AN APPROACH TO SWITCHED SYSTEMS
OPTIMAL CONTROL BASED ON
PARAMETERIZATION OF THE SWITCHING INSTANTS *

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Abstract: This paper provides a viable general approach to switched systems optimal control. Such optimal control problems require the solutions of not only optimal continuous inputs but also optimal switch hing sequences. Many practical problems only involve optimization where the number of switchings and the sequence of active subsystems are given. This is stage 1 of the tw o stage optimization methodology proposed by the authors in previous papers. In order to solve stage 1 problems, the derivatives of the optimal cost with respect to the switch hing instants need to be known. In (Xu and Antsaklis, 2001), we proposed an approach for solving a special class of such problems, namely, general switched linear quadratic problems. In this paper, the idea of (Xu and Antsaklis, 2001) is extended to general switched systems optimal control problems and an approach is proposed for solving them. The approach first transcribes a stage 1 problem into an equivalent problem parameterized by the switch hing instants and then the values of the derivatives are obtained based on the solution of a two point boundary value differential algebraic equation formed by the state, costate, stationarity equations, the boundary and continuity conditions and their differentiations. Examples are shown to illustrate the results in the paper.

Keywords: Switching systems; Optimal control; Hybrid systems; Control synthesis

1. INTRODUCTION
A switched system is a particular kind of hybrid system that consists of several subsystems and a switching law specifying the active subsystem at each time instant. Many real-world processes such as chemical processes, automotive systems, and electrical circuit systems, etc., can be modeled as switched systems.

Optimal control problems for switched systems have attracted the attention of researchers recently. For such a problem, one needs to find both an optimal continuous input and an optimal switching sequence since the system dynamics vary before and after every switching instant. The available results in the literature on such problems can be classified as theoretical (e.g., (Branicky et al., 1998; Piccoli, 1998)) and practical (see e.g., (Gokbayrak and Cassandras, 2000; Hedlund and Ranzer, 1999; Lu et al., 1993; Wang et al., 1997)). Most of the practical methods that we are aware of are using numerical methods and are based on some discretization of continuous time space and/or discretization of state space into grids and use search methods for the resultant discrete problem to find optimal/suboptimal solutions. But the discretization of time space may lead to computational combinatoric explosion and the solutions obtained may not be accurate enough. In view of this, in some previous papers by the authors...
A switching sequence $\sigma$ as defined above indicates that subsystem $i_k$ is active in $[t_k, t_{k+1}]$. We specify $\sigma \in \Sigma_{[t_0, t_f]}$ as a discrete input to the system (see (Xu and Antsaklis, 2000) for more details).

2.2 An Optimal Control Problem

Now we formulate the optimal control problem we will study in this paper. In the sequel, we denote $\mathcal{U}_{[t_0, t_f]} \triangleq \{u \mid u \in \mathcal{C}_p[0, t_f], u(t) \in \mathbb{R}^m\}$; i.e., the set of all piecewise continuous functions for $t \in [t_0, t_f]$ with values in $\mathbb{R}^m$.

**Problem 1.** Consider a switched system $\mathcal{S}$. Given a fixed interval $[t_0, t_f]$, find $u \in \mathcal{U}_{[t_0, t_f]}$ and a switching sequence $\sigma \in \Sigma_{[t_0, t_f]}$ such that $x(t)$ departs from a given $x(t_0) = x_0$ and meets $\mathcal{S}_f = \{x \mid \phi_f(x) = 0, \phi_f : \mathbb{R}^n \rightarrow \mathbb{R}^m\}$ at $t_f$ and the cost functional

$$J = \psi(x(t_f)) + \int_{t_0}^{t_f} L(x(t), u(t), t) \, dt$$

is minimized. □

In the sequel, we assume that $f, L, \phi_f$ and $\psi$ possess enough smoothness properties we need in our derivations. The way we formulate Problem 1 with a fixed final time is mainly for the convenience of subsequent studies. Note that for a problem with free end-time $t_f$, we can introduce an additional state variable and transcribe it to a problem with fixed end-time (for more details, see (Xu, 2001)).

3. TWO STAGE OPTIMIZATION

Now we review the two stage algorithm (see (Xu and Antsaklis, 2001)) in the following.

**Algorithm 1. (A Two Stage Algorithm)**

Stage 1. (a). Fix the total number of switchings to be $K$ and the sequence of active subsystems and let the minimum value of $J$ with respect to $u$ be a function of the $K$ switching instants, i.e., $J_1 = J_1(t_1, t_2, \ldots, t_K)$ for $K \geq 0$ ($0 \leq t_0 \leq t_1 \leq \cdots \leq t_K \leq t_f$). Find $J_1$.

(b). Minimize $J_1$ with respect to $t_1, t_2, \ldots, t_K$.

Stage 2. (a). Vary the order of active subsystems to find an optimal solution under $K$ switchings.

(b). Vary the number of switchings $K$ to find an optimal solution for the optimal control problem. □

Algorithm 1 has high computational costs. In the followings, we concentrate on stage 1 optimization. Note that many practical problems are in fact stage 1 problems. For example, the speeding-up of a power train in an automobile only requires switchings from gear 1 to 2 to 3 to 4. As can be seen from Algorithm 1, stage 1 can further be decomposed into two sub-steps (a) and (b) (note that a similar hierarchical decomposition
Stage 1(a)

Stage 1(a) is in essence a conventional optimal control problem which seeks the minimum value of $J$ with respect to $u$ under a given switching sequence $\sigma = ((t_0, i_0), (t_1, e_1), \ldots, (t_K, e_K))$. We denote the corresponding optimal cost as a function $J_1(t)$, where $t = (t_1, t_2, \ldots, t_K)^T$. In stage 1(a), we need to find an optimal $u$ and the corresponding minimum $J$. For stage 1(a), although different subsystems are active in different time intervals, the problem is conventional since these intervals are fixed. It is not difficult to use the calculus of variations techniques to prove the following necessary conditions.

**Theorem 1. (Necessary Conditions for Stage 1(a).)**

Consider the stage 1(a) problem for Problem 1. Assume that subsystem $k$ is active in $[t_{k-1}, t_k]$ for $1 \leq k \leq K$ and subsystem $K+1$ in $[t_K, t_{K+1}]$ where $t_{K+1} = t_f$. Let $u \in U_{\{u_{t_j}\}}$ be a continuous input such that the corresponding $x(t)$ departs from a given initial $x(t_0) = x_0$ and meets $S_f = \{x|\phi_f(x) = 0, \phi_f : \mathbb{R}^n \to \mathbb{R}^l\}$ at $t_f$. In order that $u$ be optimal, it is necessary that there exists a vector function $p(t) = [p_1(t), \cdots, p_n(t)]^T$, $t \in [t_0, t_f]$, such that the following conditions hold

(a). For almost any $t \in [t_0, t_f]$ the following state and costate equations hold

\[
\begin{align*}
\text{State eq: } \frac{dx(t)}{dt} &= \left[\frac{\partial H}{\partial p}[x(t), p(t), u(t), t]\right]^T \\
\text{Costate eq: } \frac{dp(t)}{dt} &= -\left[\frac{\partial H}{\partial x}[x(t), p(t), u(t), t]\right]^T.
\end{align*}
\]

(b) For almost any $t \in [t_0, t_f]$, the stationarity condition holds

\[
0 = \left[\frac{\partial H}{\partial u}[x(t), p(t), u(t), t]\right]^T. \tag{3}
\]

(c) At $t_f$, the function $p$ satisfies

\[
p(t_f) = \left[\frac{\partial \phi_f}{\partial x}[x(t_f)]\right]^T + \left[\frac{\partial \phi_f}{\partial x}[x(t_f)]\right]^T \lambda \tag{4}
\]

where $\lambda$ is an $l_f$-dimensional vector.

(d) At any $t_k$, $k = 1, 2, \ldots, K$, we have

\[
p(t_k-) = p(t_k+). \tag{5}
\]

**Proof:** See Chapter 6 of (Xu, 2001).

The above necessary conditions will be used in Section 5 in the development of a method for finding $\frac{\partial J_1}{\partial u}$ and $\frac{\partial^2 J_1}{\partial u^2}$. In general, it is difficult or even impossible to find an analytical expression of $J_1(t)$ using them. However, we can find the numerical solutions by solving the two point boundary value differential algebraic equation (DAE) formed by conditions (a)-(d) using numerical methods.

**Stage 1(b)**

Stage 1(b) is in essence a constrained nonlinear optimization problem with simple constraints

\[
\begin{align*}
\min_{\bar{t}} J_1(\bar{t}) \\
\text{subject to } \bar{t} &\in T
\end{align*}
\]

where $T \supseteq \{t = (t_1, t_2, \ldots, t_K)^T | t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_K \leq t_f\}$. Feasible direction methods can be applied to such problems. These methods use $\frac{\partial J_1}{\partial t}$ and $\frac{\partial^2 J_1}{\partial t^2}$. In our computations, we use the gradient projection method (using $\frac{\partial J_1}{\partial t}$) and the constrained Newton’s method (using $\frac{\partial J_1}{\partial t}$ and $\frac{\partial^2 J_1}{\partial t^2}$) (see Section 2.3 in Bertsekas (Bertsekas, 1999) for details).

**A Conceptual Algorithm**

The following conceptual algorithm provides a framework for the optimization methodologies in the sequel.

**Algorithm 2. (A Conceptual Algorithm for Stage 1)**

1. Set the iteration index $j = 0$. Choose an initial $\bar{t}$.
2. By solving an optimal control problem (Stage 1(a)), find $J_1(\bar{t})$.
3. Find $\frac{\partial J_1}{\partial \bar{t}}(\bar{t})$ and $\frac{\partial^2 J_1}{\partial \bar{t}^2}(\bar{t})$.
4. Use the gradient projection method or the constrained Newton’s method (if $\frac{\partial^2 J_1}{\partial \bar{t}^2}(\bar{t})$ is known) to update $\bar{t}$ to be $\bar{t}^j + \alpha^j d\bar{t}$ (here $\alpha^j$ is chosen using the Armijo’s rule (Bertsekas, 1999)). Set $j = j + 1$.
5. Repeat Steps (2), (3), (4), and (5), until a prespecified termination condition is satisfied.

The key elements of the above algorithm are

(a) An optimal control algorithm for Step (2).
(b) The derivations of $\frac{\partial J_1}{\partial \bar{t}}$ and $\frac{\partial^2 J_1}{\partial \bar{t}^2}$ for Step (3).
(c) A nonlinear optimization algorithm for Step (4).

In the above discussions, we have already addressed elements (a) and (c). (b) poses an obstacle for the use of Algorithm 2 because $\frac{\partial J_1}{\partial \bar{t}}$ and $\frac{\partial^2 J_1}{\partial \bar{t}^2}$ are not readily available. It is the task of the subsequent sections to address (b) and devise a method for deriving the values of $\frac{\partial J_1}{\partial \bar{t}}$ and $\frac{\partial^2 J_1}{\partial \bar{t}^2}$.

**4. AN EQUIVALENT PROBLEM FORMULATION**

In this section we transcribe a stage 1 problem into an equivalent conventional optimal control problem parameterized by the switching instants, which will be used in next section. For simplicity of notation, in the followings, we concentrate on the case of two subsystems where subsystem 1 is active for $t \in [0, t_1]$ and subsystem 2 is active for $t \in [t_1, t_f]$ ($t_1$ is the switching instant to be
determined. We also assume that $S_f = \mathbb{R}^n$ (for general $S_f$, we can introduce Lagrange multipliers and develop similar methods). We consider the following stage 1 problem.

**Problem 2.** For a switched system

\[
\dot{x} = f_1(x, u, t), \quad 0 \leq t < t_1, \\
\dot{x} = f_2(x, u, t), \quad t_1 \leq t \leq t_f.
\]

find an optimal switching instant $t_1$ and an optimal $u(t), \ t \in [t_0, t_f]$ such that

\[
J = \psi(x(t_f)) + \int_{t_0}^{t_f} L(x, u, t) \, dt
\]

is minimized. $t_0$, $t_f$ and $x(t_0) = x_0$ are given. □

We transcribe Problem 2 into an equivalent problem as follows. We introduce a state variable $x_{n+1}$ corresponding to the switching instant $t_1$. Let $x_{n+1}$ satisfy

\[
\begin{align*}
\frac{dx_{n+1}}{dt} &= 0, \\
x_{n+1}(0) &= t_1
\end{align*}
\]

Next a new independent time variable $\tau$ is introduced. A piecewise linear correspondence relationship between $t$ and $\tau$ is established as follows.

\[
t(\tau) = \begin{cases} 
  t_0 + (x_{n+1} - t_0)\tau, & 0 \leq \tau \leq 1 \\
  x_{n+1} + (t_f - x_{n+1})(\tau - 1), & 1 \leq \tau \leq 2.
\end{cases}
\]

By introducing $x_{n+1}$ and $\tau$, Problem 2 can be transcribed into the following equivalent problem.

**Problem 3.** (An Equivalent Problem). For a system with dynamics

\[
\begin{align*}
\frac{dx(\tau)}{d\tau} &= (x_{n+1} - t_0)f_1(x, u, t(\tau)), \\
\frac{dx_{n+1}}{d\tau} &= 0
\end{align*}
\]

in the interval $\tau \in [0, 1]$ and

\[
\begin{align*}
\frac{dx(\tau)}{d\tau} &= (t_f - x_{n+1})f_2(x, u, t(\tau)), \\
\frac{dx_{n+1}}{d\tau} &= 0
\end{align*}
\]

in the interval $\tau \in [1, 2]$, find an optimal $x_{n+1}$ and an optimal $u(\tau), \ \tau \in [0, 2]$ such that the cost functional

\[
J = \psi(x(2)) + \int_0^1 (x_{n+1} - t_0)L(x, u, t(\tau)) \, d\tau
\]

\[
+ \int_1^2 (t_f - x_{n+1})L(x, u, t(\tau)) \, d\tau
\]

is minimized. Here $t_f, \ x(0) = x_0$ are given. □

**Remark 1.** Problem 3 and Problem 2 are equivalent in the sense that an optimal solution for Problem 3 is an optimal solution for Problem 2 by a proper change of independent variables as in (6) and by regarding $x_{n+1} = t_1$, and vice versa. □

**Remark 2.** Problem 3 is conventional because it has fixed time instant when the system dynamics change. In fact, because $x_{n+1}$ is an unknown constant throughout $\tau \in [0, 2]$, it can be regarded as a conventional optimal control problem with an unknown parameter $x_{n+1}$. In the sequel, we regard Problem 3 as an optimal control problem parameterized by the switching instant $x_{n+1}$ with cost (11) and subsystems (7) and (9). □

5. THE DEVELOPMENT OF THE APPROACH

In this section, based on the equivalent Problem 3, we develop a method for deriving accurate numerical value of $\frac{\partial J}{\partial x_{n+1}}$. The method is based on the solution of a two point boundary value differential algebraic equation (DAE) which is formed by the state, costate, stationarity equations, the boundary and continuity conditions for Problem 3 and their differentiations with respect to the parameter $x_{n+1}$. In the following, we denote $\frac{\partial L}{\partial x}$ as row vectors and we denote $\frac{\partial L}{\partial x_{n+1}^2}$ as an $n \times n$ matrix whose $(i_1, i_2)$-th element is $\frac{\partial L}{\partial x_{n+1} i_1 i_2}$. Similar notations apply to $\frac{\partial H}{\partial x_{n+1}}$, $\frac{\partial H}{\partial x_{n+1}^2}$, etc.

Consider the equivalent Problem 3, define

\[
\begin{align*}
\hat{f}_1(x, u, \tau, x_{n+1}) &= (x_{n+1} - t_0)f_1(x, u, t(\tau)), \\
\hat{f}_2(x, u, \tau, x_{n+1}) &= (t_f - x_{n+1})f_2(x, u, t(\tau)), \\
\hat{L}_1(x, u, \tau, x_{n+1}) &= (x_{n+1} - t_0)L(x, u, t(\tau)), \\
\hat{L}_2(x, u, \tau, x_{n+1}) &= (t_f - x_{n+1})L(x, u, t(\tau)).
\end{align*}
\]

Regarding $x_{n+1}$ as a parameter, we denote the optimal state trajectory as $x(\tau, x_{n+1})$. We define the parameterized Hamiltonian as

\[
H(x, p, u, \tau, x_{n+1}) \triangleq \begin{cases} 
  \hat{L}_1(x, u, \tau, x_{n+1}) + p^T \hat{f}_1(x, u, \tau, x_{n+1}), & \text{for } \tau \in [0, 1], \\
  \hat{L}_2(x, u, \tau, x_{n+1}) + p^T \hat{f}_2(x, u, \tau, x_{n+1}), & \text{for } \tau \in [1, 2].
\end{cases}
\]

Assume that a parameter $x_{n+1}$ is given, then we can apply Theorem 1 to Problem 3. The necessary conditions (a) and (b) provide us with the following equations

State eq. \[
\frac{dx}{d\tau} = \left(\frac{\partial H}{\partial p}\right)^T = \hat{f}_1(x, u, \tau, x_{n+1})
\]

Costate eq. \[
\frac{dp}{d\tau} = -\left(\frac{\partial H}{\partial x}\right)^T = -\left(\frac{\partial f_1}{\partial x}\right)^T p - \left(\frac{\partial \hat{L}_1}{\partial x}\right)^T
\]

Stationarity eq. 0 = \[
\frac{\partial L}{\partial x_{n+1}} = \left(\frac{\partial f_1}{\partial x_{n+1}}\right)^T p + \left(\frac{\partial \hat{L}_1}{\partial x_{n+1}}\right)^T
\]

in $\tau \in [0, 1]$ and

State eq. \[
\frac{dx}{d\tau} = \left(\frac{\partial H}{\partial p}\right)^T = \hat{f}_2(x, u, \tau, x_{n+1})
\]

Costate eq. \[
\frac{dp}{d\tau} = -\left(\frac{\partial H}{\partial x}\right)^T = -\left(\frac{\partial f_2}{\partial x}\right)^T p - \left(\frac{\partial \hat{L}_2}{\partial x_{n+1}}\right)^T
\]

Stationarity eq. 0 = \[
\frac{\partial L}{\partial x_{n+1}} = \left(\frac{\partial f_2}{\partial x_{n+1}}\right)^T p + \left(\frac{\partial \hat{L}_2}{\partial x_{n+1}}\right)^T
\]

in $\tau \in [1, 2]$. Note that the optimal $p$ and $u$ are also functions of $\tau$ and $x_{n+1}$. Therefore, we denote them as $p = p(\tau, x_{n+1})$ and $u = u(\tau, x_{n+1})$. □
From the necessary condition (c) of Theorem 1, we obtain the boundary conditions

\[ x(0, x_{n+1}) = x_0, \quad p(2, x_{n+1}) = (\partial \psi / \partial x)[x(2, x_{n+1})]^T. \]  

(12) \hspace{3cm} (13)

The necessary condition (d) tells us that

\[ p(1, x_{n+1}) = p(1, x_{n+1}). \]  

(14)

(6)-(8), (9)-(11) along with (12) and (13) form a two point boundary value differential algebraic equation (DAE) parameterized by \( x_{n+1} \). For each given \( x_{n+1} \), the DAE can be solved using numerical methods. Now assume that we have solved the above DAE and obtain the optimal \( x(\tau, x_{n+1}) \), \( p(\tau, x_{n+1}) \) and \( u(\tau, x_{n+1}) \), we then have the optimal value of \( J \) which is a function of the parameter \( x_{n+1} \)

\[ J_1(x_{n+1}) = \psi(x(2, x_{n+1})) + \int_0^1 L_1(x, u, \tau, x_{n+1}) \, d\tau + \int_1^2 L_2(u, \tau, x_{n+1}) \, d\tau. \]  

(15)

Differentiating \( J_1 \) with respect to \( x_{n+1} \) provides us with

\[ \frac{dJ_1}{dx_{n+1}} = \frac{\partial \psi(x(2, x_{n+1}))}{\partial x_{n+1}} + \int_0^1 \left[ L_1(x, u, \tau) \right] \, d\tau + \int_1^2 \left[ -L_2(x, u, \tau) \right] \, d\tau + \frac{\partial L_1}{\partial x_{n+1}} \frac{dx}{dt}. \]  

(16)

So we need to obtain the functions \( \frac{\partial x(\tau, x_{n+1})}{\partial x_{n+1}} \) and \( \frac{\partial u(\tau, x_{n+1})}{\partial x_{n+1}} \) (here we assume that \( x_{n+1} \) is fixed) in order to obtain \( \frac{dJ_1}{dx_{n+1}} \). By differentiating (6)-(8) and (9)-(11) with respect to \( x_{n+1} \), we obtain

\[ \frac{\partial}{\partial \tau} \frac{\partial x}{\partial x_{n+1}} = \frac{\partial}{\partial \tau} \frac{\partial x_{n+1}}{\partial x_{n+1}} = f_1 + f_2 + \frac{\partial}{\partial \tau} \frac{\partial x_{n+1}}{\partial x_{n+1}}. \]  

(17)

\[ \frac{\partial}{\partial \tau} \frac{\partial p}{\partial x_{n+1}} = \frac{\partial}{\partial \tau} \frac{\partial p_{n+1}}{\partial x_{n+1}} = \frac{\partial}{\partial \tau} \frac{\partial x_{n+1}}{\partial x_{n+1}}. \]  

(18)

\[ \frac{\partial}{\partial \tau} \frac{\partial u}{\partial x_{n+1}} = \frac{\partial}{\partial \tau} \frac{\partial u_{n+1}}{\partial x_{n+1}} = \frac{\partial}{\partial \tau} \frac{\partial x_{n+1}}{\partial x_{n+1}}. \]  

(19)

for \( \tau \in [0, 1] \) and

\[ \frac{\partial}{\partial \tau} \frac{\partial x}{\partial x_{n+1}} = \frac{\partial}{\partial \tau} \frac{\partial x_{n+1}}{\partial x_{n+1}} = -f_2 + \frac{\partial}{\partial \tau} \frac{\partial x_{n+1}}{\partial x_{n+1}}. \]  

(20)

\[ \frac{\partial}{\partial \tau} \frac{\partial x_{n+1}}{\partial x_{n+1}} = \frac{\partial}{\partial \tau} \frac{\partial x_{n+1}}{\partial x_{n+1}} = -f_2 + \frac{\partial}{\partial \tau} \frac{\partial x_{n+1}}{\partial x_{n+1}}. \]  

(21)

\[ \frac{\partial}{\partial \tau} \frac{\partial x_{n+1}}{\partial x_{n+1}} = \frac{\partial}{\partial \tau} \frac{\partial x_{n+1}}{\partial x_{n+1}} = -f_2 + \frac{\partial}{\partial \tau} \frac{\partial x_{n+1}}{\partial x_{n+1}}. \]  

(22)

\[ \frac{\partial}{\partial \tau} \frac{\partial x_{n+1}}{\partial x_{n+1}} = \frac{\partial}{\partial \tau} \frac{\partial x_{n+1}}{\partial x_{n+1}} = -f_2 + \frac{\partial}{\partial \tau} \frac{\partial x_{n+1}}{\partial x_{n+1}}. \]  

(23)

\[ \frac{\partial}{\partial \tau} \frac{\partial x_{n+1}}{\partial x_{n+1}} = \frac{\partial}{\partial \tau} \frac{\partial x_{n+1}}{\partial x_{n+1}} = -f_2 + \frac{\partial}{\partial \tau} \frac{\partial x_{n+1}}{\partial x_{n+1}}. \]  

(24)

\[ \frac{\partial}{\partial \tau} \frac{\partial x_{n+1}}{\partial x_{n+1}} = \frac{\partial}{\partial \tau} \frac{\partial x_{n+1}}{\partial x_{n+1}} = -f_2 + \frac{\partial}{\partial \tau} \frac{\partial x_{n+1}}{\partial x_{n+1}}. \]  

(25)

It can now be observed that (6)-(8), (9)-(11) and (17)-(19), (20)-(22) along with the boundary conditions (12), (13) and (23), (24) and with the continuity conditions (14), (25) form a two point boundary value DAE for \( x(\tau, x_{n+1}) \), \( p(\tau, x_{n+1}) \), \( u(\tau, x_{n+1}) \) and \( \frac{\partial x(\tau, x_{n+1})}{\partial x_{n+1}} \), \( \frac{\partial p(\tau, x_{n+1})}{\partial x_{n+1}} \), \( \frac{\partial u(\tau, x_{n+1})}{\partial x_{n+1}} \). By solving them and substituting the result into (16), we can obtain \( \frac{dJ_1}{dx_{n+1}} \).

Remark 3. In general, we need to resort to numerical methods (e.g., shooting methods) to find the solution to the two point boundary value DAE. In particular, if all subsystems are linear in control and the cost function \( L \) is quadratic in control, then the two point boundary value DAE can hence be reduced to a two point boundary value differential equation which can be solved more easily. See Chapter 8 of (Xu, 2001) for details.

Remark 4. The approach developed in this section can be extended in a straightforward manner to the case of several subsystems and more than one switchings. The value of \( \frac{dJ_1(x_0)}{dx_{n+1}} \) can also be similarly obtained. See Chapter 8 of (Xu, 2001) for details. □
6. SOME EXAMPLES

Example 1. Consider a switched system with

subsystem 1: \[ \dot{x} = x + 2ru, \]  
(1)
subsystem 2: \[ \dot{x} = -x - 3ru. \]  
(2)
Assume \( t_0 = 0 \), \( t_f = 2 \) and the system switches at \( t = t_1 (0 \leq t_1 \leq 2) \) from subsystem 1 to 2. Find an optimal switching instant \( t_1 \) and an optimal input \( u \) such that \( J = \frac{1}{2} x(2)^2 + \frac{1}{2} \int_0^2 u^2(t) \, dt \) is minimized. Here \( x(0) = 1 \).

The method in Section 5 is used to obtain \( \frac{\partial J}{\partial x_1} \). Choose an initial nominal \( t_1 = 1.2 \). By applying Algorithm 2 with the gradient projection method, after 20 iterations we find the optimal \( t_1 \) to be \( t_1 = 0.9994 \) and the corresponding cost to be \( 1.1848 \times 10^{-7} \). Figure 1 (a) and (b) show the continuous input and the state trajectory. Note that the theoretical solutions are \( t_1^{opt} = 1 \), \( u^{opt} \equiv 0 \) and \( J^{opt} = 0 \).

![Fig. 1. Example 1: (a) The control input. (b) The state trajectory \( x(t) \).](image)

Example 2. Consider a switched system with

subsystem 1: \[
\begin{align*}
\dot{x}_1 &= -x_1 + 2x_2u \\
\dot{x}_2 &= x_2 + x_2u
\end{align*}
\]  
(3)
subsystem 2: \[
\begin{align*}
\dot{x}_1 &= x_1 - 3x_2u \\
\dot{x}_2 &= 2x_2 - 2x_2u
\end{align*}
\]  
(4)
subsystem 3: \[
\begin{align*}
\dot{x}_1 &= 2x_1 + x_2u \\
\dot{x}_2 &= -2x_2 + 2x_2u
\end{align*}
\]  
(5)
Assume \( t_0 = 0 \), \( t_f = 3 \) and the system switches at \( t = t_1 \) from subsystem 1 to 2 and at \( t = t_2 \) from subsystem 2 to 3 (\( 0 \leq t_1 \leq t_2 \leq 3 \)). Find optimal switching instants \( t_1 \), \( t_2 \) and an optimal input \( u \) such that \( J = \frac{1}{2} (x_1(3) - e_1^2)^2 + \frac{1}{2} (x_2(3) - e_2^2)^2 + \frac{1}{2} \int_0^3 u^2(t) \, dt \) is minimized. Here \( x_1(0) = 1 \) and \( x_2(0) = 1 \).

The method in Section 5 is used to obtain \( \frac{\partial J}{\partial x_1} \) and \( \frac{\partial J}{\partial x_2} \). We choose initial nominal \( t_1 = 1.1 \) and \( t_2 = 2.1 \). By applying Algorithm 2 with the gradient projection method, after 18 iterations we find that the optimal \( t_1 \) and \( t_2 \) to be \( t_1 = 1.0050 \), \( t_2 = 1.9993 \) and the corresponding cost to be \( 2.7599 \times 10^{-6} \). The corresponding continuous control and state trajectory are shown in Figure 2 (a) and (b) show the continuous input and the state trajectory. Note that the theoretical solutions are \( t_1^{opt} = 1 \), \( t_2^{opt} = 2 \), \( u^{opt} \equiv 0 \) and \( J^{opt} = 0 \), so the result we obtained is quite accurate.

![Fig. 2. Example 2: (a) The control input. (b) The state trajectory.](image)

7. CONCLUSION

In this paper, a general approach to switched systems optimal control is developed. It is mainly developed in Sections 4 and 5 and is applicable to problems with many subsystems and more than one switching. The approach is based on solving a two point boundary value DAE derived in Section 5. Derivatives of the optimal cost with respect to the switching instants can be obtained accurately and therefore nonlinear optimization algorithms can be used to find the optimal switching instants. Future research topics include the extension of the approach to systems with internally-forced switchings and systems with state discontinuities.

8. REFERENCES


