Interval observers with guaranteed confidence levels
Application to the activated sludge process

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Abstract. This paper deals with the design of robust observers for uncertain models, with
application to the activated sludge process. We assume that the model is such that the bacterial
growth rate is unknown, and the measurements and influent flow rates are disturbed. However
we suppose that the upper and lower bounds of the uncertainties are known. Under appropriate
hypotheses, we are able to build interval observers giving dynamic bounds containing the variables
to estimate. Besides, if we know a priori the probability density of each uncertain parameter,
we can synthesize interval observers with guaranteed confidence levels and provide a confidence
density for the state. We apply this approach to the sludge process and compare the results to the
probability density estimation obtained with Monte-Carlo simulations.

Keywords: uncertain models, non-linear estimation, interval observers, wastewater treatment, ac-
tivated sludge process, biological models.

1. INTRODUCTION

One of the main limitations to the improvement of
monitoring and optimization of bioreactors is probably
due to the difficulty to measure chemical and biologi-
cal variables. Indeed there are very few sensors which
are at the same time cheap and reliable and that can be
on-line used. The measurement of some biological va-
riables (biomass, cellular quota, etc) is sometimes very
difficult and can necessitate complicated and sophisti-
cated operations.

The development of observers addresses this issue
by estimating the internal state of a bioreactor. It relies
both on a model of the system and on the available on-
line measurements [1]. Nevertheless these methods are
often disappointing when dealing with bioprocesses
since observers are based on models which are often
rough approximations and also because the used mea-
surements are often corrupted by a high level of noise.
As a consequence, it can be difficult to interpret the
observer predictions.

To cope with the uncertainties which characterize the
biological systems, a first class of observers was pro-
posed by [2] on the principle of unknown input ob-
servers [3, 4]. The biological kinetics was considered
as an unknown input and the observer construction did
not use it. This approach provided more robustness to
the observers, but managed less easily the uncertainty
on the mass inputs or on the yield coefficients.

A complementary approach called “interval obser-
vers” was then proposed [5, 6, 7], based on the prin-
ciple of cooperative systems [8]. This approach as-
umes a bounded uncertainty and provides tools to de-
termine the bounds in which the state must lie. It is
therefore a very robust approach, but it can be too conser-
vative. In this paper, we use this approach to try to de-
termine a more precise information related to the pro-
bability density of the state. The idea is to use statis-
tical models on the probability densities of the unk-
owns (initial conditions, parameters, inputs, measu-
rements) in order to estimate the probability density of
the state. In general, this is a very difficult problem,
and we propose therefore to approximate this proba-
bility density by a confidence level approach where we
can give an upper bound for the probability of the state
to lie in a given interval.

The paper is organized as follows. We fist recall the
principles of the interval observers. Then we define
the notion of confidence levels and we explain how to
compute the intervals associated to a confidence level.
We illustrate the method on a bioprocess used to pro-
cess the wastewater: the activated sludge. We compare
then the results with a Monte-Carlo approach and we
show that the intervals associated to confidence levels provide a good estimate of the probability density of the state.

2. THE ACTIVATED SLUDGE PROCESS

The activated sludge process used for biological wastewater treatment consists of two tanks (FIG. 1). The main plant component, the aerator, is an aerobic biological reactor in which the substrate is biodegraded by suspended micro-organisms. This bioreactor is supposed to be continuously stirred, so that the substrate concentration is homogeneous. This substrate (organic matter) is consumed by a biomass which agglomerates into flocks: the activated sludge. We assume that the total solid part is separated by sedimentation of these flocks in the settler linked to the aerator. A fraction of the sludge collected in the settler is recycled to the aeration tank, whereas the remaining sludge is wasted to the sludge collected in the settler. We assume that the biomass concentration in the aeration tank, and exchanges are separated by sedimentation of these flocks in the settler linked to the aerator. A fraction of the sludge collected in the settler is recycled to the aeration tank, whereas the remaining sludge is wasted to the sludge collected in the settler. We assume that the total solid part is separated by sedimentation of these flocks in the settler linked to the aerator. A fraction of the sludge collected in the settler is recycled to the aeration tank, whereas the remaining sludge is wasted to the sludge collected in the settler.

FIG. 1 – Functional diagram of the activated sludge process.

In the aerator, we consider that a single bacterial population $x$ is growing on one limiting organic substrate $s$. We assume that it is the only biological reaction of the process. We also suppose that the organic matter does not settle in the sedimentation tank. The mass balance of the various constituents leads to the simplified model of activated sludge process [2, 5]:

\[
\begin{align*}
\dot{x} &= \mu(x) x - (1 + q_r) D(t) x + q_r D(t) x_r \\
\dot{s} &= -\frac{\mu(x)}{Y_s} x - (1 + q_r) D(t) s + D(t) s_{in}(t) \\
\dot{x}_r &= w(1 + q_r) D(t) x - w(q_w + q_r) D(t) x_r \\
\end{align*}
\]

with the following notations:

\[D(t) = \frac{Q_{in}}{V_a} ; q_r = \frac{Q_r}{Q_{in}} ; q_w = \frac{Q_w}{Q_{in}} ; w = \frac{V_a}{V_s}\]

where $x$, $s$ and $x_r$ are the model state variables representing respectively the biomass, the substrate and the recycled biomass concentrations. $Q_{in}$, $Q_{out}$, $Q_r$, $Q_w$ are respectively the influent, effluent, recycle and waste flow rates. $V_a$ and $V_s$ are the constant aerator and settler volumes. $s_{in}$ represents the influent substrate concentration. $Y_s$ corresponds to the yield coefficient of the growth of biomass on substrate. The initial conditions are respectively $x_0$, $s_0$, $x_{r0}$.

3. INTERVAL OBSERVERS

3.1. Recall on the interval observers [6]

We assume here that the disturbances and uncertainties are bounded, and that these bounds are known. We derive the dynamic bounds on the state variables from the bounds on the uncertainties. Thus, we compute two estimates bounding the state variables: an upper bound and a lower bound.

We consider the system:

\[
S \{ \dot{x} = Ax + B(x(t)); \quad x(t_0) = x_0 \}
\]

with $x \in \mathcal{X} \subset \mathbb{R}^n$, $A \in \mathcal{M}^{n \times n}(\mathbb{R})$ and $B \in \mathcal{M}^{n \times p}(\mathbb{R})$. Indeed, this system will correspond in the sequel to a reduced order observer, and $\phi \in \mathbb{R}^p$ will be the measured output of the whole system.

We suppose that the input is uncertain with known bounds $\phi^-$, $\phi^+$ such that:

\[
\phi^-(t) \leq \phi(t) \leq \phi^+(t), \quad \forall t \in \mathbb{R}^+
\]

Remark: The inequalities applied to vectors must be considered term by term.

Under this assumption, we build two asymptotic observers, using the detectability of the system [1, 2].

Definition 1 Let us consider the system $(S^- , S^+)$ with:

\[
\begin{align*}
S^- \{ \dot{x}^- &= Ax^- + B^-(\phi^-(t),\phi^+(t)) \\
\dot{x}^- (t_0) &= x^-_0
\}
S^+ \{ \dot{x}^+ &= Ax^+ + B^+(\phi^-(t),\phi^+(t)) \\
\dot{x}^+(t_0) &= x^+_0
\}
\end{align*}
\]

is an interval estimator for the system $(S)$ if for any compact set $\mathcal{X}_0 \subset \mathcal{X}$, the coupled system $(S, S^-, S^+)$ verifies for any initial condition $x(t_0) \in \mathcal{X}_0$:

\[
\forall t \geq t_0, \quad x^-(t) \leq x(t) \leq x^+(t)
\]

Function $B^+$ (respectively $B^-$) is such that:

\[
B^-(\phi^-(t),\phi^+(t)) \leq B(\phi(t)) \leq B^+(\phi^-(t),\phi^+(t))
\]

We remark that we need an estimate of $x_0$ and $x_{r0}$ at initial time $t_0$, but this estimate can be very loose.

Let us define the upper error $E^+ = \dot{x}^+ - x$ and the lower error $E^- = x - \dot{x}^-$. 

Lemma 1 If the matrix $A$ is cooperative (i.e. has positive off-diagonal elements), then:

\[
E^+(t_0) \geq 0 \Rightarrow E^+(t) \geq 0, \quad \forall t \geq t_0.
\]
We prove this lemma using the comparison theorem for cooperative systems [8]. More intuitively, it can be noticed that the vector field is repulsive on the boundaries. We have similar properties for the lower error, and consequently for the total error $E(t) = E^+(t) + E^-(t)$. The following theorem is a particular case of a theorem in [6].

**Theorem 1** If $A$ is stable and cooperative, and if we have an initial estimation: $x_0^\ast \leq x(t_0) \leq x_0^\ast$, then system $(S^-)$ is a bounded estimator for the model $(S)$.

Moreover, if the total error is bounded by a positive vector $M$:

$$B^+(\phi^-(t),\phi^+(t)) - B^-(\phi^-(t),\phi^+(t)) \leq M$$

then the total error $E(t)$ is asymptotically lower (term by term) than the non-negative vector:

$$E_{\text{max}} = -A^{-1}M$$

(2)

The proof of this theorem is a consequence of Lemma 1, and (2) follows from the differential vector inequality between $E$ and $E_{\text{max}}$.

**3.2. Application to the sludge process**

We assume the following hypotheses for model (1):

- the specific growth rate $\mu(\cdot)$ is unknown.
- the substrate concentration $s$ is the only measurable state variable, and the measurement is noisy: $y(t) = s(t) + b(t)$.
- we know the bounds on the uncertainty of measurement:
  \[ \forall \ t \geq t_0, \ b^- \leq b(t) \leq b^+ \]
  which implies:
  \[ s^- = y(t) - b^+ \leq s(t) \leq s^+ = y(t) - b^- \]
- the bounds on the inflow $s_{in}(t) = s_{in}^0(t) + \delta s_{in}$ are known:
  \[ s_{in}^- \leq s_{in}(t) \leq s_{in}^+ \]
- bounds on initial values $x_0$ and $x_{r0}$ are known.

A similar case has already been discussed in [10] without any noise on measurements. We apply now the results presented in , with a slight difference due to the additional scalar term $D(t)$: it is easy to see that it does not change anything since this term is always positive.

Firstly, we build an asymptotic observer [2, 11] for the set of equations (1) in order to eliminate the unknown function $\mu(\cdot)$ by the following change of variable:

$$Z = X + \begin{bmatrix} Y_s \\ 0 \end{bmatrix} s \text{ with } Z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \ X = \begin{bmatrix} x \\ x_r \end{bmatrix}$$

and we obtain the reduced system:

$$\dot{Z} = D(t)[AZ + B(s,t)], \ Z_0 = \begin{bmatrix} x_0 + Y_s s_0 \\ x_{r0} \end{bmatrix}$$

(4)

$$A = \begin{bmatrix} -(1 + q_r) q_r w(1 + q_r) -w(q_w + q_r) \end{bmatrix};$$

$$B(s,t) = \begin{bmatrix} Y_s s_{in}(t) \\ -Y_s w(1 + q_r).s \end{bmatrix}$$

We build the two estimators for system (4):

$$\dot{Z}^+ = D(t)[AZ^+ + B^+(s^-, s_{in}^0)]; \ Z_0^+ = X_0^+ + \begin{bmatrix} Y_s \\ 0 \end{bmatrix} s_0^-$$

$$\dot{Z}^- = D(t)[AZ^- + B^-(s^+, s_{in}^0)]; \ Z_0^- = X_0^- + \begin{bmatrix} Y_s \\ 0 \end{bmatrix} s_0^+$$

(3)

with

$$B^+(s^-, t) = \begin{bmatrix} s_{in}^0(t) \\ -w(1 + q_r).s(t) \end{bmatrix}.Y_s$$

$$B^-(s^+, t) = \begin{bmatrix} s_{in}^0(t) \\ -w(1 + q_r).s(t) \end{bmatrix}.Y_s$$

Matrix $A$ is stable and cooperative, therefore hypotheses of Theorem 1 are fulfilled. As a result the estimators (5) define an interval observer for the system (1). On Figure 2 we obtain estimations for the unmeasured state variables $x$ and $x_r$ (Fig. 2).

The specific growth rate $\mu(\cdot)$ chosen for the simulation purpose follows the Monod law [12]:

$$\mu(s) = \mu_{\text{max}} \frac{s}{M + s};$$

Besides, the numerical values used in the simulation are presented in (Tab. 1). We note $N(m, \sigma^2)$ the Gaussian distribution law with a mean $m$ and a standard deviation $\sigma$.

**4. INTERVAL OBSERVERS WITH GUARANTEED CONFIDENCE LEVELS**

**4.1. Confidence levels**

Now, we assume that we know the probability density of the uncertainties and disturbances. Here, we consider that the probability densities for each parameter are independent. More precisely we assume that we know the probability density of the following quantities:

- The model parameters,
- The input disturbances,
- The disturbances associated to the measurements,
- The probability density associated to the initial condition.

For sake of brevity, we will talk about parameter uncertainties for these four types of uncertainties and disturbances.
Note however that the experimental determination of the probability distribution of the uncertainties is often a difficult task requiring a high number of experiments. For illustration purpose we will consider Gaussian distributions, but the proposed algorithm can be used for any (unimodal) probability density function.

**Assumption 1** We assume that the uncertain parameters $p_j$, $j \in [1; k]$ have independent unimodal probability densities $f_{p_j}$ on k intervals $I_j$. Let us note $P_j = P(p_j \in [p_j^-; p_j^+])$ the probability for each parameter to be in a given interval $[p_j^-; p_j^+]$.

Now we will index the intervals $[p^-; p^+]$ by a confidence level $\chi$ corresponding to the probability for $p$ to be in this interval. This confidence level is thus defined as follows:

$$\chi = \prod_{j=1}^{k} P_j = P(p \in [p^-; p^+])$$

We will see that $\chi$ is a lower bound for the probability of the state $x$ to lie in an interval $[x^-; x^+]$. As a consequence, $\chi$ will be referred as a “guaranteed confidence level”. We will consider the particular interval $[x^-; x^+]$ provided by the interval observer, and we will therefore estimate the probability $P(x(t) \in [x^-; x^+])$. This computation is complicated; and we will only provide a lower bound of this probability. Indeed, if the uncertainties $p_j$ are in the interval $[p_j^-; p_j^+]$, then the previous section ensures that:

$$x(t) \in [x^-; x^+]$$

It follows that we have the following property:

$$P(x(t) \in [x^-; x^+]) \geq \chi$$

We will thus propose a way to choose a set of bounds $p^-$ and $p^+$, associated to a confidence level $\chi$. Of course, the choice of the bounds associated to a confidence level is not unique and other bounds could be considered.

**Definition 2** Under Assumption 1, we can choose the (finite) bounds $p_j^-(\chi)$ and $p_j^+(\chi)$ of the interval associated to the confidence level $\chi$ as follows:

i) $\mathcal{X}_\chi = \bigcap_{p_j^-} f_{p_j}(p_j^-) \cdots = \bigcap_{p_j} f_{p_j}(p_j^-) \cdots f_{p_k}(p_k^-) \cdots f_{p_{k+1}}(p_{k+1})$.

ii) $\forall j \in [1; k], \ f_{p_j}(p_j^-) = f_{p_j}(p_j^-)$

**Definition 3** Under these assumptions and those inherent to the synthesis of interval observers, we can define a new class of observers: the interval observers with guaranteed confidence level $\chi$.

Choosing a confidence level $\chi$ gives us $k$ fixed bounds $p_j^-(\chi)$ and $p_j^+(\chi)$ associated with the $k$ parameters $p_j$. We will now consider several possible values $\chi_i$ of $\chi$, and for each of these $\chi_i$ values we will build an interval observer $[x^-; x^+]$ associated with $\chi_i$. The interval observer $[x^-; x^+]$ would then be built using the bounds $p_j^-(\chi_i)$ and $p_j^+(\chi_i)$.
based on the bounds \([p^-(\chi_i); p^+(\chi_i)]\). The probability to have \(p\) in \([p^-; p^+]\) is \(\chi_i\), and therefore, the probability that \(x(t)\) is in \([x^-(p^-(\chi_i), p^+(\chi_i), t); x^+(p^-(\chi_i), p^+(\chi_i), t)]\) is larger than \(\chi_i\).

If \(r\) is large enough, we use \(x^-(p^-(\chi_i), p^+(\chi_i), t)\) and \(x^+(p^-(\chi_i), p^+(\chi_i), t)\) to estimate the so called confidence density function.

We will see in the next section that (after a renormalization to ensure that the total probability is 1) the confidence density function can be interpreted as a probability density function.

4.2. Application to the activated sludge process

We apply these interval observers with guaranteed confidence levels to the activated sludge process under the same operating conditions as before. Besides, we suppose that uncertainty on the model is mainly due to four parameters: an offset \(\delta s_{ini}\) on the input \(s_{ini}\), a noise \(b(t)\) on the measurement of substrate \(s\) and the initial conditions on biomasses: \(x_0\) and \(x_{r0}\). We assume that the uncertainties are characterized by Gaussian distributions (TAB. 2).

![Fig. 3 – The \(r = 21\) interval observers with guaranteed confidence levels.](image)

We run the simulation for \(r = 21\) values of \(\chi_i\), from \(\chi = 0\) to \(\chi = 0.99\), with a step of 0.05. Thus we build \(r = 21\) interval observers with guaranteed confidence levels (FIG. 4). Then we interpolate the results of these \(r\) interval observers with confidence levels to obtain the estimations of the biomasses bounds with various guaranteed confidence levels (FIG. 5).

Thus, at any time, we get the confidence density for the unmeasured variables \(x\) and \(x_r\).

The parameters used in these simulations are presented in (TAB. 2).

5. COMPARISON

In order to compare the confidence density computed from the interval observers to the actual probability density, we perform a Monte Carlo analysis. Thus, we run 30000 simulations associated to 30000 different values for the set of parameters \(\delta s_{ini}, x_0\) and \(x_{r0}\), according to their distribution (TAB. 2). Note that the disturbance on the output does not intervene here.

Thus we can compare the results of the two computations: on (FIG. 6) for the Monte Carlo computation, on (FIG. 7) for the interval observers with guaranteed confidence levels.

![Fig. 4 – Interval bounds on estimated biomasses \(x\) and \(x_r\) at different times, with respect to the \(t\) confidence levels.](image)

<table>
<thead>
<tr>
<th>Param.</th>
<th>Units</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>(b)</td>
<td>mg.L(^{-1})</td>
<td>(N(0,1))</td>
</tr>
<tr>
<td>(x_0)</td>
<td>mg.L(^{-1})</td>
<td>(N(500,150))</td>
</tr>
<tr>
<td>(x_{r0})</td>
<td>mg.L(^{-1})</td>
<td>(N(700,200))</td>
</tr>
<tr>
<td>(\delta s_{ini})</td>
<td>mg.L(^{-1})</td>
<td>(N(0,20))</td>
</tr>
<tr>
<td>(s_{ini}^2(t))</td>
<td>mg.L(^{-1})</td>
<td>(250 + 50 \sin(\frac{2\pi t}{30}))</td>
</tr>
</tbody>
</table>

![Tab. 2 – Parameter distribution used to build the interval observers with guaranteed confidence levels.](image)

![Fig. 5 – Probability density estimated by Monte Carlo computation.](image)
The comparison between the average value and standard deviations issued from the Monte Carlo analysis and from the interval observers with guaranteed confidence levels computation shows that the results are very close (same average, but different standard deviations). It ensures us that the two computations lead roughly to the same results: the interval observers with confidence levels give a good approximation of the probability densities for the variables to estimate. The computation of the set of interval observers is much faster.

However, the interval observers with guaranteed confidence levels computation provide a distribution which is more spread, because it is based on a worst case approach.

6. Conclusion

We have used a set of interval observers, which are based upon the deterministic bounds on uncertain parameters, by using the knowledge of their probability density to build confidence levels. Let us emphasize the following points:

- It is necessary to provide a probability density for each unknown parameter. They are supposed to result from an experimental analysis. Here we have assumed Gaussian distributions, but any kind of distribution law could be chosen.
- These observers cope with measurements noise, even if the noise distribution is not Gaussian.
- In this paper, we guarantee only an asymptotic rate of observer convergence, but in some cases, it is possible to tune this rate [9].
- These observers could improve monitoring of bioreactors since they characterize the spread of the state estimate.

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References


