ROBUST STABILITY OF UNCERTAIN SYSTEMS
VIA PARAMETER - DEPENDENT LYAPUNOV FUNCTION

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Abstract: The robust stability problem for uncertain, linear, state-space models is considered.
When a fixed Lyapunov function is used to provide an admissible perturbation set, the
obtained variation bounds can be too conservative. The main purpose of this investigation is to
define the conditions, under which it is always possible to construct a parameter-dependent
Lyapunov function for a class of uncertain systems. The contribution to robustness study is due
to a new sufficient condition for robust stability. The advantages of this approach are
illustrated by examples and comparison with results, obtained by known procedures is made.

Keywords: linear control systems, Lyapunov stability, uncertain dynamic systems, robust
stability, matrix equations.

1. INTRODUCTION

The usefulness of any system analysis or design approach is crucially dependent on the accuracy of
it’s mathematical model. Speaking practically, any real system is subjected to parameter variations,
leading to identification errors, often called model uncertainties. It is obvious, that discrepancies
between the model and the real system may result in degradation in the system’s functioning. Any control
system should be designed to be insensitive, i.e. a robust one, against uncertainties in the plant’s model.
By no doubt, the most important concern in this regard is that of robust stability.

This research considers the problem of determining admissible variation sets for an uncertain vector
parameter, included in the model of a linear, dynamic system to reflect the influence of various perturbation
factors and modelling inaccuracies. Lyapunov’s stability theory is a key-tool for the purpose and a
subjective account of some of the main results in the use of quadratic functions in robust analysis for
uncertain systems is presented in (Corless, 1993). Many of the available results (Martin, 1990; Patel
and Toda, 1980; Yedavalli, 1985; Zhou and Khargonekar, 1987, etc), provide norm bounded
admissible perturbation sets and these are proved to be very conservative due to the symmetry of norms.
Certain parameters may admit much larger perturbations, than presented by the norm bounds.
Asymmetric stability bounds on the uncertain parameters are obtained, e.g. in (Gao and Antsaklis,
1993; Mansour, 1998; Wang, et al., 1991), showing clearly their superiority to norm-based ones.
Although better, they can still be too conservative and thus the problem of asymmetric admissible
perturbation sets extension is posed. The main shortcoming, shared by approaches of the kind is due
to the fact, that for the analysis of an uncertain system, possibly time-varying, a fixed Lyapunov’s
function, or simply a Lyapunov’s matrix is used. What’s more - since it’s choice is made arbitrarily as
a rule, the obtained admissible variation sets are yet rather conservative and many actually stable systems
are treated according to them as unstable ones.

The main purpose of this investigation is directed towards the question, whether it is possible to
construct a Lyapunov’s matrix, which depends on the uncertain parameter. In the literature, one can find
some results that analyse affinely perturbed linear systems with affine parameter dependent Lyapunov’s
matrices (Amato, et al., 1997; Chockalingham, et al., 1995; Gahinet, et al., 1996). Considerable number of

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approaches are currently in use (see, e.g., the references in (Barmish and Kang, 1993)), but each of them has its own demerits. Among them one should mention the results obtained in (Feron, 1995; Haddad and Bernstein, 1994, 1995; Mori and Kokame, 2000).

The contribution of this work consists mainly in defining the conditions under which such a function always exists for the class of uncertain systems considered here. This helps to achieve much more adequate reflection of the real system’s nature, eliminates the arbitrary - subjective choice for a Lyapunov’s matrix and finally results in defining much less conservative admissible sets.

The rest of the paper is organized as follows. The statement of the problem is given in section 2. Some aspects concerning linear matrix and vector equations, which are closely related to the main result, are presented in section 3. The main result is obtained in section 4. It contains a new characterization of the sets of stable and positive - definite matrices and an extension of these important results for the case considered here. Various solution’s aspects are widely discussed in section 5. Two examples, illustrating the abilities of the suggested approach are solved in section 6.

2. THE ROBUST STABILITY PROBLEM

Consider the state - space model of a linear system

\[ \frac{dx}{dt} = Ax, \quad x \in \mathbb{R}^n, \]

where \( A \) is a constant and stable \( n \times n \) real matrix, i.e. \( A \in S, \quad S \equiv \{ A : \lambda \in \sigma (A) \Rightarrow Re \lambda < 0 \} \). The set of eigenvalues of \( A \) is denoted by \( \sigma (A) \). The stability problem for any system modelled by eq.(1) refers to stability of its state coefficient matrix, which will be considered in this regard from now on. Let \( p, \ p \geq 1 \), real scalar parameters, varying in some unknown intervals (zero included), perturb some known entries of \( A \). This corresponds to the presence of structured parametric uncertainty in a stable matrix and can be modelled as \( A + \Delta \), where

\[ \Delta = \sum_{i=1}^{p} \alpha_i A_i, \quad \alpha_i \preceq \alpha_i^* \preceq -\alpha_i^- \]

The constant and known matrix \( A_i \) defines the influence structure for the \( i \)-th uncertain parameter \( \alpha_i \) over one or more entries of \( A \). Define the uncertain vector \( \alpha \) as: \( \alpha = (\alpha_1, \alpha_2, ..., \alpha_p)^T \). The robust stability problem for this class of uncertain matrices is stated as: determine an admissible vector set \( \Omega^* \), such that

\[ \alpha \in \Omega^* \Leftrightarrow \alpha^* \preceq \alpha \preceq \alpha^- \Rightarrow A + \Delta \in S, \]

where \( \alpha^+ \) and \( \alpha^- \) are constant vectors, which should be determined. All vector inequalities are intended element - by - element.

The general scheme, followed by all approaches aimed at solving the above stated problem by using fixed Lyapunov’s function, consists in the following. According to Lyapunov’s stability theorem

\[ (A + \Delta)^T P + PA = -Q, \quad Q \in P, \quad P \in P, \]

\[ \Delta = WAW^{-1}, \quad \text{rank } W = n, \quad \text{where } P \text{ denotes the set of symmetric, positive definite matrices. Since } A \text{ is a stable matrix, the solution } P \text{ (called Lyapunov’s matrix)} \text{ exists uniquely, for any fixed right-hand side matrix } Q \in P. \text{ Then, } A + \Delta \in S, \text{ if } \Delta P + P \Delta < Q. \text{ By imposing various restrictions on } \alpha, \text{ it is always possible to find an admissible vector set } \Omega^*, \text{ such that robust stability for the uncertain matrix is sufficiently guaranteed.} \]

Contrary to this, the purpose here is to determine a matrix \( R(\Delta) \in P \), such that:

(i) \( (A + \Delta)^T R(\Delta) + R(\Delta)(A + \Delta) < 0, \quad \alpha \neq 0, \)

(ii) \( \Delta^T R(0) + R(0)\Delta = -Q. \quad \alpha \equiv 0, \)

which means, that only for \( \Delta \equiv 0 \), the Lyapunov’s matrix \( R(0) \equiv P \) is a fixed one.

3. LINEAR MATRIX AND VECTOR EQUATIONS

Consider a linear in the unknown matrix \( X \) equation

\[ XY + Y^T X = Z, \quad Y \in \mathbb{R}^{nxn}, \quad Z \in \mathbb{R}^{nxn}, \]

and the mapping vec\( R^{nxn} \rightarrow R^{n^2} \) given by

\[ (6.1) \quad X \rightarrow \text{vec } X \equiv x = (x_y), \]

\[ (6.2) \quad Z \rightarrow \text{vec } Z \equiv z = (z_y). \]

For a given linear transformation \( T: R^{nxn} \rightarrow R^{n^2} \), there exists an unique matrix \( M(Y) \), such that eq.(5) can be put in a vector form as:

\[ (7) \quad M(Y)x = z, \quad M(Y) \in R^{n^2 \times nxn}, \]

for all \( X \). Equation (5), respectively eq.(7), has an unique solution for any right-hand side, if and only if (Horn and Johnson, 1991):

\[ (8) \quad \sigma(Y) \cap \sigma(-Y) \equiv \emptyset \]

The transformation of eq.(5) into eq.(7) is not practically justified in the general case. Fortunately, there exists a case, when the order of eq.(7) can be significantly decreased. Suppose, that \( Z = -Z^T \) in eq.(5). It can be easily shown, that the solution matrix \( X \) is a skew-symmetric one as well, i.e.
X = −X^T with entries \( x_{ij} = 0 \), for \( i = j \) and \( x_{ij} = −x_{ji} \), otherwise. The associated with matrices X and Z vectors (6.1) and (6.2) become:
\[
x = (0x_{12}x_{13}...x_{1n} − x_{12}0...x_{2n} − x_{13}...−x_{1n} −x_{2n}...0)^T,
\]
\[
z = (0 z_{12}z_{13}...z_{1n} − z_{12}0...z_{2n} − z_{13}...−z_{1n} −z_{2n}...0)^T.
\]

For the sake of simplicity and without any loss of generality, a change in subscripts for \( i < j \) is suggested, according to which the couples \((i,j)\) are transformed as follows:
\[
(1,2) → 1, (1,3) → 2,...,(1,n) → n−1,
(2,3) → n,...,(n−1,n) → k, k = 1/2n(n−1).
\]

The solution and right-hand matrices can be presented as:
\[
X = ∑_{i=1}^{k} x_{i} I_{i}, \quad Z = ∑_{i=1}^{k} z_{i} I_{i}, \quad I_{i} = −I_{i}^T,
\]
where \( I_{i} \) is a respective matrix with only two non-zero, symmetrically positioned entries (1 and -1).

Equation (5) is rewritten as:
\[
(9) \quad ∑_{i=1}^{k} x_{i} Y_{i} = ∑_{i=1}^{k} z_{i} I_{i}, \quad Y_{i} = I_{i} Y + Y^{T} I_{i} = −Y_{i}^T.
\]

This presentation of eq.(5), for \( Z = −Z^{T} \) clearly shows, that it can be put in a vector form as:
\[
(10) \quad M(Y)x = z, \quad M(Y) ∈ R^{k×k},
\]
\[
x = (x_{1}x_{2}...x_{k})^T, \quad z = (z_{1}z_{2}...z_{k})^T.
\]

Therefore, taking into account the specific structure of any skew-symmetric matrix helps to decrease significantly the order of eq.(7), which is important for the practical application of the present research.

4. MAIN RESULT

Recall Lyapunov’s theorem (4). The next two theorems show in an alternative way the close relation between sets \( S \) and \( P \), realized through another matrix set \( S^{−} ≡ \{ S; S^T + S < 0 \} \) and make possible to characterize uniquely the sets of stable and positive definite matrices.

**Theorem 4.1.** A matrix \( X ∈ S \), if and only if
\[
(11) \quad X = YZ, \quad Z ∈ P, \quad Y ∈ S^{−}.
\]

**Proof.** Let \( X ∈ S \). For any \( Q ∈ P \), there exists a matrix \( G ∈ P \), such that \( XG + GX^T = 2Q \). Then
\[
XG + Q = −(XG + Q)^T = F = −F^T \Rightarrow
X = (−Q + F)G^{-1} = YZ,
\]
as required. Let (11) holds for some matrix \( X \). Therefore, \( XZ^{-1} + Z^{-1}X^T < 0 \), which is possible, if and only if \( X ∈ S \).

**Theorem 4.2.** A matrix \( Z = Z^T ∈ P \), if and only if
\[
(12) \quad Z = YX, \quad X ∈ S, \quad Y ∈ S^{−}.
\]

**Proof.** Let \( Z ∈ P \). Then for any matrix \( Y ∈ S^{−} \),
\[
ZY^{-1}Z + ZY^{-1}Z = X^T Z + ZX < 0,
\]
which is possible, if and only if \( X = Y^{-1}Z ∈ S \), or \( Z = YX \) is the required presentation.

Let (12) holds for some matrix \( Z = Z^T \). Then \( X^{-T}Z + ZX^{-1} < 0 \). Since \( X ∈ S \), the unique solution \( G \) to the Lyapunov’s equation \( X^{-T}G + GX^{-1} = X^{-T}Z + ZX^{-1} \) is a symmetric, positive definite matrix. Obviously \( G = Z \) and consequently \( Z ∈ P \).

These theorems play a basic role in the derivation of the main result, which is to a great extent their application for the case of an uncertain matrix robustness study, considered here.

**Theorem 4.3.** The uncertain matrix \( A + Δ ∈ S \), if
\[
(i) \quad −Y ∈ S, \quad Y = I + ΔA^{-1}
\]
\[
(ii) \quad FY = R = R^T, \quad −F ∈ S^{−},
\]
\[
(iii) \quad A^TF ∈ S^{−}.
\]

**Proof.** Let (i) holds. If \( R \), as defined in (ii), is a symmetric matrix and since \( R = −F(Y) \), according to Theorem 4.2., then it is also a positive definite one, i.e. \( R ∈ P \). Symmetry and positive definiteness are preserved by multiplying \( R \) by \( A \) and \( A^T \) from the right and from the left, respectively. Therefore,
\[
A^TF(A + Δ) = (A + Δ)^TF^T A = R_1 > 0,
\]
or \( A + Δ = (A^TF)^{-1}R_1 \). If (iii) is valid, according to Theorem 4.1. it follows, that \( A + Δ ∈ S \).

5. SOLUTION ASPECTS

(a1) Requirements (i), (iii), Theorem 4.3. Their satisfaction consists for the general case in the solution of a linear matrix inequality problem with respect to vectors \( α \) and \( x \), i.e.
\[
α ∈ Ω_{1} ⇔ α_{1} ≤ α ≤ α_{1}^+ ⇒ −[I + ∑_{i=1}^{P} α_{i}LA_{i}A_{i}^{-1}L_{i}] ∈ S^{−},
\]
\[
x ∈ Ω_{S} ⇔ x^− ≤ x ≤ x^+ ⇒
\]
(a2) Requirement (ii), Theorem 4.3. It is important now, to answer the question: how one can determine a matrix $F$, such that $R = RF$ for $-F \in S^-$. Let matrix $F$ be chosen as $F = X + P$, where $X = -X^T$ is an unknown matrix. The condition for symmetry can be rewritten as eq.(5), where $Z = A^{-T} \Delta^T P - P \Delta A^{-1} = -Z^T$. Therefore, the solution $X$ is also a skew-symmetric matrix. First of all, it should be underlined, that this choice for $F$ guarantees that $-F \in S^-$. Secondly, due to (i), Theorem 4.3., condition (8) for an unique solution $X$, for any right-hand side matrix is always satisfied for $\alpha \in \Omega_1$.

(a3) Computation of matrix $M(Y(\alpha))$ and vector $z(\alpha)$. Consider the uncertain matrix $\Delta$ in eq. (2) and eq.(9). For this special case, one has:

$$Y_i = 2I_i + \sum_{j=1}^{p} \alpha_j D_{ij} = -Y_i^T,$$

$$Z = \sum_{j=1}^{p} \alpha_j Z_j = \alpha = -Z^T,$$

$$D_{ij} = I_j A_j A^{-1} + A^{-T} A_j I_j,$$

$$Z_j = A^{-T} A_j P - P A_j A^{-1}.$$

The constant matrices $D_{ij}$ and $Z_j = -Z_j^T$, $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, p$, can be easily computed. The respective vector form (eq.(10)) of eq.(9) is

$$M[Y(\alpha)]x = [2I + M(\alpha)]x = z(\alpha).$$

The entries of $M(z)$ and $z$ are some linear functions of the uncertain vector $\alpha$.

Comments(1). The Lyapunov’s matrix, by means of which robustness is studied in this case is

$$R(\Delta) = FY = (X + P)(I + \Delta A^{-1}) \in P.$$

When Lyapunov’s stability theorem (4) is applied for $A + \Delta$ and $W = A$, one can easily verify, that

$$(A + \Delta A^{-1})^T R(\Delta) + R(\Delta)(A + \Delta A^{-1}) = (I + \Delta A^{-1})^T V(I + \Delta A^{-1}) < 0,$$

where $V = A^T (X + P) + (X^T + P)A < 0$, in accordance with Theorem 4.3. For all $\Delta \neq 0$, $R(\Delta)$ depends on the uncertain part. The only case when $R(\Delta)$ is a fixed, constant matrix is for $\Delta = 0$, since $x=0$ and $R(0) = P$.

(2). Requirement (iii), Theorem 4.3. imposes a restriction on the solution vector $x$. When eq.(13) is taken into consideration, one has to solve the vector inequality

$$2x^- \leq 2x = z(\alpha) - M(\alpha)x \leq 2x^+.$$ 

The solution to it, $\alpha \in \Omega_2$ is obtained by checking all extreme cases (so called corner vectors) for $\alpha$ and $x \in \Omega_x$. The solution to the overall problem (3) is given by $\alpha \in \Omega^- \equiv \Omega_1 \cap \Omega_2$.

(3). Matrix $M(Y)$ does not depend on the choice for matrix $Q$, respectively matrix $P$. Therefore, when a particular case is studied, changes may occur only in vector $z(\alpha)$.

(4). In the special case, when $L \Delta = T^\Delta L A$, where $T^\Delta$ denotes an upper (lower) triangular matrix, for some nonsingular matrix $L$, problem (i), Theorem 4.3. has an exact solution. It can be proved, that $M(Y)$ is also a triangular matrix, but due to lack of space, this is omitted.

6. EXAMPLES

6.1. Consider a second order uncertain system

$$A + \Delta = \begin{bmatrix} -1 + \alpha_1 & -1 + \alpha_2 \\ 1 - \alpha_3 & 0 \end{bmatrix},$$

$$A + \Delta \in S \iff \alpha_1 < 1, \alpha_2 < 1, \alpha_3 < 1.$$ 

Matrix $Y$ is computed as

$$Y = \begin{bmatrix} 1 - \alpha_2 & \alpha_1 - \alpha_2 \\ 0 & 1 - \alpha_3 \end{bmatrix},$$

$$-Y \in S \iff \alpha_2 < 1, \alpha_3 < 1.$$ 

Let $Q = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ in eq.(4). The scalar solution $(k=1)$ to eq.(11) is

$$x = (-2\alpha_1 + \alpha_2 + \alpha_3)(2 - \alpha_2 - \alpha_3)^{-1}.$$ 

Requirement (iii), Theorem 4.3. is met, if $x > -1$. The admissible subset $\Omega_2$ can be defined easily as $\alpha_1 < 1$. The final solution to this example is given by $\alpha_1 < 1, \alpha_2 < 1$ and $\alpha_3 < 1$, which is just the exact one.

6.2. Consider a third order uncertain system

$$(A^T P + \sum_{j=1}^{p} \gamma_j A^T I_j) \in S^-,$$

where $P \in \mathbb{P}$ is the unique solution to eq.(4) and $L$ is any nonsingular matrix. There exist various techniques for the purpose, e.g. (Boyed and Ghaoui, 1994; Gao and Antsaklis, 1993; Wang, et al. 1991, etc.)
\[
A + \Delta = \begin{bmatrix}
-1 + \alpha_1 & 0 & -1 \\
0 & -3 + \alpha_2 & 0 \\
-1 & -2 & -4 \\
\end{bmatrix},
\]

\[A + \Delta \in S \iff \alpha_1 < 1.75 \text{ and } \alpha_2 < 3.\]

For \(L = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = L^T, LL^T = I \) and \(Y = I + A^{-1} \Delta, \) one has

\[LYL^T = \begin{bmatrix}
1 - 0.333\alpha_2 & 0 & 0 \\
-0.0952\alpha_2 & 1 - 0.5714\alpha_1 & 0 \\
-0.0952\alpha_2 & 0.1429\alpha_1 & 1 \\
\end{bmatrix},\]

\[Y \in S \iff \alpha_1 < 1.75, \alpha_2 < 3.\]

For \(Q = (A^T A)^{1/2} \) in eq.(4), the vector form (eq.(10)) of eq.(9) is

\[
\begin{bmatrix}
-2 - 0.5714\alpha_1 & -0.333\alpha_2 & -0.0161\alpha_1 + 0.0573\alpha_2 \\
0 & 2 - 0.5714\alpha_1 & 0.0673\alpha_1 \\
0 & -0.0952\alpha_2 & 2 - 0.333\alpha_2 \\
\end{bmatrix}x = \begin{bmatrix}
0.503\alpha_1 + 0.357\alpha_2 \\
0.553\alpha_1 + 0.265\alpha_2 \\
0.1346\alpha_1 + 0.2356 \\
0.1236\alpha_2 \\
\end{bmatrix} \leq 12.
\]

It was found, that requirement (iii), Theorem 4.3. is guaranteed, if

\[
\begin{bmatrix}
-0.09 & -0.01178 \\
-0.06 & -0.01178 \\
0.03 & -0.01178 \\
\end{bmatrix} \leq x \leq \begin{bmatrix}
0.053\alpha_1 + 0.357\alpha_2 & \leq 1.66666 \\
0.553\alpha_1 + 0.265\alpha_2 & \leq 1.8; \\
0.1346\alpha_1 & \leq 0.2356; \\
0.199\alpha_2 & \leq 0.6; \\
0.1236\alpha_2 & \leq 1.2.
\end{bmatrix}
\]

The main contribution to system’s robustness study is due to Theorem 4.3., which extends the important results get by Theorem 4.1. and 4.2. for the class of systems considered here. The applicability of this approach and its superiority over some available ones is illustrated by two examples. It is believed, that the philosophy of the present approach can be used to define necessary and sufficient condition for robust stability for a nominal state matrix, influenced by a given structured perturbation uncertainty in terms of a parameter-dependent Lyapunov matrix.

\section*{REFERENCES}


