$$

$$\tilde{x}_i = f_i(\tilde{\eta}_2, \ldots, \tilde{\eta}_{i+1}, \tilde{x}_i, \ldots, \tilde{x}_j, v, w) + g_i(\tilde{\eta}_2, \ldots, \tilde{\eta}_{i-1}, \tilde{x}_i, \ldots, \tilde{x}_{j-1}, v, w) \tilde{x}_{i+1}, i = 1, \ldots, r$$
where \( \bar{x}_{n+1} = \bar{u} \) and \( \hat{f}, \hat{g}, i = 1, \ldots, r \) are sufficiently smooth vanishing at the origin, and \( \bar{g}, i = 1, \ldots, r \), still satisfy assumption A4.

By Theorem 2.8, the global robust output regulation problem for system (3.1) will be solved if the equilibrium of system (3.2) at \( (\bar{x}, \bar{x}, \bar{\eta}) = (0, 0, 0) \) can be made globally stable for all trajectories \( \nu(t) \) of the exosystem, and all \( w \in W \). To handle the global stabilization problem of (3.2), let us quote the following result in (Chen and Huang, 2001) which can be considered as a corollary of Lemma 11.4.1 of (Isidori, 1999) which in turn is based on the work of (Jiang et al, 1997).

### 3.1 Theorem:
Consider the system

\[
\dot{z} = \Phi(z, x, \mu), \quad \dot{x} = \Phi(z, x, \mu) + \Psi(z, x, \mu)u
\]

(3.3)

in which \( (z, x) \in R^n \times R \), \( \Phi(z, x, \mu) \) and \( \Psi(z, x, \mu) \) are polynomials in \( z \) and \( x \), and \( \Phi(0, 0, \mu) = 0, \Phi(0, 0, \mu) = 0 \) for \( \mu \in P \subset R^m \), with \( P \) a prescribed compact set containing the origin of \( R^n \). Suppose the following:

(i) for each \( \mu \) the subsystem \( \dot{z} = \Phi(z, x, \mu) \) is input-state-stable and, in particular, a class \( K_\infty \) function \( \kappa(\cdot) \), which is locally Lipschitz, and independent of \( \mu \), is known such that the response \( z(\cdot) \) to any bounded \( u(\cdot) \) satisfies

\[
\|z(t)\| \leq \max \left\{ \beta \|z(0)\|, \kappa(\|x(\cdot)\|_\infty) \right\}
\]

(3.4)

for some class \( KL \) function \( \beta(\cdot, \cdot) \).

(ii) there exists a numbers \( b_0 > 0 \) such that \( \Psi(z, x, \mu) \geq b_0 \) for all \( (z, x) \) and all \( \mu \in P \).

Then, there exists a smooth function \( k(x) \), with \( k(0) = 0 \), such that, under the control law

\[
u = k(x) + v
\]

(3.5)

the closed-loop system (3.3) and (3.5), viewed as a system with input \( v \) and state \( (z, x) \), is ISS and, in particular, a class \( K_\infty \) function \( \bar{\kappa}(\cdot) \), which is locally Lipschitz and independent of \( \mu \), is known such that the response \( Z(\cdot) = (z^T(\cdot), x(\cdot))^T \) to any bounded \( v(\cdot) \) satisfies

\[
\|Z(t)\| \leq \max \left\{ \bar{\beta} \|Z(0)\|, \bar{\kappa}(\|v(\cdot)\|_\infty) \right\}
\]

(3.6)

for some class \( KL \) function \( \bar{\beta}(\cdot, \cdot) \).

Also the following can be obtained

### 3.2 Theorem:
Consider the system

\[
\dot{z} = f(z, u, \mu), \quad \dot{x} = Ax + g(z, u, \mu)p(z, u, \mu)
\]

(3.7)

in which \( (z, x, u) \in R^{m_1} \times R^{m_2} \times R, \mu \in P, f(0, 0, \mu) = 0 \) and \( p(0, 0, \mu) = 0 \). Moreover \( p(z, u, \mu) \) is polynomial in \( z, u, A \) is Hurwitz and \( \|g(z, u, \mu)\| \leq g \) for all \( z \) and all \( \mu \in P \) with some positive real number \( g \). Suppose for each \( \mu \) the subsystem \( \dot{z} = f(z, u, \mu) \) is ISS with \( z \) as state and \( u \) as input, and, in particular, a locally Lipschitz class \( K_\infty \) function \( \kappa(\cdot) \), independent of \( \mu \), is known such that

\[
\|z(t)\| \leq \max \left\{ \beta \|z(0)\|, \kappa(\|u(\cdot)\|_\infty) \right\}
\]

for some class \( KL \) function \( \beta(\cdot, \cdot) \). Then the system is also ISS with \( Z = (z^T, x^T)^T \) as state and \( u \) as input, in particular, a class \( K_\infty \) function \( \bar{\kappa}(\cdot) \), independent of \( \mu \), is known such that

\[
\|Z(t)\| \leq \max \left\{ \bar{\beta} \|Z(0)\|, \bar{\kappa}(\|u(\cdot)\|_\infty) \right\}
\]

(3.8)

for some class \( KL \) function \( \bar{\beta}(\cdot, \cdot) \). Moreover, \( \bar{\kappa}(\cdot) \) is also locally Lipschitz.

To make use of Theorems 3.1 and 3.2, suppose

\(
A_5: \quad \dot{z} = f(z + z(v, w), \bar{x}_1 + x_1(v, w), v, w)
\)

\(\dot{x} - f(z(v, w), x_1(v, w), v, w)

is input-state-stable with \( \bar{x}_1 \) as input, and, in particular, a locally Lipschitz class \( K_\infty \) function \( \kappa(\cdot) \), independent of \( v \) and \( w \), is known such that the response \( z(\cdot) \) to any bounded \( \bar{x}_1(\cdot) \) satisfies

\[
\|z(t)\| \leq \max \left\{ \beta \|z(0)\|, \kappa(\|\bar{x}_1(\cdot)\|_\infty) \right\}
\]

(3.9)

### Algorithm:

**Step 1:**

Define a subsystem out of (3.2) as follows,

\[
\begin{align*}
\tilde{\eta}_2 &= (M_2 + N_2 \Psi_2) \tilde{\eta}_2 + N_2 \bar{x}_2 \\
\tilde{z} &= f(z, \bar{x}_1, v, w) \\
\tilde{x}_1 &= f_1(\tilde{\eta}_2, \bar{x}_1, v, w) + \tilde{g}_1(z, v, w) \bar{x}_2
\end{align*}
\]

(3.10)

Performing the coordinate transformation

\[
\eta_2 = \eta_2 - N_2 \bar{g}_1^T(\bar{x}, v, w) \bar{x}_1
\]

(3.11)

gives

\[
\begin{align*}
\tilde{\eta}_2 &= M_2 \tilde{\eta}_2 + \tilde{g}_2(\bar{x}, \bar{x}_1, v, w) \\
\tilde{x}_1 &= f_1(\tilde{\eta}_2, \bar{x}, v, w) + \tilde{g}_1(z, v, w) \bar{x}_2
\end{align*}
\]

(3.12)

where

\[
\begin{align*}
\bar{f}_1(\tilde{\eta}_2, \bar{x}_1, v, w) &= f_1(\tilde{\eta}_2, \bar{x}_1, v, w) \\
\tilde{g}_2(\bar{x}, \bar{x}_1, v, w) &= \bar{g}_1(\bar{x}, v, w) \bar{x}_1
\end{align*}
\]

(3.13)

and \( \tilde{g}_2(\bar{x}, \bar{x}_1, v, w) \) is in the form

\[
\tilde{g}_2(\bar{x}, \bar{x}_1, v, w) = \tilde{g}_1(\bar{x}, v, w) \bar{x}_1
\]

(3.14)

Thus, (3.6) has been put into the following block lower triangular form,

\[
\begin{align*}
\tilde{\eta}_2 &= M_2 \tilde{\eta}_2 + \tilde{g}_2(\bar{x}, \bar{x}_1, v, w) \\
\tilde{z} &= f(\bar{x}, \bar{x}_1, v, w) \\
\tilde{x}_1 &= f_1(\tilde{\eta}_2, \bar{x}_1, v, w) + \tilde{g}_1(z, v, w) \bar{x}_2
\end{align*}
\]

(3.15)

Next let \( Z_1(\tilde{z}, \tilde{\eta}_2)^T \). Then system (3.15) can be rewritten as

\[
\begin{align*}
\dot{Z}_1 &= \phi_1(Z_1, \bar{x}_1, v, w) \\
\dot{\bar{x}}_1 &= \phi_1(Z_1, \bar{x}_1, v, w) + \psi_1(Z_1, v, w) \bar{x}_2
\end{align*}
\]

(3.16)

where

\[
\begin{align*}
\phi_1(Z_1, \bar{x}_1, v, w) &= \left[ \tilde{f}(\bar{x}, \bar{x}_1, v, w) \right] \\
\psi_1(Z_1, v, w) &= \tilde{g}_1(\bar{x}, v, w)
\end{align*}
\]
Viewing \((v, w)\) as \(\mu\), the augmented system (3.8) takes the same form as (3.3) with \(\ddot{z}_i\) as input.

Under assumption A3, \(\ddot{y}_i(\ddot{z}, \ddot{x}, v, w)\) is a polynomial in \(\ddot{z}, \ddot{x}\). By theorem 3.2, \(\ddot{Z}_i = \psi_j(Z_i, \ddot{x}_i, v, w)\) is ISS with \(\ddot{x}_i\) as input, and in particular, an estimate of the form (3.4) holds with \(\kappa_{\dot{\theta}}(\cdot)\) locally Lipschitz at the origin, and independent of \(v, w\).

By Theorem 3.1, there exists a sufficiently smooth function \(\delta_i(x)\) such that the coordinate transform \(\ddot{x}_i = \hat{x}_i, \ddot{x}_2 = \hat{x}_2 - \alpha_i(\dot{x}_i)\) converts system (3.2) into the following

\[
\ddot{\dot{x}}_i = F_i(\ddot{z}, \ddot{x}_2, v, w) + \ddot{g}_i(\ddot{z}, \ddot{x}_2, \ddot{x}_3, \ddot{x}_4, v, w)\ddot{x}_3 + \frac{\delta_i(x)}{\dot{x}_i}
\]

Moreover, the subsystem governing \(\ddot{z}_i\) is ISS with state \(\ddot{z}_i\) and input \(\ddot{x}_i\) and, in particular, a class \(K_\infty\) function \(\kappa_{\dot{\theta}}(\cdot)\), locally Lipschitz at the origin and independent of \(v, w\), is known such that the response \(\ddot{z}_i(\cdot)\) to any bounded \(\ddot{x}_i(\cdot)\) satisfies

\[
\left\|\ddot{z}_i(t)\right\| \leq \max\left\{\bar{\Psi}_i(\|\ddot{x}_i(0)\|), \bar{\kappa}_{\dot{\theta}}(\|\ddot{x}_i(\cdot)\|)\right\}
\]

for some class KL function \(\bar{\Psi}_i(\cdot)\).

**Step j, \(j = 2, \ldots, r - 1\):**

Assume at the end of the \((j - 1)th\) step system, one obtains a system of the form

\[
\ddot{\dot{z}}_{j-1} = F_{j-1}(\ddot{z}_{j-1}, \ddot{x}_j, v, w) + \ddot{g}_{j-1}(\ddot{z}_{j-1}, \ddot{x}_j, v, w)\ddot{x}_{j-1} + \ddot{\dot{x}}_{j-1}
\]

where

\[
\ddot{\dot{x}}_{j-1} = \left(\ddot{z}_j^T, \ddot{\dot{x}}_{j-1}, \ddot{x}_{j-1} \right)^T = \ddot{g}_j(\ddot{\dot{x}}_{j-1}, \ddot{x}_{j-1}), \ddot{x}_{j-1} = \ddot{g}_j(\ddot{x}_{j-1}, \ddot{x}_{j-1})
\]

Furthermore, the subsystem governing \(\ddot{z}_{j-1}\) is ISS with state \(\ddot{z}_{j-1}\) and input \(\ddot{x}_j\) and, in particular, a class \(K_\infty\) function \(\kappa_{\dot{\theta}}(\cdot)\), locally Lipschitz at the origin, and independent of \(v, w\), is known such that the response \(\ddot{z}_{j-1}(\cdot)\) to any bounded \(\ddot{x}_j(\cdot)\) satisfies

\[
\left\|\ddot{z}_{j-1}(t)\right\| \leq \max\left\{\bar{\Psi}_i(\|\ddot{x}_j(0)\|), \bar{\kappa}_{\dot{\theta}}(\|\ddot{x}_j(\cdot)\|)\right\}
\]

for some class KL function \(\bar{\Psi}_i(\cdot)\).

Now define a subsystem out of (3.9) as follows:

\[
\ddot{\dot{z}}_{j+1} = F_{j+1}(\ddot{z}_{j+1}, \ddot{x}_{j+1}, v, w) + \ddot{g}_{j+1}(\ddot{z}_{j+1}, \ddot{x}_{j+1}, v, w)\ddot{x}_{j+1} + \ddot{\dot{x}}_{j+1}
\]

The coordinate transformation

\[
\ddot{\dot{x}}_{j+1} = \ddot{\dot{x}}_{j+1} - \ddot{\dot{g}}_{j+1}(\ddot{z}_{j+1}, \ddot{x}_{j+1}, v, w)\ddot{x}_{j+1} = \ddot{g}_{j+1}(\ddot{x}_{j+1}, \ddot{x}_{j+1}, v, w)
\]

where \(\ddot{g}_{j+1}(\ddot{z}_{j+1}, \ddot{x}_{j+1}, v, w)\) is in the form

\[
\ddot{g}_{j+1}(\ddot{z}_{j+1}, \ddot{x}_{j+1}, v, w) = \ddot{g}_{j+1}(\ddot{z}_{j+1}, \ddot{x}_{j+1}, v, w)
\]

for some function \(p_{j+1}\) which is polynomial in \(\ddot{z}_{j+1}\) and \(\ddot{x}_j\).

Next let \(Z_j = (z_{j-1}, \ddot{z}_{j+1})^T\). Then system (3.10) can be rewritten as

\[
\ddot{Z}_j = \phi_j(Z_j, \ddot{x}_j, v, w)
\]

where

\[
\ddot{z}_j = \ddot{g}_j(Z_j, \ddot{x}_j, v, w)
\]

Again viewing \((v, w)\) as \(\mu\), the augmented system (3.12) takes the same form as (3.3) with \(\ddot{x}_{j+1}\) as input.

Under assumption A3 one can find that

\[
\ddot{g}_j(Z_j, \ddot{x}_j, v, w)
\]

is polynomial in \((\ddot{z}_{j-1}, \ddot{x}_j)\). By theorem 3.2, \(\ddot{Z}_j = \phi_j(Z_j, \ddot{x}_j, v, w)\) is ISS with \(\ddot{x}_j\) as input, and in particular, an estimate of the form (3.4) holds with a class \(K_\infty\).
function $\kappa(\cdot)$ locally Lipschitz at the origin, and independent of $v$ and $w$. By Theorem 3.1, there exists a sufficiently smooth function $\alpha_j(\tilde{x}_j)$ such that the coordinate transform $\tilde{x}_{j+1} = \tilde{x}_{j+1} - \alpha_j(\tilde{x}_j)$ converts system (3.9) into the following

\[ \tilde{y}_j = (M_j + N_j \Psi_j) \tilde{x}_j + N_j \tilde{x}_j, \quad i = j + 2, \cdots, r + 1 \]
\[ \tilde{x}_{j+1} = F_j(\tilde{x}_j, \tilde{x}_{j+1}, v, w) \]
\[ \tilde{x}_j = F_j(\tilde{x}_j, \tilde{x}_{j+1}, v, w) \]

(3.13)

where

\[ \tilde{z}_j = \left( Z_j^T, \tilde{x}_j \right)^T, \quad F_j(\tilde{z}_j, \tilde{x}_{j+1}, v, w) \]
\[ = \left[ \phi_j(\tilde{Z}_j, \tilde{x}_j, v, w) + \psi_j(\tilde{Z}_j, \tilde{x}_j, v, w) \left( \alpha_j(\tilde{x}_j) + \tilde{x}_{j+1} \right) \right] \]

Moreover, the subsystem governing $\tilde{z}_j$ is ISS with state $\tilde{z}_j$ and input $\tilde{x}_{j+1}$, and, in particular, a class $K_\infty$ function $\tilde{k}_j(\cdot)$, locally Lipschitz at the origin, and independent of $v$ and $w$, is known such that the response $\tilde{z}_j(\cdot)$ to any bounded $\tilde{x}_{j+1}(\cdot)$ satisfies

\[ \|z_j(t)\| \leq \max \left\{ \beta_j \left( \|z_j(0)\| \right), k_j(\|\tilde{x}_{j+1}(\cdot)\|_w) \right\} \]

for some class $KL$ function $\beta_j(\cdot, \cdot)$.

Step $r$:

At the end of step $r - 1$, one has obtained a system of the form (3.13) with $j = r - 1$, and $\tilde{x}_{r+1} = u$. Again, one can perform a coordinate transformation of the form (3.11) with $j = r$ to obtain a triangular system of the form (3.12) with $j = r$. Thus, Theorem 3.2 shows that $Z_r = \phi_r(Z_r, \tilde{x}_r, v, w)$ is ISS with $\tilde{x}_r$ as input, and in particular, an estimate of the form (3.4) holds with a function $\kappa_r(\cdot)$ independent of $v$ and $w$. Moreover, the function $\kappa_r(\cdot)$ is locally Lipschitz. By Theorem 3.1, there exists a globally defined feedback control $\tilde{u} = \alpha_r(\tilde{x}_r)$ such that the equilibrium of system (3.2) is globally asymptotically stable for all $v(t) \in V$ and $w \in W$. As a result, (3.2) is globally asymptotically stable under controller

\[ \tilde{u} = \alpha_r(\tilde{x}_r), \quad \tilde{x}_r = \tilde{x}_r - \alpha_{r-1}(\tilde{x}_{r-1}), \quad \tilde{x}_1 = \tilde{x}_1 \]

The results above are summarized as follows.

**3.3 Theorem:** Under assumptions A2 to A4, the global robust output regulation problem for system (3.1) can be solved by an explicitly constructed dynamic state feedback controller.

**REFERENCES**


