ENERGY OF FRACTIONAL ORDER TRANSFER FUNCTIONS

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Abstract: The objective of the paper is to compute the impulse response energy of a fractional order transfer function having a single mode. The differentiation order \( n \), defined in the sense of Riemann-Liouville, is allowed to be a strictly positive real number. A necessary and sufficient condition is established on \( n \), in order for the impulse response to belong to the Lebesgue space \( L_2[0, \infty) \) of square integrable functions on \([0, \infty)\).

Keywords: fractional order differentiation, impulse response energy, transfer function, dynamical system, fractional calculus

1. INTRODUCTION

Although as yet relatively unused in physics, the concept of differentiation to an arbitrary order (also called fractional differentiation) was defined in the 19th century by Riemann and Liouville. Their main concern was to extend differentiation by using not only integer but also non-integer (real or complex) orders. The \( n^\text{th} \) order derivative of fractional order is defined as (Samko, 1993):

\[
(D^n x)(t) = \frac{1}{\Gamma(n-m)} \left( \frac{d}{dt} \right)^{n-m} \int_0^t \frac{x(\tau)d\tau}{(t-\tau)^{n-m}}
\]

where \( t > 0, n > 0 \) and \( m = \lfloor n \rfloor + 1 \). \( \lfloor n \rfloor \) means integer part of \( n \).

Studies on real systems such as thermal or electrochemical (Battaglia et al., 2000, Cois et al., 2000), reveal inherent fractional differentiation behaviour. The use of classical models (based on integer order differentiation) is thus inappropriate in representing these fractional systems. Thus, a further class (called fractional) of mathematical models has been provided since 1983 by (Oustaloup) using the concept of fractional differentiation. These models are based on either a fractional differential equation or a fractional state-space representation, namely (Matignon, 1994, Oustaloup, 1995, Cois et al., 2001):

\[
\begin{align*}
\left\{ D^n x(t) \right\} &= A x(t) + B u(t) \\
y(t) &= C x(t) + D u(t)
\end{align*}
\]

where \( n \) is a real, integer or non integer, number.

The modal decomposition of such a representation leads to express the system output as a linear combination of elements called eigenmodes governed by the following equation:

\[
D^n x(t) + \lambda x(t) = u(t),
\]

where \( \lambda \) denotes a system eigenvalue. This latter equation can also be written (see (Oldham and Spanier, 1974) for example) in the Laplace domain as:

\[
B_n(s) = \frac{X(s)}{U(s)} = \frac{1}{s^n + \lambda}
\]

One of the inherent system characteristics in control engineering is its impulse response energy. For instance, if the energy is finite, it can be concluded that a system belongs to \( L_2[0, \infty) \), space of squared integrable functions.

Our concern in this paper is to compute the impulse response energy of \( B_n(s) \) whatever the differentiation order \( n \) is. Up to our knowledge, the method presented herein is original since no work
was found in the literature concerning the computation of impulse response energies of fractional order systems.

Based on the method proposed herein, other types of fractional order transfer functions can be treated.

2. FRACTIONAL STATE-SPACE REPRESENTATION

2.1. Definition and main properties

Definition. Fractional state-space representation is defined, as for classical state-space representation, by two equations (Matignon 1994, Oustaloup 1995):

- a state equation where the state vector is differentiated to a real, that is integer or non-integer, order. This vector has different properties compared to a classical state vector and is termed fractional state vector;
- an output equation, as in classical representation:

\[
\begin{align*}
(D^n x)(t) &= A x(t) + B u(t) \\
y(t) &= C x(t) + D u(t)
\end{align*}
\] (2)

The \( n \) order derivative of the fractional state vector \( x(t) \) is defined by (Samko et al., 1993):

\[
(D^n x)(t) = \frac{1}{\Gamma(n-m)} \frac{d^m}{dt^m} \int_0^t \frac{x(\tau)}{(t - \tau)^{n-m}} d\tau,
\] (3)

with \( \tau > 0, n > 0 \) and \( m = \lfloor n \rfloor + 1 \).

The Laplace transform of this derivative (Oldham and Spanier, 1974), considering null initial conditions, is:

\[
\mathcal{L}\{D^n x(t)\} = s^n X(s), \quad \text{where } X(s) = \mathcal{L}\{x(t)\}.
\] (4)

Scalars \( u(t) \) and \( y(t) \) are system input and output respectively. \( x(t) \) is the fractional state vector. \( A, B, C \) and \( D \) are suitable dimension matrices.

2.2. Modal decomposition

The modal decomposition of complex-fractional systems is obtained as for classical systems. Applying a similarity transformation to representation, one can obtain a new realization, where matrix \( A \) becomes Jordan matrix \( J \):

\[
\begin{align*}
(D^n x)(t) &= J x_J(t) + B_J u(t) \\
y(t) &= C_J x_J(t) + E_J u(t)
\end{align*}
\] (5)

The system eigenvalues are on the diagonal of matrix \( J \). Using relation (4) and considering null initial conditions, the Laplace transform of the complex-fractional state equation is:

\[
s^n x_J(s) = J x_J(s) + B_J u(s).
\] (6)

Then, from relation (5), system output is:

\[
y(t) = \mathcal{L}^{-1} \left[ C_J \left( s^n I - J \right)^{-1} B_J \right] \otimes u(t) + E_J u(t),
\] (7)

where \( \otimes \) denotes the convolution product. Matrix \( J \) is a Jordan matrix, so \( \left( s^n I - J \right)^{-1} \) can be expressed as:

\[
\begin{bmatrix}
\frac{1}{s^n - \lambda_1} & 0 & \cdots & 0 \\
0 & \frac{1}{s^n - \lambda_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \frac{1}{s^n - \lambda_l}
\end{bmatrix}
\] (8)

with

\[
\lambda_l = \arg \left\{ \lambda_j \right\} + \frac{2\pi}{n} \left( k + \frac{n}{2} \right), \quad k = 0, 1, \ldots, n - 1,
\] (9)

System output \( y(t) \) is thus a vector composed of a linear combination of partial fractions called eigenmodes:

\[
L^{-1} \left[ \frac{1}{s^n - \lambda_j} \right] \otimes u(t) \quad \text{where } q_i \text{ is an integer number computed from the algebraic multiplicity of each eigenvalue } \lambda_j.
\]

2.3. Output analytical expression

Important results, obtained by Oustaloup and Matignon permit the analytical expression of fractional system output. Using the Mellin-Fourier inverse transformation and the residue theorem, the fractional system output is:

\[
y(t) = \mathcal{L}^{-1} \left[ \frac{1}{s^m} \right] \otimes u(t) + \sum_{k=1}^{\text{pole number}} \frac{P_k}{\lambda_k q_i - 1} \left( \frac{1}{n} \right)^p_k e^{\theta_k t}
\]

\[
+ \frac{1}{\Gamma(n-m+1)} \int_0^\infty \frac{e^{-\xi t} \sum_{k=0}^{q_i-1} (-1)^k \left( \frac{\theta_k}{n} \lambda_k \right)^k x^{\theta_k}(x/k) \sin[n\theta_k (x/k)]}{x^{2n-2m} - 2\lambda_x x^m \cos[n\theta_k + \lambda_x^2]^{\theta_k}} dx
\]

where the poles \( p_k \) of \( H(s) \) are defined by:

\[
\left| p_k \right| = \left| \lambda_k \right|^{1/n} \quad \text{with } \theta_k = \frac{\arg \left\{ \lambda_k \right\}}{n} + \frac{2\pi}{2n} \left( k + \frac{n}{2} \right), \quad -\frac{n}{2} < k < \frac{n}{2} - \frac{\arg \left\{ \lambda_k \right\}}{2\pi}
\] (10)

and where \( Q_k(x,y) \) is a 2 variable polynomial defined.
by:

\[
\begin{cases}
Q_0(x,y) = x \\
KQ_\epsilon(x,y) = (xy + x - y)Q_{\epsilon - 1}(x,y) + xy \frac{\partial}{\partial y} Q_\epsilon(x,y)
\end{cases}
\] (12)

Equation (10) shows the usual decomposition of fractional systems into two parts (Oustaloup, 1983):
- the exponential mode, resulting from the computation of residue(s) on each pole of \(H_l(s)\) which generates an exponential behavior;
- the aperiodic multimode, the main characteristic of fractional systems, resulting from an integral along the negative real axis.

2.4. Stability condition

BIBO stability (Bounded Input-Bounded Output) is considered. A sufficient condition of this stability is given by:

\[
\int_0^{\infty} |H(s)|^2 ds < K < \infty,
\] (13)

where \(H(s)\) is the system transfer function. Using the structural decomposition of fractional systems (10), the stability condition is given by (Matignon, 1994):

\[
\arg(\lambda_l) > \frac{\pi l}{2}, \text{ for } l = 1, \ldots, \dim(x),
\] (14)

where the \(\lambda_l\) are the eigenvalues of matrix \(A\).

3. MAIN RESULT

The main result of the paper is announced in the following theorem:

**Theorem**

Let \(B_n(s) = \frac{1}{s^n + \lambda}\) be a Laplace-domain transfer function defined for every \(n \in \mathbb{R}^+\) and for every \(\lambda \in \mathbb{R}^+\); then the Euclidean norm of \(B_n(s)\) squared is:

\[
\|B_n(j\omega)\|^2 = \frac{\frac{1}{n^2}}{\sin\left(\frac{\pi}{n}\right)} \cot\left(\frac{\pi}{n}\right)
\]

if \(\frac{1}{2} < n < 2\) and \(n \neq 1\) (15a)

\[
\|B_n(j\omega)\|^2 = \frac{1}{2\lambda}
\]

if \(n = 1\) (15b)

\[
\|B_n(j\omega)\|^2 = \infty
\]

if \(0 \leq n \leq \frac{1}{2}\) or \(n \geq 2\) (15c)

**Proof**

See appendix.

As a direct consequence, the following corollary is deduced.

**Corollary**

Let \(b_n(t)\) be the impulse response of \(B_n(s) = \frac{1}{s^n + \lambda}\), then \(b_n(t) \in L_2(0, \infty)\) if and only if \(\frac{1}{2} < n < 2\).

Where \(L_2[0, \infty]\) is Lebesgue space of squared integrable functions on the interval \([0, \infty]\).

**Remark**

It is worth mentioning that the discontinuity at \(n = 1\) in formula (15a) is removed by formula (1b). In other words, according to (1a) and (1b):

\[
\lim_{n \to 1^-} \|B_n(j\omega)\|^2 = \lim_{n \to 1^+} \|B_n(j\omega)\|^2 = \|B_1(j\omega)\|^2
\]

4. PLOT AND INTERPRETATION

It can be verified that \(\|B_n(j\omega)\|^2\) is minimal for \(n = 1\), by solving \(\frac{d}{dn} \|B_n(j\omega)\|^2 = 0\).

\(\|B_n(j\omega)\|^2\) is plotted versus \(n\) in the following figure:

![Fig 1 – Energy versus differentiation order](image)

As can be seen the energy of \(B_n(j\omega)\) tends to infinity as \(n\) tends to 2 which is due to the poles of \(B_n(j\omega)\). Plotting pole locus versus \(n\), it can be seen that the 2 complex conjugate poles of \(B_n(s)\) tend to the \(j\omega\) axis as \(n\) tends to 2. Hence, the corresponding impulse response becomes more and more oscillatory. When \(n\) reaches 2, the system becomes pseudo-stable and hence its energy infinite.

Beyond \(n = 2\), the system is instable, because the number of poles of \(B_n(s)\) is greater or equal to four, two of which at least are instable.

On the other hand, when the transfer function order is less than one, (Oustaloup, 1995, p.155 eq.4.83) shows that \(B_n(s)\) has no pole and that the time domain impulse response of \(B_n(s)\) is written as:
Moreover, it was shown that $B_n(s)$ belongs to $L_2[0,\infty[$ if and only if $1/2 < n < 2$. Actual work goes towards the generalization of this result to any fractional transfer function.

![Fig. 2 – Pole locus versus differentiation order](image1)

![Fig. 3 – Impulse responses for three values of $n$ with $\lambda = 1$](image2)

6. REFERENCES


7. APPENDIX - PROOF OF THE THEOREM

In this correspondence, the impulse response energy of the fractional order transfer function $B_n(s) = (s-\lambda)^n$ was computed for all values of $n$. Energy of $B_n(s)$ can be written either in the time or
frequency domain by applying Parseval’s theorem which is valid only for stable systems. The stability of \( B_n(s) \), depending on the number of poles, was widely studied in (Oustaloup, 1995). The conclusions are reported below in terms of the poles of the transfer function \( B_n(s) \):

- If \( 0 < n < 1 \) ➔ no pole,
- if \( n = 1 \) ➔ 1 pole,
- if \( 1 < n < 2 \) ➔ 2 complex conjugate poles,
- if \( n = 2 \) ➔ 2 complex conjugate poles on the imaginary axis. The system is pseudo-stable and hence its energy infinite.
- if \( n > 2 \) ➔ at least 4 poles two of which at least have positive real parts. i.e. the system is unstable.

\( B_n(s) \) is a multivalued complex function if \( n \neq 1 \), hence it is important, at this stage to cut the complex plane \( \mathbb{C} \) on \([-\infty, 0] \). The arguments of \( s \) are allowed to vary in the interval: \([-\pi, \pi] \).

Consequently, the norm of \( B_n(s) \) will be computed only for \( 0 < n < 2 \) and \( n \neq 1 \).

The energy of \( b_n(t) \) is defined in the time domain as:

\[
\|b_n(t)\|^2 = \int_{-\infty}^{\infty} |b_n(t)|^2 dt
\]

where \( \overline{b_n(t)} \) is the complex conjugate of \( b_n(t) \).

Applying Parseval’s theorem, for stable systems, yields the following:

\[
\|B_n(j\omega)\|^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{1}{\omega^2 + \lambda^2} \right|^2 d\omega
\]

or due to the symmetry in the frequencies:

\[
\|B_n(j\omega)\|^2 = \frac{1}{\pi} \int_{0}^{\infty} \frac{1}{\omega^2 + \lambda e^{-j\pi/2}} d\omega
\]

Letting \( \omega = x \), and \( d\omega = \frac{1}{n} x^{n-1} dx \), yields the following integral to solve:

\[
\|B_n(j\omega)\|^2 = \frac{1}{n\pi} \int_{0}^{\infty} x^{n-1} \left( x^2 + 2\lambda \cos\left( \frac{\pi n}{2} \right) x + \lambda^2 \right) dx
\]

The remaining part of the appendix is devoted to presenting the method used for solving (16).

By analogy to (16) define a complex valued function \( G(z) \) as:

\[
G(z) = \int_{0}^{1} \left( z - \lambda e^{j\pi/2} \right)^{n-1} \left( z - \lambda e^{-j\pi/2} \right)\frac{1}{z^n} dz
\]

Set:

\[
I = \int_{0}^{1} G(z) dz
\]

The integral (16) is now equivalent to

\[
\|B_n(j\omega)\|^2 = \frac{1}{n\pi} I.
\]

For sake of simplicity a new plane cut is defined in the complex \( \mathbb{C} \) plane:

- it excludes from \( \mathbb{C} \) the axis \([0, +\infty[\),
- arguments of complex numbers are allowed to vary in \([0, 2\pi[\). Hence, the second pole of \( G(z) \) reads \( \lambda e^{j\pi/2 - \pi/2} \).

To compute \( I \), it is preferred to evaluate the integral of \( G(z) \) along the contour \( \Gamma \) of fig.3 and the residues inside.

\[
I = \int_{\Gamma} G(z) dz
\]

Applying residue theorem, the integral along \( \Gamma \) reads:

\[
I = 2\pi j \sum \text{Res}(G(z), \text{poles})
\]

A.1 Residues inside the closed contour

\[
I_r = 2\pi j \sum \text{Res}(G(z), \text{poles})
\]

A.2 Integral along the small circle \( \gamma \)

Set: \( z \in \gamma \Leftrightarrow z = e^{j\theta}, \theta \) varies from \( 2\pi \) to \( 0 \),

\[
dz = ej e^{j\theta} d\theta
\]
\[ I_e = \oint G(z)dz \]
\[ I_e = \int_0^{2\pi} \frac{e^m}{\lambda e^{\frac{\lambda}{2}} - e^{\frac{\lambda}{2}}} d\theta \]

If \( e \to 0 \):
\[ I_e = \frac{me}{\lambda} \left[ 1 - e^{\frac{2\pi}{n}} \right] \]

Consequently, \( \lim_{e \to 0} I_e = 0 \) if \( n > 0 \) which condition is always satisfied.

### A.3 Integral along the big circle \( \gamma_R \)

\[ I_R = \oint G(z)dz \]
\[ z \in \gamma_R \implies z = R e^{\lambda}, \ \theta \text{ varies from } 0 \text{ to } 2\pi, \ dx = R e^{\lambda} d\theta \]
\[ I_R = \int_0^{2\pi} \frac{R e^{\lambda}}{R e^{\lambda} - \lambda e^{\frac{\lambda}{2}}} d\theta \]

If \( R \to \infty \):
\[ I_R = \frac{1}{n - 2} R^{\frac{1}{n} - 1} \left( e^{\frac{1}{n} - 1} - 1 \right). \]

Hence,
\[ \lim_{R \to \infty} I_R = 0 \text{ if } n > \frac{1}{2} \]

and \( \lim_{R \to \infty} I_R = \infty \text{ if } \frac{1}{2} \geq n > 0 \)

### A.4 Integral along positive and negative segments

Let \( \gamma_+ \), be the segment \( z = x, \ \varepsilon < x < R, \ dx = dx \).
\[ I_+ = \oint G(z)dz \]

Note that:
\[ \lim_{\varepsilon \to 0} I_+ = I \text{ where } I \text{ is the desired integral of eq. (17).} \]

Let \( \gamma_+ \) be the segment \( z = x e^{2\pi j}, \ R > x > \varepsilon, \ dx = dx \).

### A.5 Deducing \( I \)

\[ I_r = I_g + I_e + I_+ + I_- = 2\pi \sum \text{Res}(G(z), \ pôles) \]

Replacing each term by its computed value yields:
\[ I = \pi \lambda^2 \left( \frac{\lambda}{2} - 1 \right) \left( e^{\frac{\lambda}{2}} - e^{\frac{\lambda}{2}} \right) \]

Or after some tedious simplifications:
\[ I = -\pi \lambda^2 \left( \frac{1}{n^2} - 1 \right) \frac{\cot \left( \frac{\pi n}{2} \right)}{\sin \left( \frac{\pi n}{2} \right)} \]

Back to the norm: \( \| B_n(j\omega) \|^2 = \frac{1}{n\pi} I \)

\[ \| B_n(j\omega) \|^2 = -\lambda^2 \left( \frac{1}{n^2} - 1 \right) \frac{\cot \left( \frac{\pi n}{2} \right)}{n \sin \left( \frac{\pi n}{2} \right)} \]

when \( \frac{1}{2} < n < 2 \) and \( n \neq 1 \)

\[ \| B_n(j\omega) \| = \infty \text{ for } 0 < n \leq \frac{1}{2}. \]

And due to the instability of \( B_n(x) \) when \( n \geq 2 \):
\[ \| B_n(j\omega) \| = \infty \text{ for } n \geq 2. \]

### A.6 Special case \( n = 1 \)

In this case the integral to solve is much easier:
\[ \| B_n(j\omega) \|^2 = \frac{1}{2\pi} \int \frac{d\omega}{(\omega + \lambda j)(\omega - \lambda j)} \]

The solution is obtained by evaluating residues inside a closed contour without any plane cut. The result is straightforward:
\[ \| B_n(j\omega) \|^2 = \frac{1}{2\lambda} \]