ORDER EXTENSION OF NONLINEAR SYSTEMS FOR OBSERVER DESIGN UNDER REDUCED OBSERVABILITY

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Abstract: Nonlinear unforced systems of order \( n \) with reduced observability properties are considered. This reduction concerns the fact that the \( r \)-observability map is injective only for \( r > n \). An immersion into a higher order system containing the original system coordinates is proposed, such that estimation of the original states of the system is simplified. Approximate or event-based observers are proposed when some properties are lost for the extension, e.g. smoothness or continuity. Observer design is illustrated by examples and a simplified model of an aerobic bioreactor.

Keywords: observers, observability, nonlinear systems, singular points, immersion.

1. INTRODUCTION

Most methods for constructing observers for nonlinear systems require at least detectability of the Taylor linearization at some operating points. This property is necessary for the existence of smooth or exponential observers (Xia and Gao 1988, Krener 1994, Xia and Zeitz 1997) and enables the design of local observers, where the convergence can be specified by arbitrary eigenvalue assignment.

Other approaches are needed if the system has reduced observability properties. Although generically an unforced observable nonlinear single output system of order \( n \) has a globally injective \( r \)-observability map with \( r \leq 2n + 1 \) (Gauthier et al. 1991), many nonlinear observer design methods assume \( r = n \), i.e. unique determination of the state from knowledge of the output and \( n - 1 \) time derivatives.

This paper considers observer design when \( r > n \) is finite. The problem has already been studied (Levine and Marino 1986, Gauthier et al. 1991, Jouan and Gauthier 1996), but the observer construction is somehow limited to the cases when the \( r \)-observability map is an embedding. Here, the condition is relaxed and order extensions of the \( n \)th order system are proposed in order to simplify observer design. These order extensions include the coordinates of the original system.

The paper is organized as follows. First the observability assumptions for the nonlinear system are given. Next, the order extension with respect to reduced observability is presented, discussing the possibility of designing smooth, continuous, or approximate high gain observers in these new coordinates. It is then explained how event-based observers may be used to overcome some difficulties in the observer design, e.g. non-
smoothness. Several examples illustrate the procedures. Finally, a simplified model of an aerobic bioreactor is treated and an observer design is made using the proposed strategy.

Throughout the paper, the following notation will be used to represent the time derivatives of the output: \( y_0 := y \) and \( y_k := \frac{d^k y}{d t^k} \).

2. OBSERVABILITY ASSUMPTIONS

Consider the following unforced single output nonlinear system
\[
\dot{x} = f(x), \quad x(0) = x_0, \quad y = h(x), \quad (1)
\]
where the vector field \( f \) and the mapping \( h \) are smooth. The state space \( \mathcal{X} \subset \mathbb{R}^n \) is assumed to be compact and positively invariant.

Its observation space is defined to be (Nijmeijer and van der Schaft 1990)
\[
\mathcal{O} = \text{span}\{h, L_f h, \ldots, L_{f}^{r-1} h(x)\}, \quad (2)
\]
where \( L_f^k h \) is the \( k \)-th Lie derivative along the vector field \( f \). System (1) is observable if its observation space separates the points of \( \mathcal{X} \). In this paper, observable systems which satisfy the following finiteness criterion are considered.

**Definition 1.** System (1) is called \( r \)-observable if its \( r \)-observability map with \( r \geq n \)
\[
\mathbf{q}_r : x \mapsto [h(x), L_f h(x), \ldots, L_{f}^{r-1} h(x)]^T \quad (3)
\]
is globally injective. Furthermore, the integer \( r \) is minimal, i.e. the \((r-1)\)-observability map is not globally injective.

Injectivity implies the existence of a unique inverse function \( \mathbf{q}_r^{-1} \) when defined on the range \( \mathcal{Y}_r := \mathbf{q}_r(\mathcal{X}) \); this inverse is continuous if \( \mathcal{X} \) is compact. However, as soon as \( r > n \) and the domain is extended, it is no longer unique. It is here assumed that an inverse function \( \mathbf{q}_r^{-1} \) may be defined in an open set of \( \mathbb{R}^n \) containing \( \mathcal{Y}_r \) and coinciding with \( q^{-1} \), in \( \mathcal{Y}_r \). To illustrate this, consider the following example.

**Example 2.**
\[
\dot{x}_1 = x_2(x_2 + a), \quad \dot{x}_2 = x_2, \quad y = x_1, \quad (4)
\]
with \( a \neq 0 \). The 2-observability map is not injective because two values of \( x_2 \) correspond to the same \( y_1 := \hat{y} \), but the 3-observability map
\[
[y_0, y_1, y_2]^T = [x_1, x_2^2 + ax_2, 2x_2^2 + ax_2]^T
\]
forms a system of consistent and solvable linear equations in \( x_1, x_2, \) and \( x_2^2 \). Thus it is injective and an expression for its inverse is given by
\[
x = \mathbf{q}_3^{-1}(y_0, y_1, y_2) = [y_0, \frac{1}{a} (2y_1 - y_2)]^T. \quad (5)
\]
This inverse is smooth and its domain can naturally be extended to the whole \( \mathbb{R}^3 \). In fact, it easily shows that \( \mathbf{q}_r : \mathbb{R}^2 \rightarrow q_r(\mathbb{R}^2) \) is a global diffeomorphism, i.e. smooth, bijective and with a smooth inverse \( \mathbf{q}_r^{-1} : q_r(\mathbb{R}^2) \rightarrow \mathbb{R}^2 \).

In contrast, another expression for the inverse may also be given by \( x_1 = y_0, x_2 = \text{sign}(2y_1 - y_2)/a \cdot \sqrt{y_2 - y_1} \). Although it is equivalent to (5) in \( \mathcal{Y}_3 \), outside of \( \mathcal{Y}_3 \) it is only defined when \( y_2 \geq y_1 \) and is discontinuous at \( 2y_1 = y_2 \).

Using the \( r \)-observability map \( \mathbf{q}_r \) as coordinate transformation, i.e. \( z = \mathbf{q}_r(x) \), leads to the \( r \)-observability form
\[
\dot{\hat{z}} = [z_2 \cdots z_r \varphi(z)]^T, \quad z_0 = \mathbf{q}_r(x_0), \quad (6)
\]
\[
y = z_1, \quad \text{with} \quad \varphi(z) = L_f^r h \circ q_1^r(z), \quad (7)
\]
where all nonlinearities are concentrated in the function \( \varphi(z) \). After choosing an inverse \( q_1^r \), an explicit expression for \( \varphi \) can be constructed using (7). However, this is not the only possibility since any expression for \( \varphi \) that satisfies (7) for all \( z \in \mathcal{Y}_r \) may be used.

High gain observer design methodologies, such as those of Gauthier et al. (1991) and Hou et al. (2000) rely on the assumption that \( \varphi \) can be chosen Lipschitz continuous and its domain can be extended to the whole \( \mathbb{R}^r \). This is possible when \( \mathbf{q}_r \) is an embedding and \( \mathcal{X} \) is compact (Gauthier et al. 1991). The observer has the following structure
\[
\dot{\hat{z}} = [\dot{z}_2 \cdots \dot{z}_r \varphi(\hat{z})] + l(\theta) \cdot [y - \hat{z}_1], \quad (8)
\]
\[
\dot{\hat{x}} = q_1^r(\hat{z}), \quad (9)
\]
with \( l(\theta) \) a correction term depending on some high gain \( \theta \gg 1 \) (see, e.g. Atassi and Khalil’s (2000) paper). This observer has a dynamic part based on the \( r \)-observability form (6) and an algebraic part which transforms back into the original coordinates using an inverse transformation \( q_1^r \). For Example 2, using (7) with (5) leads to \( \varphi(z) = 4 \left( \frac{1}{a} (2z_1 - z_2) \right)^2 + 2z_1 - z_2 \), a smooth function defined in \( \mathbb{R}^3 \).

If the Jacobian of the \( r \)-observability map does not have full rank everywhere and therefore no smooth inverse \( q_1^r \) can be constructed, but a continuous \( q_1^r \) is possible, i.e. \( \mathbf{q}_r \) is a semi-diffeomorphism (Xia and Zeitz 1997), then the choice of the inverse \( q_1^r \) and the properties of \( \varphi \) represent additional ingredients that play a significant role for observer design. This is illustrated by

**Example 3.**
\[
\dot{x}_1 = x_1 + x_2^2, \quad \dot{x}_2 = \frac{1}{a} x_2^2, \quad y = x_1. \quad (10)
\]
The 3-observability map is semi-diffeomorphic, because \( y_2 = y_1 + x_2^2 \), so an inverse \( q_3^r \in C^0 \) is given by the non-smooth map
\[ x_1 = y_0, \quad x_2 = (y_2 - y_1)^{1/3}. \]

Since \( L^2_h(x) = x_1 + x_2^2 + x_2 + 1/3 \), using (7) yields
\[
\varphi(z) = z_1 - z_2 + z_3 + (z_3 - z_2)^2/3 + \frac{1}{2}(z_3 - z_2)^4, \quad (11)
\]
which is not Lipschitz continuous. However, since \( x_2^2 = y_1 - y_0 \) and \( x_2^2 = y_2 - y_1 \) is also true, direct substitution in \( L^2_h(x) \) yields
\[
\varphi(z) = z_3 + \frac{1}{2}(z_2 - z_1)^2, \quad (12)
\]
which allows the design of a continuous extended order observer (Xia and Zeitz 1997), because now the r-observability form has a Lipschitz continuous right hand side (12). The approximate high gain observer (Moreno and Vargas 2000) relaxes this condition, e.g. for (11), but then the observer error can no longer converge to zero.

In any case, the r-observability form (6) used for observer design is an extended order system in which the original system (1) is immersed. The set \( \mathcal{Y}_r \) is invariant under (6), as every trajectory starting in \( \mathcal{Y}_r \) will remain there for all \( t \geq 0 \). For observer design purposes it is important that such an immersion can be done, because the dynamics of (6) are modified by a correction term and trajectories are no longer confined to \( \mathcal{Y}_r \), even if they do start there.

Other immersions into higher order systems have already been proposed, e.g. into observable linear systems (Levine and Marino 1986). In the present contribution, immersions into r-observable systems that preserve the original coordinates of (1) are considered.

### 3. ORDER EXTENSIONS

In order to be precise about extended order systems, consider the following definition.

**Definition 4.** The \((n + m)\)th order system
\[
\xi = F(\xi), \quad \xi(0) = \xi_0, \quad y = H(\xi), \quad (13)
\]
where \( \xi \in \Xi \), an open set of \( \mathbb{R}^{n+m} \), is called an \((n + m)\)th order extension of system (1) if there exists an injective mapping \( E: \mathcal{X} \rightarrow \mathbb{R}^{n+m}, \) such that \( E(\mathcal{X}) \in \Xi \) is positively invariant, i.e. \( \xi_0 \in E(\mathcal{X}) \implies \xi(t) \in E(\mathcal{X}) \) \( \forall t \geq 0. \) Furthermore, trajectories of system (1) are mapped uniquely by \( E \) onto trajectories of system (13), i.e. there exists a unique homomorphic time transformation \( \rho: \tau \mapsto t, \) such that \( \xi(t) = \xi \circ \rho(\tau) = E(\mathcal{X}(\tau)), \) \( \tau \) being “time” in (1). Finally, \( H \circ E(\mathcal{x}) = h(\mathcal{x}) \) for all \( \mathcal{x} \in \mathcal{X}. \)

**Remark 5.** The extended r-observability form (6) with \( \varphi \) defined in an open set \( \mathcal{Z} \subset \mathbb{R}^r \) is an \( r \)th order extension of system (1) with \( E(\mathcal{x}) = q_r(\mathcal{x}) \).

For an \( r \)-observable nonlinear system (1), this paper proposes to choose a function \( \psi: \mathcal{X} \rightarrow \mathbb{R}^m \) with \( m = r - n \), such that
\[
E(\mathcal{x}) = [x^T, \psi^T(\mathcal{x})]^T. \quad (14)
\]
Partition the state vector \( \xi \) of (13) as \( \xi^T = [\xi_a^T, \xi_b^T] \) with \( \xi_a \in \Xi_a \subset \mathbb{R}^n \) and \( \xi_b \in \Xi_b \subset \mathbb{R}^m. \)

Then the dynamics of \( \xi_a \) resemble those of \( x \) in the original system (1) when \( \xi_a(0) = x_0 \) and \( \xi_b \) represent additional auxiliary states.

Of course, there are many degrees of freedom in the design of (13). However, impose the restriction of r-observability. If additionally its r-observability map \( q_r \) is diffeomorphic, then the high gain observer (8)–(9) can also be implemented as
\[
\dot{\hat{z}} = F(\hat{z}) + \left[ \frac{\partial q_r}{\partial z} \right]^{-1} I(\theta) \cdot [y - h(\hat{z})], \quad (15)
\]
\[
\hat{x} = [I_n, 0_{m \times n}] \hat{z}, \quad (16)
\]
where \( \partial q_r / \partial z \) is the Jacobian of \( q_r \), and \( I_n \) is the identity matrix of order \( n. \) It also makes sense to set the initial condition of this observer as
\[
\hat{x}(0) = [\hat{x}_a^T, \psi^T(\hat{x}_0)]^T. \quad (17)
\]

If the r-observability map is only semi-diffeomorphic and the differentiability of \( q_r \) is lost only at some points, then event-based observers (Vargas et al. 2001) may be used to bypass the singular points of the Jacobian matrix in (15).

A suggestion on how to build the extended system (13) is given by the following proposition.

**Proposition 6.** Choose the components of \( \psi \) as
\[
\psi_i(\mathcal{x}) = f_i^{k+i-1}(h(\mathcal{x})), \quad i = 1, 2, \ldots, m, \quad (18)
\]
with some \( k \leq n - 1. \) Then the following system is a \((n + m)\)th order extension of system (1)
\[
\xi_a = \tilde{\xi}_a(\xi), \quad \xi_b = A \xi_b + B \phi(\xi), \quad y = H(\xi), \quad (19)
\]
where
\[
A = \left[ \begin{array}{cc} 0_n & I_{m-1} \\ \mathbf{0} & 0 \end{array} \right], \quad B = \left[ \begin{array}{c} 0_m \\ \mathbf{0} \end{array} \right],
\]
\[
\tilde{\xi}_a(\xi) = \bar{f}(\xi), \quad \phi \circ E = L_t^{k+m} h, \quad H \circ E = h. \quad (20)
\]

**Proof.** \( H \circ E(\mathcal{x}) = h(\mathcal{x}) \) is ensured by construction. Suppose \( \xi_a(0) = \mathcal{x}_0 \) and \( \xi_b(0) = \psi(\mathcal{x}_0). \) Because of the first condition in (20), it is clear that \( \xi_a(t) \) will remain in \( \mathcal{X} \) for all \( t \geq 0. \) Since \( E(\mathcal{X}) = \mathcal{X} \times \psi(\mathcal{X}) \), it remains to show that under the above initial condition restriction, \( \xi_b(t) \in \psi(\mathcal{X}) \) for all \( t \geq 0. \) Recall \( y_b(t) = L_t^h(\xi_b(t)), \) so if \( \xi_b(0) = L_0^{k+m} h(\mathcal{x}_0), \) then \( \xi_b(t) = L_t^{k+m} h(\mathcal{x}(t)) \) implies that \( \xi_b(t) = y_b(t) \) for all \( t \geq 0. \) The chain of integrators \( \xi_{b_i} = \xi_{b_{i+1}} \) then implies \( \xi_{b_i}(t) = y_{b_{i+1}}(t) \) for \( i = 1, \ldots, m-1 \) if \( \xi_b(0) = L_0^{k+m} h(\mathcal{x}_0). \) Hence \( \xi_b(t) \in \psi(\mathcal{X}). \) The time transformation \( \rho \) is the identity. □
Proposition 6 does not suppose any observability properties, but it is desired that the extension (19) is $r$-observable, i.e. that the $r$-observability map

$$q_r : \xi \mapsto \left[ H(\xi) \ L_{\psi} H(\xi) \ \cdots \ L_{\psi}^{r-1} H(\xi) \right]^T$$

(21)
is injective, and furthermore the inverse $q_r^{-1} \in C^0$ is continuous. This puts some additional restrictions on how $\bar{f}$, $\phi$, and $H$ should be chosen, as well as on the integer $k$ to decide on the choice of $\psi$ in (18). Unfortunately, their design is based more on heuristics, than on a constructive algorithm, but information gathered while checking for invertibility of the $r$-observability map may provide a useful guideline.

For Example 2 choose $\psi(x) = L_1^3 h(x) + x_2 a x_2$. An obvious choice of $\bar{f}$, $H$, and $\phi$ leads to the 3rd order extension

$$\dot{\xi}_1 = \xi_3, \ \dot{\xi}_2 = \xi_2, \ \dot{\xi}_3 = 2\xi_3 - a\xi_2, \ \ y = \xi_1.$$  \hspace{1cm} (22)

This is an observable linear system, so a smooth exponential observer can be designed, such as the high gain observer (15)–(17).

For Example 3, choosing $\psi = L_1^3 h$ and $\bar{f}_1 = \xi_3$, a 3-observable 3rd order extension is given by

$$\dot{\xi}_1 = \xi_3, \ \dot{\xi}_2 = \frac{1}{2}\xi_2^2, \ \dot{\xi}_3 = \xi_3 + \xi_2^2, \ \ y = \xi_1.$$  \hspace{1cm} (23)

This extension has a semi-diffeomorphic 3-observability map, and the characteristic nonlinearity $\varphi(z) = z_3 + \frac{1}{2}(z_3 - z_2)^{4/3}$ of the corresponding 3-observability form (6) is Lipschitz continuous.

A continuous observer design (8)–(9) is possible. Notice that if $\bar{f}_2 = \frac{1}{2} \dot{\xi}_3$ were chosen instead, the same injective 3-observability map is obtained, but $\varphi(z)$ in the corresponding 3-observability form results non-Lipschitz.

4. EVENT-BASED OBSERVERS

For cases like the last one discussed, where it is not possible to design a high gain observer in observability coordinates due to the lack of Lipschitz continuity in $\varphi$, approximate high gain observers have been proposed (Moreno and Vargas 2000). Here the non-Lipschitz nonlinearity $\varphi$ is approximated by a Lipschitz function $\bar{\varphi}$ and a high gain observer is designed in observability coordinates. Then the continuous inverse $q_r^{-1}$ is used as the algebraic part of the observer.

Instead of using approximate high gain observers, event-based observers (Vargas et al. 2001) could also be implemented, with the advantage of being designed on the same coordinate system as the order extension (13). Furthermore, they are also practical when the order extension (13) fails to be smooth. In order to explain this, consider the following example.

Example 7.\[
\dot{x}_1 = x_2^2p, \quad \dot{x}_2 = x_1, \quad y = x_1, \quad (24)\]

with $p \geq 1$ is 4-observable. From $y_1 = y = x_2^p$, two possible values for $x_2$ are obtained. These two values can be distinguished via $y_2 = 2px_2^{2p-1}x_1$, but only if $y_0 \neq 0$. Consider another derivative: $y_3 = 2p(2p-1)x_1^2x_2^{2(p-1)} + 2px_2^{2p-1}$. Since $x_2^p \geq 0$ for all $x_2$, then it is valid to consider

$$x_2^{(p-1)} = (x_2^p)^{(2p-1)/2p} = (y_1)^{(p-1)/p},$$

and because $4p - 1$ is odd, an inverse of the 3-observability map is given by

$$x_1 = y_0,$$

$$x_2 = \left(\frac{1}{2}y_3 - (2p-1)\int_0^1 y_1^2(y_1^{p-1}/p) \frac{dy_1}{y_1}\right)^\frac{1}{2p}.$$  \hspace{1cm} (25)

Choosing $\psi_1 = L_1^3 h$ and $\psi_2 = L_1^2 h$, the extension is easily constructed as

$$\dot{\xi}_1 = \xi_3, \ \dot{\xi}_2 = \xi_1, \ \dot{\xi}_3 = \xi_4, \ \dot{\xi}_4 = \phi(\xi),$$  \hspace{1cm} (26)

$$y = \xi_1.$$  \hspace{1cm} (27)

with

$$\phi(\xi) = 2p \left( (2p-1)\xi_3^{(p-1)/p} + \xi_2^{2p-1} \right).$$  \hspace{1cm} (28)

The 4-observability map is semi-diffeomorphic, but the extended system does not have a Lipschitz continuous right hand side, so uniqueness of solutions is not even guaranteed. Use of an approximate high gain observer design would require approximating the resulting discontinuous function $\varphi$ in its observability form (6).

In order to introduce the event-based observer as a solution, several definitions are needed first.

Definition 8. Consider the $r$-observability map $q_r(\xi)$ of an $r$-observable dynamical system (13) of order $r$. A bad point $\xi^*$ is a point in the state space $\Xi$ where the Jacobian $[\partial q_r / \partial \xi]$ is singular or not defined.

Note that by the inverse function theorem and $q_r$, being injective, bad points $\xi^*$ are actually points where the inverse $q_r^{-1}$ is not differentiable.

Assume that the complement of the set of all bad points $\Xi^*$ is dense in $\Xi$, i.e. that $\Xi^*$ is a “small” subset of $\Xi$. Define also some distance function $\delta^* : \Xi \rightarrow \mathbb{R}_+$ to the set of bad points with $\delta^*(\xi) \geq 0 \ \forall \xi \in \Xi$ and $\delta^*(\xi^*) = 0$ only for $\xi^* \in \Xi^*$.

Definition 9. Given a solution trajectory $\xi(t)$ of the system (13), a set of bad points $\Xi^*$ and a distance function $\delta^*$, define for some $\epsilon > 0$, an event $\Delta_i^* = [t_{in}^i, t_{out}^i], \ i = 1, 2, \ldots$, as the time interval when $\xi(t)$ passes through the set $\Xi_i^* := \{\xi \in \Xi : \delta^*(\xi) < \epsilon\}$. 
An event-based observer considers the times when events occur and suspends the use of a conventional observer during this time. Instead, it implements a simulation of the system, while allowing the trajectory \( \hat{x}(t) \) to remain continuous with respect to time \( t \). If the dynamics of the system do not have finite escape times, then the observer error \( \hat{x} - x \) will remain bounded during the event \( \Delta_i^* \). After \( t = t_i^\text{out} \), the conventional observer is again used. If the time until the next \( t_i^\text{in} + 1 \) is large enough, the observer error will again begin to converge to zero.

It is quite straightforward to design a high gain event-based observer for system (13). The correction term in (15) can be designed when \( t \not\in \Delta_i^* \) because the Jacobian \( \partial \xi / \xi \) is invertible and the observability normal form is Lipschitz. During the events, the correction term is set to zero, i.e.

\[
\dot{\hat{\xi}} = F(\hat{\xi}) + L(\hat{\xi}, \theta) \cdot \begin{bmatrix} u - H(\hat{\xi}) \end{bmatrix},
\]

\[L(\hat{\xi}, \theta) = \begin{cases} 0 & \text{if } \hat{\xi} \in \Xi^e, \\ \left[ \frac{\partial q_3}{\partial \xi} \right]^{-1} & \text{otherwise}. \end{cases}
\]

The trajectories \( \hat{\xi}(t) \) of the observer are continuous functions of time \( t \) since they are defined by a piecewise continuous vector field.

**Remark 10.** Evidently, event-based observers are approximate observers, but even the assumption of smallness of the set of bad points \( \Xi^e \) is not enough to guarantee some convergence. The time between events must be large with respect to the duration of the events. This obviously restricts the system, since trajectories should not spend too much time in the boundary layer.

For Example 7, the set of bad points is characterized by \( \xi_2 = 0 \) (where the inverse \( q_3^{-1} \) is not differentiable) and by \( \xi_3 = 0 \) (where \( F \) is not Lipschitz). Thus define

\[
\Xi^e = \{ \xi \in \Xi : |\xi_2| < \varepsilon_2, |\xi_3| < \varepsilon_3 \},
\]

with \( \varepsilon_2, \varepsilon_3 > 0 \). The event-based high gain observer is built as in (29), choosing the initial condition as in (17), plus the simple algebraic part

\[
\hat{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \hat{\xi}.
\]

5. APPLICATION EXAMPLE

The approach outlined so far will be used to design an observer for a simple model of the reaction phase of a batch aerobic bioreactor used to degrade toxic compounds in wastewater,

\[
\begin{align*}
\dot{X} &= \mu(S)X - K_d X, \\
\dot{S} &= -\frac{1}{\beta} \mu(S)X, \\
X(0) &= X_0, \\
S(0) &= S_0,
\end{align*}
\]

where \( X \) is the biomass concentration, \( S \) is the substrate concentration, \( Y \) and \( K_d \) are parameters, and the specific growth rate \( \mu \) is given by the Haldane equation

\[
\mu(S) = \frac{\mu_{\text{max}} S}{S^2 + K_s S + K_I},
\]

which is positive for every \( S > 0 \) and non-injective. It is difficult to measure both concentrations on-line, but it is possible to obtain reliable measurements from the oxygen concentration, whose dynamics is given by

\[
\dot{O} = -\frac{1}{V_{so}} \mu(S)X + K_{oa}(O_{\text{sat}} - O),
\]

with \( Y_{so} \) and \( K_{oa} \) parameters and \( O_{\text{sat}} \) the oxygen saturation constant in the medium. Suppose it is only of interest to estimate the substrate concentration \( S \), e.g. to determine when the toxic reaches a minimum. Under usual operating conditions, the increase in biomass during one cycle is not significant and furthermore, it is kept almost constant by drawing extra biomass periodically. This allows to consider \( X \) as a known parameter. Defining \( x_1 := O_{\text{sat}} - O \) and \( x_2 := S \), a simplified model is given by

\[
\begin{align*}
\dot{x}_1 &= k_1 \frac{x_2}{\beta(x_2)} - a_1 x_1, \\
\dot{x}_2 &= -k_2 x_1 \frac{x_2}{\beta(x_2)}, \\
y &= x_1,
\end{align*}
\]

where \( \beta(x_2) = \frac{1}{2} x_2 + k_3 x_2 + k_4 \) (37) with parameters \( a_1 \) and the \( k_i, i = 1, \ldots, 4 \). This system is not 2-observable, but the 3-observability map is injective. To see this, notice first that (35) can also be written as

\[
\dot{x}_2 = -k_2 (\dot{x}_1 + a_1 x_1) - k_2 (y_1 + a_1 y_0) \]

by substituting \( y_0 \) and \( y_1 \) for \( x_1 \) and \( \dot{x}_1 \), respectively. Applying this substitution to (34), differentiating and using (38), the following consistent linear system of equations with unknowns \( x_2 \) and \( \beta \) parameterized by \( y_0, y_1 \), and \( y_2 \), is obtained:

\[
\begin{align*}
0 &= k_1 x_2 - (y_1 + a_1 y_0) \beta, \\
k_1 (y_1 + a_1 y_0) &= (y_1 + a_1 y_0)^2 (x_2 + k_3) \\
&\quad - \frac{1}{k_2} (y_2 + a_1 y_1) \beta.
\end{align*}
\]

A solution can be found for \( x_2 \), so an inverse 3-observability map \( q_3^e \) is given by

\[
\begin{align*}
x_1 &= y_0, \\
x_2 &= \frac{k_1 - k_3 (a_1 y_0 + y_1)}{(a_1 y_0 + y_1) - \frac{k_1 (a_1 y_0 + y_2)}{k_2 (a_1 y_0 + y_1)}}.
\end{align*}
\]

To build the order extension, choose \( \psi_1 = L_{\text{fit}} \), such that \( \hat{f}_1 = \xi_3 \). The function \( \phi \) in (19) can be obtained from (42) by substituting \( x_1, \xi_3 \), and \( \xi_3 \) for \( y_0, y_1 \), and \( y_2 \), respectively. Finally
This 3rd order extension has an injective 3-order extension with the polynomial $P(\xi)$ defined by

$$P(\xi) = \frac{k_2}{k_1} (a_1 \xi_1 + \xi_3)^3 (\xi_2 + k_3) - a_1 \xi_2 \xi_3 - k_2 (a_1 \xi_1 + \xi_3)^2.$$  \hfill (47)

This 3rd order extension has an injective 3-observability map $\mathbf{q}_3(x)$, whose Jacobian matrix is given by

$$\begin{bmatrix} \partial \mathbf{q}_3 \partial z \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & \frac{\partial P/\partial \xi_1}{\xi_2} \\ 0 & 0 & \frac{\partial P/\partial \xi_2}{\xi_2} - \frac{P(\xi)}{\xi_2^2} \\ 0 & 1 & \frac{\partial P/\partial \xi_3}{\xi_2} \end{bmatrix}.$$  \hfill (48)

This matrix is not defined at $\xi_2 = 0$, which should be outside the domain of definition (it makes no sense to operate the bioreactor without substrate). On the other hand, it is singular at

$$(a_1 \xi_1 + \xi_3)^2 (k_3 (a_1 \xi_1 + \xi_3) - k_1) = 0.$$  \hfill (49)

An event-based observer (29) can be designed by defining for example

$$\Xi^*_\epsilon = \left\{ \xi \in \Xi : |a_1 \xi_1 + \xi_3| < \epsilon_1, |k_3 (a_1 \xi_1 + \xi_3) - k_1| < \epsilon_2 \right\}$$

for $\epsilon_1, \epsilon_2 > 0$. Notice however, that in the invariant manifold $\xi_3 = \psi(R_+)$, it happens that $a_1 \xi_1 + \xi_3 = k_3 x_{2z}/\beta(x_2)$, which is never zero if $x_{2z} > 0$. It is also not difficult to see that then also $k_3 (a_1 \xi_1 + \xi_3) - k_1 \neq 0$, so that at least on the invariant manifold where the original system’s trajectories move there are no bad points.

6. CONCLUSIONS

An heuristic methodology to construct order extensions for $r$-observable unforced systems has been proposed. It is based on finding an inverse of the $r$-observability map and enables the design of extended order observers which do not require an explicit algebraic part, i.e. the observer’s first $n$ states may be viewed as estimates of the original system’s states. The extension may lose some of the original system properties, e.g. smoothness or Lipschitz continuity. Approximate or event-based observers can be used to overcome this.

Event-based observers pose an interesting alternative to other methods requiring an explicit transformation into observability normal form. They could also enable a further reduction of the needed order $r$ of the observer if they are implemented on an $m$th order extension with $m < r - n$, where the set $\Xi^*$ of bad points for the observability map is redefined to include also points where the observability map $\mathbf{q}_n+m$ is not injective. The complement of this set should be dense in $\Xi$.

A simple aerobic bioreactor model shows the applicability of the strategy. A more interesting challenge considers forced systems, where bad inputs may be encountered, for which observability is lost altogether. In fact, the bioreactor model (30)–(33) with inputs is such a system. Designing event-based observers could lead to possible answers to cope with this problem and this is currently being investigated.

7. REFERENCES


