HYBRID FUZZY CONTROL OF A CLASS OF NONLINEAR SYSTEMS

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Abstract: In this paper, a hybrid fuzzy controller based on a fuzzy proportional-integral plus conventional derivative control structure is proposed to control a class of single-input single-output nonlinear systems. The proposed controller is capable of generating both linear and nonlinear outputs so that the closed-loop system is equivalent to a feedback connection of a linear system and a nonlinear element in the error state space. It is shown that absolute stability of the closed-loop system can be guaranteed if the nonlinear output of the fuzzy controller is bounded by virtue of the circle criterion. The proposed controller is used to control an inverted pendulum and the performance is shown to be superior.

Keywords: hybrid control, fuzzy logic controller, nonlinear system, circle criterion

1. INTRODUCTION

Although proportional-integral-derivative (PID) controllers are still widely used in industrial applications, they could not fulfill much more complicated and demanding control tasks of today. Many fuzzy PID controllers are developed under such a consideration and inherit both the advantages of PID controllers and fuzzy logic controllers (FLC’s). When fuzzy inferences are used to compute the PID gains, they are called gain scheduling (GS) type fuzzy PID controllers (Zhao et al., 1993). On the other hand, when FLC’s are designed to be nonlinear PID controllers or used in the PID-like structure together with conventional controllers, they are classified as direct action (DA) type (Mann et al., 1999).

Because of their resemblance to a conventional PID controller, fuzzy PID controllers are more appealing to practising engineers. Despite the rapid theoretical development and wide applications of fuzzy PID controllers, it is still very hard to compare different types of fuzzy PID controllers under one benchmark test and identify their advantages and disadvantages by mathematical means. There are several considerations that lead to the choice of the proposed control structure used in this paper. Firstly, since the fuzzy controller itself has many design parameters, using it to tune conventional PID gains may result in more tuning problems. Some self-learning methods like neural network and evolutionary algorithms like genetic algorithm (GA) are able to achieve better tuning of controller gains. This is the main reason why only the DA type fuzzy PID controller is considered here. Secondly, construction of the fuzzy rule base is the most difficult aspect in the design of FLC’s. Fuzzy rules can be either extracted from the knowledge of experienced human operators by trial and error or derived systematically based on some constraints. As a matter of fact, most of the fuzzy rules in the literature are defined.
in linear or linear-like manner, since only these fuzzy controllers can be interpolated mathematically and are transparent to designers. Thirdly, since a single-input single-output (SISO) FLC can be easily interpolated by a single-variable polynomial, the power of fuzzy control as a knowledge-based method diminishes unless a very complicated fuzzy rule base is used. Hence, multi-input fuzzy controllers are considered in this paper. Finally, besides the conventional fuzzy PI/PD controllers (Astrom and Hagglund, 1995; Ying, 1993), a good choice is to place fuzzy controllers into a hybrid PID structure framework such as fuzzy P plus conventional ID of (Li, 1998), fuzzy PI plus conventional D of (Er et al., 2000). All these issues have been discussed in the authors’ previous work (Er et al., 2000; Sun and Er, 2001), where a fuzzy PI plus conventional D controller was developed for both linear and nonlinear systems.

Motivated by the previous work, a generalized fuzzy PID (GFPID) controller whose structure is similar to the hybrid PID structure in proposed. A multi-input fuzzy controller is designed to work together with a conventional feedback controller. The inputs to the fuzzy controller are the error signals. Without loss of generality, triangular membership function and center of gravity defuzzification are used in this paper. However, other combinations of fuzzy components can also be used as long as the nonlinear output of the fuzzy controller can be extracted from the overall output and certain bound conditions satisfied. Following the systematical design approach developed in this paper, close-loop stability can be guaranteed by using the circle criterion of (Khalil, 1992).

This paper is organized as follows. In the next section, a fuzzy controller is designed and it is shown that it can generate both linear output and bounded nonlinear output. The proposed fuzzy controller is used together with a conventional feedback controller to construct a GFPID controller which is capable of controlling a class of SISO nonlinear systems. In Section 3, the closed-loop system is shown to be absolutely stable by using the circle criterion. In Section 4, an application of the proposed controller to control an inverted pendulum system demonstrates its effectiveness. Finally, concluding remarks are drawn in Section 5.

2. DESIGN OF GFPID CONTROLLER

2.1 Hybrid fuzzy control structure

Conventional PID controllers have several variations in terms of structure. Many industrial applications employ the following structure:

\[ u_{PID} = k_P e + k_I \int e dt + k_D \frac{dy}{dt} \]  \hspace{1cm} (1)

where \( y \) is the system output, \( e = y_d - y \) is the tracking error, \( y_d \) is the desired reference input, \( u_{PID} \) is the controller output, and \( k_P, k_I \) and \( k_D \) are the proportional, integral gain and derivative gain, respectively.

If the conventional PI controller is replaced by a fuzzy controller, a fuzzy PI plus conventional D controller is obtained (Er et al., 2000), which is shown have a better performance than conventional PID controllers for linear systems. Encouraged by the good results of the previous work, the fuzzy PI plus conventional D controller was modified to control a tactical missile model in (Sun and Er, 2001). The model was linearized to a second-order quasi-linear model by using the input-output pseudo-linearization technique of (Tsourdos et al., 1999). Simulation results demonstrate excellent performance of the controller and advantages of the hybrid fuzzy PID structure. In order to extend the design methodology to more general cases, we need to formalize the definition of the controller structure. Instead of being constrained to PID-like controllers, a GFPID is composed of fuzzy plus conventional controllers as follows:

\[ \bar{u}_{PID} = k_F u_f - k_c u_c \]  \hspace{1cm} (2)

where \( \bar{u}_{PID} \) is the controller output, \( u_f \) and \( u_c \) are the output of a fuzzy controller and a conventional controller respectively, and \( k_F \) and \( k_c \) are fuzzy output scaling factor and conventional gain respectively.

As depicted in Fig. 1, the fuzzy controller uses the tracking error and its derivatives as input signals, \( e = [e, \dot{e}, \ldots, \dot{e}^{(n-1)}]^{T} \in \mathbb{R}^{n \times 1} \). The conventional controller is constructed by using states feedback and assuming that the plant knowledge is available. For notation simplicity, only column vectors will be used in this paper.

2.2 Construction of the FLC

To avoid any trial and error approach to be used in our systematic design, many design parameters
are set by common sense or analytically calculated while maintaining enough flexibility of the controller structure. Most FLC’s are designed in a linear or linear-like manner, which means at least the membership functions are symmetric and the fuzzy rules are symmetric or skew-symmetric. If the width and position of membership functions are determined one by one, mathematical interpolations would be very difficult. Hence, we define membership functions linearly as depicted in Fig. 2. Without loss of generality, triangular membership functions are used. The term $d_i$, $i = 0, \ldots, n-1$ is the input width and $d$ is the output width. Since $d = 1/\sum_{i=0}^{n-1} (1/d_i)$, the number of output membership functions is determined by the number of input membership functions used. It is natural to define a linear fuzzy rule base as follows:

$$R : IF \text{ } e \text{ } is \text{ } LV_0 \text{ AND } \dot{e} \text{ } is \text{ } LV_1 \ldots \text{ THEN } u_f \text{ } is \text{ } LU = \sum_{i=0}^{n-1} LV_i$$

where $LV_i$, $i = 0, \ldots, n-1$ are the linguistic values for input signals and $LU$ is the linguistic value for the output signal.

For each input signal, two adjacent membership functions are used at each time. Hence, for each set of input signals, a total of $2^n$ rules are fired. Assuming that the nearest reference linguistic value of input $e^{(i)}$ is $LV_i = r$ (as depicted in Fig. 2), then the distance from the reference value to the actual input signal always satisfies $-d_i/2 < \delta e^{(i)} < d_i/2$. By calculating the output solution with MIN inference and center of gravity defuzzification, the following statement is derived.

**Lemma 1.** The fuzzy controller constructed as above can generate both linear and nonlinear outputs. The linear output is a linear combination of error signals and the nonlinear output is always bounded.

**Proof:** Instead of directly showing an extremely long solution for the $n$-input case, a simple two-input example is used to illustrate the concept. General solutions can be easily deduced from this example. Consider the special case of a two-input example as depicted in Fig. 3, where $e_1, e_2$ are the input signals, $\delta e_1, \delta e_2$ are the distances to their respective reference linguistic value $i, j$ and $0 < \delta e_1/d_1 < \delta e_2/d_2 < 0.5$. Thus, fuzzy values of the four fired membership functions are:

$$\mu_i = 1 - \delta e_1/d_1 \quad \mu_{i+1} = \delta e_1/d_1$$

$$\mu_j = 1 - \delta e_2/d_2 \quad \mu_{j+1} = \delta e_2/d_2$$

and $\mu_{i+1} < \mu_{j+1} < \mu_j < \mu_i$. Since $d = d_1d_2/(d_1 + d_2)$, the output of the fuzzy controller is given by

$$u_f = d(i + j) + d(3\delta e_1/d_1 + \delta e_2/d_2)$$

$$= e_1 + e_2 - \frac{1}{d_1 + d_2}(d_1e_1 + d_2e_2) + \zeta(\delta e_1, \delta e_2) \tag{3}$$

where $\zeta(\cdot)$ is a rational polynomial of $\delta e_1$ and $\delta e_2$ and is bounded. Considering all other combinations of $\delta e_1/d_1$ and $\delta e_2/d_2$, we have

$$u_f = e_1 + e_2 + \phi(e_1, e_2, \delta e_1, \delta e_2) \tag{4}$$

where $\phi(\cdot)$ is derived from the last two terms of (3) and is a nonlinear function with respect to the input signals $e_1$ and $e_2$. The sector condition of $\phi(\cdot)$ is therefore given by

$$\alpha_1 e_1 + \alpha_2 e_2 \leq \phi(e_1, e_2) \leq \beta_1 e_1 + \beta_2 e_2 \tag{5}$$

where the design parameters $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$. It follows from (4) and (5) that the statement of Lemma 1 is true for two-input fuzzy controllers. Following the same approach, general solutions for the output of a $n$-input fuzzy controller can be shown to be

$$u_f = k^Te + \phi(k^Te) \tag{6}$$

and the sector condition of the nonlinear term is given by

$$\alpha^T e \leq \phi(k^T e) \leq \beta^T e \tag{7}$$

where $k = [k_0, k_1, \ldots, k_{n-1}]^T \in \mathbb{R}^{n \times 1}$ is the input scaling factor of the fuzzy controller and $\alpha, \beta \in \mathbb{R}^{n \times 1}$ are design parameters. This completes the proof.
3. STABILITY ANALYSIS OF GFPID CONTROLLER

3.1 Nonlinear system control

Many practical systems can be expressed as a class of SISO nonlinear systems in the companion form which is given by

\[ x^{(n)} = f(x) + g(x)u \]
\[ y = x \]

where \( u \in \mathbb{R} \) is the scalar input, \( y \in \mathbb{R} \) is the scalar output, \( x = [x, \dot{x}, \ldots, x^{(n-1)}]^T \in \mathbb{R}^{n \times 1} \) is the state vector, and \( f(x), g(x) \in \mathbb{R} \) are nonlinear functions of the states.

Since the tracking errors are \( e^{(n)} = y_d^{(n)} - y^{(n)} = y_d^{(n)} - x^{(n)} \in \mathbb{R} \) and \( e = [e, \dot{e}, \ldots, e^{(n-1)}]^T \in \mathbb{R}^{n \times 1} \), system (8) can be easily transformed to the following system in error state-space, which can be regarded as a feedback connection of a linear system and a nonlinear element (depicted in Fig. 4):

\[ \dot{e} = Ae + bu_e \]
\[ z = c^T e \]
\[ u_e = -\psi \]

where \( u_e \) and \( z \) are the input and output of the linear system, respectively. The linear system matrices are

\[ A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -k_0 & -k_1 & -k_2 & \cdots & -k_{n-1} \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \]

and \( c \in \mathbb{R}^{n \times 1} \), while the nonlinear element is given by

\[ \psi(z) = f + gu - x^{(n)}_d - k^T e \]

for \( k = [k_0, k_1, \ldots, k_{n-1}]^T \).

Assume that the plant knowledge is known, a generalized fuzzy PID controller is given by

\[ u = \frac{1}{g}u_f - \frac{1}{g}(f - x^{(n)}_d + \gamma^T e) \]

where \( \gamma \in \mathbb{R}^{n \times 1} \) can be constructed. If \( k \) is used as the input scaling factor of the fuzzy controller, the nonlinear element of the closed-loop system in the error state space can be obtained by substituting (6) and (12) into (11) and is given by

\[ \psi(z) = \phi(k^T e) - \gamma^T e \]

where the nonlinearity of the FLC is transformed to the nonlinearity of the closed-loop system. The sector condition of \( \psi(z) \) is contained in the following lemma.

Lemma 2. By choosing a proper design parameter \( \gamma \) in (12) according to the design parameters \( \alpha, \beta \) in (7) and the input scaling factor of the fuzzy controllers \( k \) in (6), the nonlinear element \( \psi(z) \) of the closed-loop system (9) in the error state space is sector bounded.

Proof: Substituting (7) into (13), we have

\[ \alpha^T e - \gamma^T e \leq \psi(c^T e) \leq \beta^T e - \gamma^T e \]

By using the pseudo-inverse of \( c \), where \( c^+ = (cc^T)^{-1}c \in \mathbb{R}^{n \times 1} \) and \( c^+c^T = I \), (14) can be transformed to

\[ (\alpha - \gamma)^T c^+ c^T e \leq \psi(c^T e) \leq (\beta - \gamma)^T c^+ c^T e \]

Let \( \tau_1 = (\alpha - \gamma)^T c^+ c^T \in \mathbb{R} \) and \( \tau_2 = (\beta - \gamma)^T c^+ c^T \in \mathbb{R} \), the sector condition can be simplified to

\[ \tau_1 \leq \frac{\psi(z)}{z} \leq \tau_2 (z \neq 0) \]

By choosing proper design parameters, \( \tau_1 \) and \( \tau_2 \) can be guaranteed to be non-negative. Since \( \psi(z) \) passes through the origin, the nonlinear element \( \psi(z) \) is sector bounded.

In this paper, a special case of \( \psi(z) \), which belongs to the sector \([0, \tau]\) is considered by following the circle criterion of absolute stability analysis. By choosing \( \gamma = \alpha \) in (15), the lower bound \( \tau_1 = 0 \) and the upper bound \( \tau = \tau_2 = (\beta - \alpha)^T c^+ c^T \in \mathbb{R} \), sector condition (16) becomes

\[ \psi(z)[\psi(z) - \tau z] \leq 0 \]

It should be noted that since \( \psi(z) \) in (11) contains the term \( k^T e \), the sector bound \( \tau \) will be affected by the conventional gain \( k \).
3.2 Stability analysis

The system given in (9) and Fig. 4 have a forward path which is a linear system (A, b, c) and a feedback part ψ(z) which is a memoryless nonlinearity. If the feedback only contains a constant gain, i.e., if ψ(z) = κz, then the stability of the closed-loop system can be simply determined by examining the eigenvalues of the closed-loop system matrix [A − κbeT]. However, with nonlinear feedback function ψ(z), stability analysis is much more difficult as shown in (Slotine and Li, 1991). In this case, only the question about whether the system is stable or unstable will be answered. In other words, only conditions for absolute stability can be established (Kuo, 1995).

One of the basic requirements for absolute stability of system (9) is that the linear system should be controllable. It is not difficult to choose a vector k so that the polynomial sn + kn−1sn−1 + ... + k1s + k0 has all its roots strictly in the left-half complex plane. Hence, matrix A is stable and (A, b) is controllable.

Since A is stable, there exist a symmetric positive definite matrix P ∈ Rn×n, a vector q ∈ Rn×1 and a constant ε > 0 such that

\[ PA + A^T P = -qq^T - εP \]
\[ Pb = τc - \sqrt{2q} \]  
(18)

where the nonlinear element ψ(z) belongs to the sector [0, τ].

Consider the following Lyapunov function candidate in quadratic form

\[ V(e) = e^T Pe \]  
(19)

The time derivative along the trajectories of system (9) is given by

\[ \dot{V}(e) = e^T (PA + A^T P)e - 2e^T Pbψ(z) \]  
(20)

which can be shown to be upper bounded by subtracting the non-positive term in (17), i.e.

\[ \dot{V}(e) \leq e^T (PA + A^T P)e - 2e^T Pbψ(z) - 2ψ(z)[ψ(z) − τcT e] \]
\[ = e^T (PA + A^T P)e - 2e^T (τc − Pb)ψ(z) - 2ψ^2(z) \]
\[ = -εe^T Pe - W(e) \]  
(21)

Since

\[ W(e) = e^T qq^T e - 2\sqrt{2ε}e^T qψ(z) + 2ψ^2(z) \]
\[ = [q^T e - \sqrt{2ε}ψ(z)]^T [q^T e - \sqrt{2ε}ψ(z)] \geq 0 \]  
(22)

we have \( \dot{V}(e) \leq 0 \) and the closed-loop system is stable by the Lyapunov theorem.

4. SIMULATION RESULTS

To demonstrate the effectiveness of the proposed controller, the GFPID controller is employed to control an inverted pendulum system in this section. The inverted pendulum system composes of a pole and a cart (as depicted in Fig. 5). The cart is allowed to move horizontally in order to keep the pole at the upright position from different initial angles or let the angular position of the pole follows a particular trajectory. The system dynamics is given by

\[ \dot{\theta} = \frac{(m_p + m_c)g \sin \theta - m_p \dot{\theta}^2 \cos \theta \sin \theta - F \cos \theta}{(m_p + m_c)(4/3 - m_p \cos^2 \theta)} \]  
(23)

where \( \theta \) is the angular position, \( \dot{\theta} \) is the angular velocity and F is the external force applied to the cart. The gravity acceleration \( g \) is 9.8m/s2, mass of the cart \( m_c \) is 1.0kg, mass of the pole \( m_p \) is 0.1kg and the length of the pole l is 0.5m.

Figs. 6 and 7 show that the cart-pole system can be stabilized by the GFPID controller from the initial condition (0 rad) to the equilibrium point (0 rad). Note that the angular position \( \theta \) and the angular velocity \( \dot{\theta} \) of the pole are the input signals of the fuzzy controller since the reference input is at the origin 0. The balancing of the pole can be easily affected by external disturbances because of its light weight. A white noise is used to represent external disturbances acting on the system. The proposed controller regulates the system well as evidenced from Fig. 8. With the initial angular position of 0.1 rad, the output of the closed-loop system is able to follow the trajectory as shown in Fig. 9. Simulation results support our new approach of designing a hybrid fuzzy PID controller.

5. CONCLUSIONS

In this paper, a hybrid fuzzy controller based on the success of the fuzzy PI plus conventional D controller is proposed. A linearly defined fuzzy
logic controller is placed into a hybrid PID structure framework and it works together with a conventional feedback controller. By showing that the nonlinear output generated by the linear fuzzy controller is bounded, stability of the closed-loop system can be guaranteed using the circle criterion. Simulation results show that the proposed controller performs well for a class of SISO nonlinear systems. If the plant knowledge can be expressed as fuzzy rules, the conventional feedback controller will be replaced by another fuzzy controller so that a single fuzzy control structure can be obtained. This work is still under investigation.

REFERENCES


